

CAM CANS: COMPUTER-AIDED MATHEMATICS AND PACKING RIGHT CIRCULAR CYLINDERS

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1. INTRODUCTION

At Tarleton State University, one of the learning objectives for the mathematics program is the use of technology. This learning objective is introduced in Calculus I, reinforced in Calculus II, and mastered in both Calculus III and Differential Equations. The technology learning objective covers the use of both graphing calculators and computer algebra systems (CAS) software. This report describes a particular lab, using *Mathematica*, that partially addresses the technology learning objective in Calculus III.

2. OVERVIEW OF CALCULUS III AND CAS RESOURCES AT TARLETON STATE UNIVERSITY

Calculus III is taught on a five-day-a-week, 15-week semester. The students are predominately sophomore and junior mathematics, engineering, or science majors. The course is designed as the culmination of an early transcendental calculus sequence using chapters 12-16 of Stewart [1] or the equivalent material in OpenStax [2]. Through the Texas A&M system, the students have free access to the CAS software *Mathematica* in all open-access computer labs as well as the ability to install the software on their personal computers or to use an on-line version.

Calculus III is an advanced course that deals with complex concepts and techniques, including partial derivatives, multiple integrals, and vector calculus. As an advanced course, it can be challenging to teach since the concepts and techniques involved can be abstract. Students often find the material difficult to visualize, and instructors may struggle to keep students engaged and motivated. However, research has shown that incorporating applied optimization problems, like Lagrange Multipliers, in combination with technology like a CAS, can improve student engagement, deepen their understanding of the material, and develop their problem solving skills.

In this lab, we solve the classic can problem of finding the dimensions of a right-circular cylinder of fixed volume and minimal surface area, by using Lagrange Multipliers. We use a CAS to graph the cans of optimal dimensions determined the constraints, layout of the cans in a box, and the cost function. By applying these

concepts to a real-world problem, we are able to see the practical applications of Calculus III concepts in a tangible way.

One of the main benefits of integrating technology and a CAS in teaching Calculus III is that it allows students to visualize and interact with the material in a more dynamic way. Technology provides tools and software that can help students better understand complex concepts and techniques, allowing them to experiment with different scenarios and explore solutions more easily while reducing computational errors. The activity utilizing the Manipulate command [3] in *Mathematica* provides students with an exploratory experience. Thus, they can develop a deeper understanding and intuition of the material, focus on the concepts, and apply these concepts in real-world situations.

3. ASPECTS OF THE TECHNOLOGY LAB

3.1. One Can, No Box: The problem of finding the optimal right-circular cylinder of fixed volume may easily be solved using Lagrange Multipliers. If we consider a right-circular cylinder of height h and base radius r , then the problem is to minimize the surface area: $S(r, h) = 2\pi r^2 + 2\pi r h$, subject to the constraint that the volume is fixed: $\pi r^2 h = 1$ (1 unit of volume). We will use a volume of 2π for convenience in the following example of a single can and no box. The constant used for the volume will not affect the ratio of h/r for the optimal can, but will affect the values of r and h .

A visual representation of this optimization problem is that we are looking for values of r and h where a level curve of surface $S(r, h)$ is tangent to the constraint curve. In *Mathematica*, we can illustrate this relationship by the graphic shown in Figure 1.

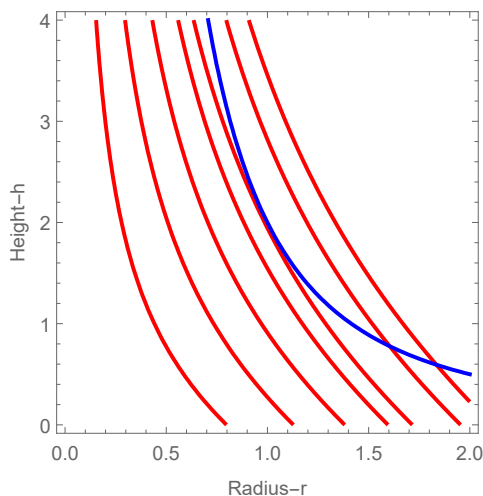


FIGURE 1. Level curves of S -red, constraint-blue.

We see that the point of tangency is when $r = 1$ and $h = 2$, indicating that the optimal can has height equal to the diameter of the base. This relationship is easily confirmed in the lab by using the Derivative and Solve commands in *Mathematica* to

do the algebra and calculus of the Lagrange Multiplier process. A graph of this type of can is shown in Figure 2.

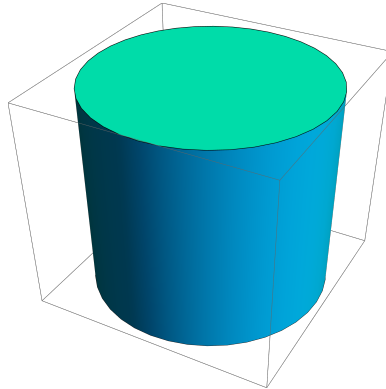


FIGURE 2. Can where height equals diameter of the base.

3.2. Multiple Cans, A Box, and Money: Now we will consider the situation of multiple cans packaged in a rectangular box and include costs for the the material of the cans and for the box. We will consider the problem of stacking cans in rows, columns and layers. Figure 3 is an example with 4 rows, 3 columns and 2 layers of cans where the height is equal to the diameter of the base of each can.

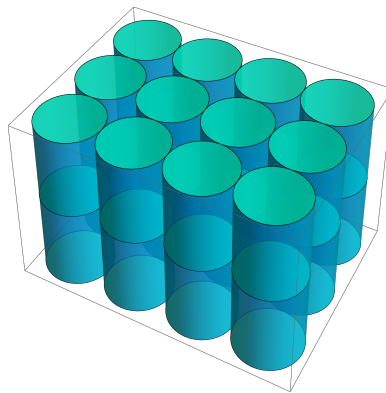


FIGURE 3. Stacking Cans.

