COMPUTER AIDED MATHEMATICS IN ORDINARY DIFFERENTIAL EQUATIONS

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1. Introduction

At Tarleton State University, one of the learning objectives for the mathematics program is the use of technology. This learning objective is introduced in Calculus I, reinforced in Calculus II, and mastered in both Calculus III and Differential Equations. The technology learning objective covers the use of both graphing calculators and computer algebra systems (CAS) software. This report describes a particular set of labs, using *Mathematica*, that partially addresses the technology learning objective in Differential Equations.

2. Overview of Differential Equations and CAS resources at Tarleton State University

Differential Equations is taught on a three-day-a-week, 15-week semester. The students are predominately sophomore and junior mathematics, engineering, or science majors. The course is designed with a prerequisite of an early transcendental calculus sequence using the single variable part of Stewart [1] or the equivalent material in OpenStax [2]. One of the books used by these authors for the Ordinary Differential Equations (ODE) course is by Boyce and DiPrima [3]. Through the Texas A&M system, the students have free access to the CAS software *Mathematica* in all open-access computer labs as well as the ability to install the software on their personal computers or to use an on-line version.

Differential Equations is an advanced course that deals with complex concepts and techniques such as first-order and multi-order ODE's, initial value problems, exact equations, series solutions, Laplace Transforms and an introductions to systems of equations. As an advanced course, it can be challenging to teach since the concepts and techniques involved can be abstract. Students often find the material difficult to visualize, and instructors may struggle to keep students engaged and motivated. However, research has shown that incorporating applied problems, like numerical approximations to solutions and graphing, in combination with technology like a CAS, can improve student engagement, deepen their understanding of the material, and develop their problem solving skills.

In this set of labs, we apply classic numerical techniques to an initial values problem that presents some challenging behavior. While the problem can be solved exactly, the purpose of the labs is to provide the students with tools that can be applied to a diverse range of problems and point out some of the possible difficulties that can arise in numerical approximations. In the latter part of the labs, students apply these techniques and graphs to investigate the dynamics of solutions to systems of ordinary differential equations.

3. Aspects of the Technology Labs

3.1. **Pedogogy.** For the technology learning objective, the mathematical concepts of numerical approximations and the use of CAS are introduced in the classroom. The in-class computer and projector system are used, with *Mathematica*, to look at built-in differential equation solvers. These will be used as exact solutions or "highly accurate" approximation for the purpose of comparing introductory numerical techniques for approximating solutions.

Programing in *Mathematica* using Modules [4] is explored first using Euler's Method (Lab 1), then Modified Euler's Method (Lab 2) and finally higher order, fixed step Runge-Kutta methods and systems of ordinary differential equations (Lab 3). Example's and basic code are discussed in the classroom. The students are then asked to implement the methods on their own and test their implementation on a problem designed to highlight some of the complications that can arise when approximating solutions.

3.2. **The Initial Value Problem.** For the assignment, the students are asked to investigate an initial value problem that exhibits complex behavior. The chosen problem is

$$\frac{dy}{dx} = e^x \cos(y), \;\; y(0) = -\frac{3}{2}, \;\; 0 \le x \le 4.$$

Students are tasked with: noting the equilibrium solutions, generating a directions field on an appropriate window, approximate the solution using several step sizes (chosen to be relatively large), graph the results and comment on the how the approximations change as the step size decreases. They are then tasked with comparing the "best" of the approximations with the exact solution. The product produced by the students is a professional report using the various aspects of the *Mathematical* notebook interface. This report should include not only the command code used, but also text of the students observations. The students may collaborate with each other on the labs, but must produce individual reports.

3.3. **The Results.** Technology should not be necessary to find the equilubrium ($y = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, ...$). *Mathematica* can be used to generate the directions field, shown in Figure 1. Based on the direction field, the solution to the initial value problem should initially increase and then become asymptotic to the equilibrium $y = \frac{\pi}{2}$.