We will use a , b , and c to represent the number of rows, columns and layers of cans. We assume that it costs 1 monetary unit per unit area for the cans and p units per unit area for the box. The total surface area for all of the cans combined will be

$$abc(2\pi r^2 + 2\pi rh).$$

The total surface area of the box will be the sum of the six faces of the box (top, bottom and the four sides), which depends on the radius, height and configuration of the cans giving a total surface area of

$$2(2ar)(2br) + 2(2ar + 2br)(ch) = 8abr^2 + 4c(a + b)rh.$$

So the cost function for the cans-in-a-box problem becomes:

$$\text{cost}(r, h, a, b, c, p) = abc(2\pi r^2 + 2\pi r h) + p(8abr^2 + 4c(a + b)rh).$$

The constraint for this optimization problem is still that the volume of each can is fixed at 1 unit: $\pi r^2 h = 1$.

In the lab, a Module [4] is created to solve this constrained optimization problem and the Manipulate command [3] is used to graph the cans and calculate the ratio h/r . Two examples of the results of this process are shown in Figure 4 and Figure 5.

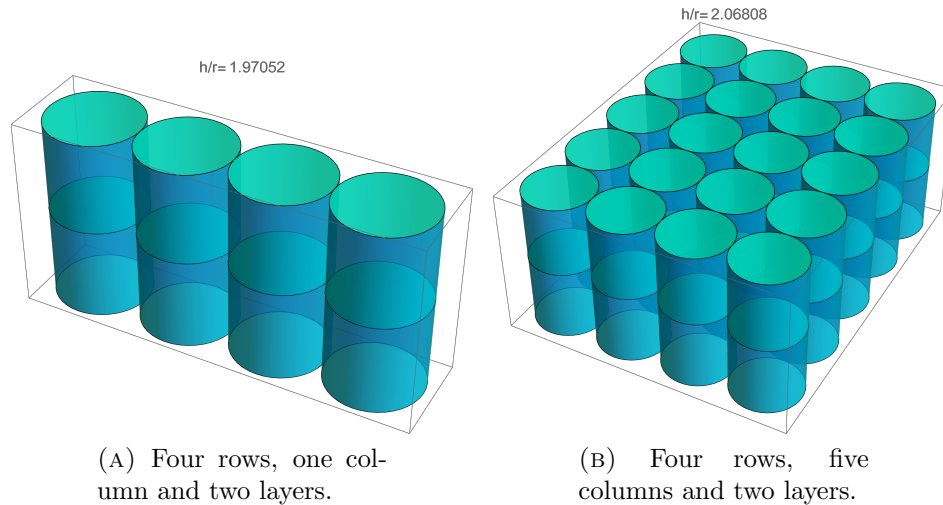


FIGURE 4. Cost of the box is $p = 1/10$.

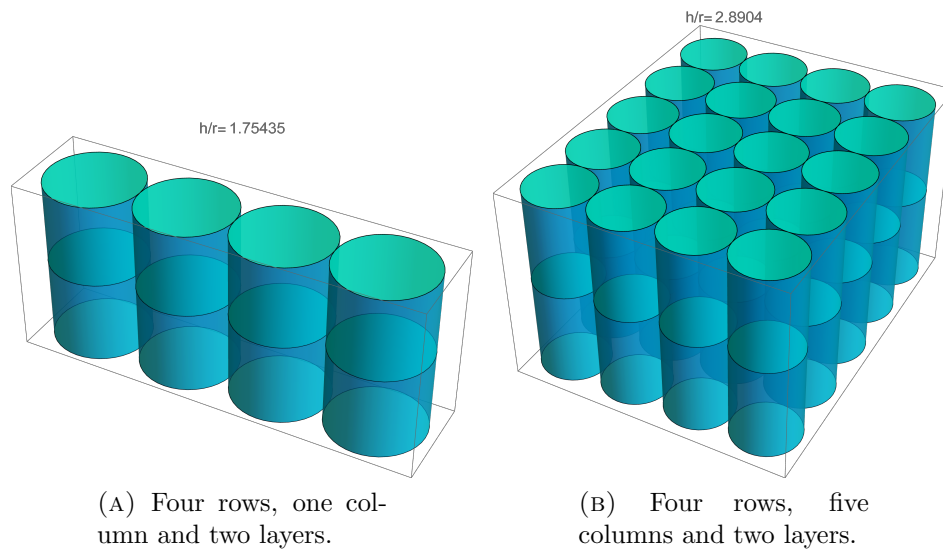


FIGURE 5. Cost of the box is $p = 2$.

In Figure 4, the cost of the box material is one-tenth the cost of the cans. Therefore, the problem tries to produce h/r ratios close to 2 (the optimal can). In Figure 5, the cost of the box is twice the cost of the cans, so the box is optimized and the h/r tries to produce a square box.

4. IMPACT OF TECHNOLOGY LABS AND CONCLUSION

The use of technology labs gives the students in Calculus III the opportunity to spend more time on topic, explore more complicated and real world problems, as well as becoming familiar with professional level software used in industry. Based on two decades of student observations, the authors note that there is a positive correlation between successfully completing technology labs and success in Calculus III.

Technology labs may be viewed using the Wolfram player [5]. This lab and more examples may currently be obtained by contacting the lead author and should soon appear in the Calculus III portion of his website [6]. All technology labs on the author's website may be freely used. Suggestions for improvements are welcome. The authors would like to thank the mathematics faculty at Tarleton State University for their input over the years that have contributed to these labs.

REFERENCES

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