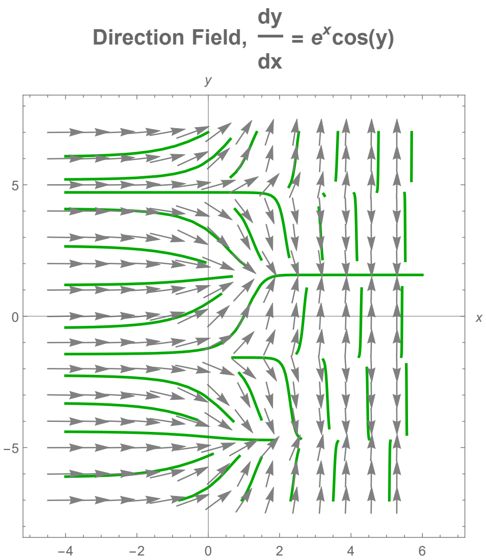

FIGURE 1. Direction Field with Steam Lines in green.

The students are then tasked with applying Euler's Method to approximate the solution using step sizes of 1, 0.5, 0.25 and 0.125. the graphs of these approximations are shown in Figure 2. As the step size decreases, the approximation appears to approach the actual solution. But for larger step sizes, the approximation seems to jump away from the true solution as x increases.

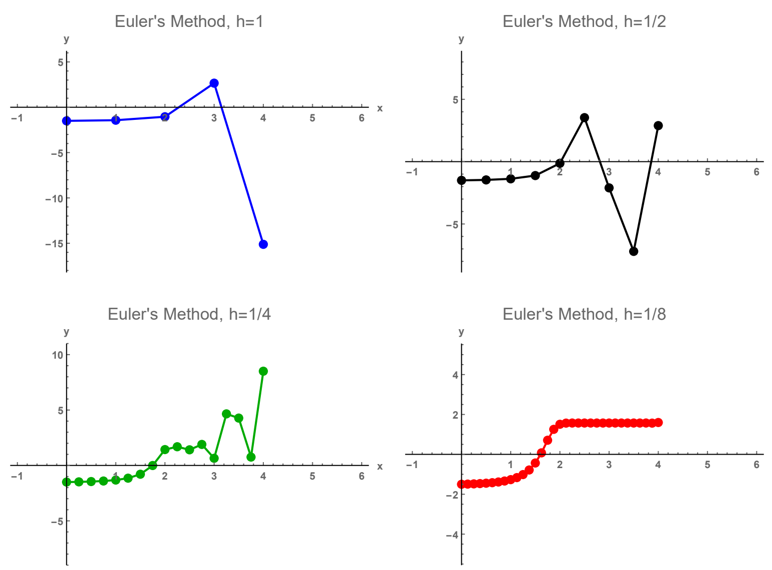

FIGURE 2. Approximations using Euler's Method with various step sizes (h).

In Lab 2, Modified Euler's Method produces the approximations shown in Figure 3. In this case, the theoretically more accuare Modified Euler's method does not appear to do as good a job as Eluer's Method did in Lab 1. In fact, when comparing to the "exact" solution, Modified Euler's Method does do a better job of approximating the solution on the approximate interval $0 \leq x \leq 2.5$ (see Figure 4 and 5). It is only when the direction field indicates near vertical slopes that the method breaks down.

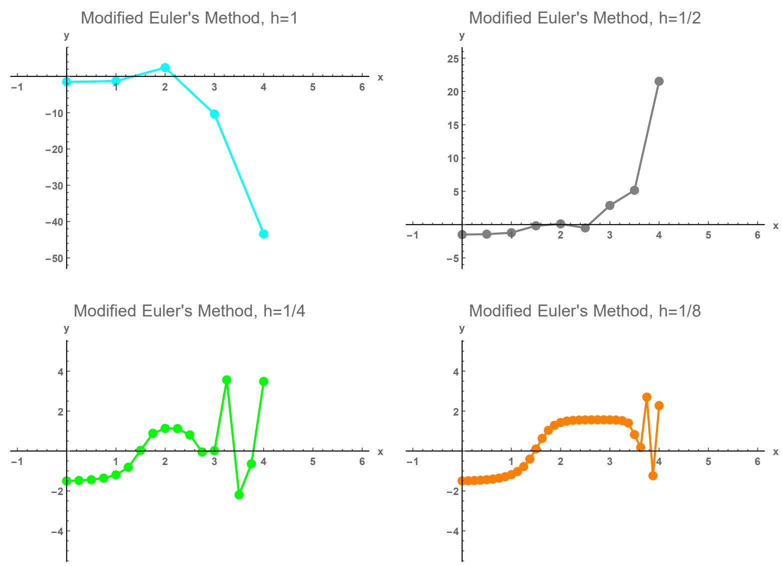

FIGURE 3. Approximations using Modified Euler's Method with various step sizes (h).

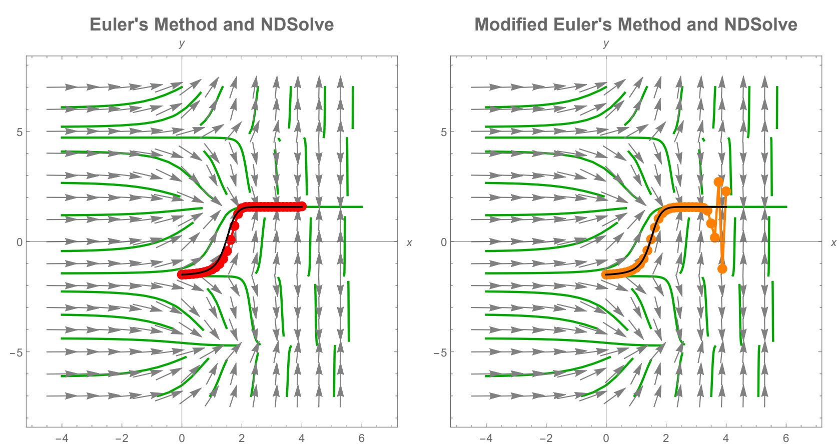

FIGURE 4. Comparison with "exact" solution for $h = 0.125$.

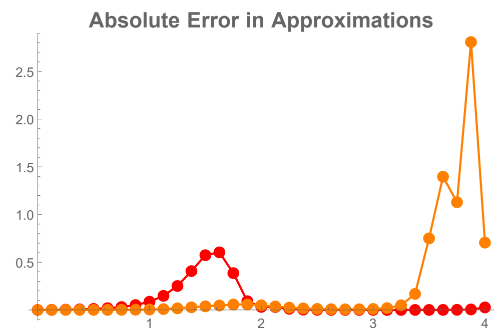

FIGURE 5. Modified Euler's Method does better on $1 \leq x \leq 2$, but Euler's Method was more stable for $x > 3$ (where the tangents were nearly vertical).

In Lab 3, the higher order Runge-Kutta methods exhibit this same behavior of being more accurate for a longer interval in x, but losing stability as x approaches the value of 4. Of course, reducing the step size further would increase the accuracy on all three of the methods, but the problem of stability will still appear for larger x values. Lab 3 also includes examples and tasks on implementing the higher order Runge-Kutta methods on systems of first order ODE's with initial conditions.

4. Impact of Technology Labs and Conclusion

The use of technology labs gives the students in a Differential Equations course the opportunity to spend more time on topic, explore more complicated and real world problems, as well as becoming familiar with professional level software used in industry. Based on two decades of student observations, the authors note that there is a positive correlation between successfully completing technology labs and success in the course.

Technology labs may be viewed using the Wolfram player [5]. These labs and more examples may currently be obtained by contacting the lead author and should soon appear in the Differential Equations portion of his website [6]. All technology labs on the author's website may be freely used. Suggestions for improvements are welcome. The authors would like to thank the mathematics faculty at Tarleton State University for their input over the years that have contributed to these labs.

References

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