CONCEPTUAL DEVELOPMENT OF LIMITS WITH DELTA-EPSILON? PRECISELY!

Beth Riggs
Tarleton State University
Department of Mathematics
Box T-0470
Stephenville, TX 76402
eriggs@tarleton.edu

Nancy Summer
Tarleton State University
Department of Mathematics
Box T-0470
Stephenville, TX 76402
summer@tarleton.edu

Abstract

Students often struggle with the precise definition of the limit in Calculus I. They seem to get caught up in the symbols and the algebraic manipulation associated with absolute value, and the conceptual understanding can be overshadowed by the mechanics. Even those students who are successful on tests may just be mimicking procedures and not really understand the concept.

It is important for students to visualize the relationship between output tolerance limits and the corresponding input values allowed to maintain those tolerances. However, many students are intimidated by the notation and lose sight of the practical applications of precise limits. This paper describes a scaffolded approach to teach this content using an "Experience First, Formalize Later" (EFFL) pedagogical structure (Wilcox, 2023) by presenting the students with a concrete example initially and allowing them to build tools to define and explore those limits. Students utilize technology where appropriate, along with honing their problem-solving skills.

Introduction

Many of the students that we encounter in Calculus 1 are engineering majors who need to see the relevance and application of the content to problems they will be expected to solve in their chosen career fields. Granted, this need to see connections is not limited to engineering majors – students of all majors benefit when they can see how content and applications connect. By using visualization technology to investigate aspects of the problem, students can explore numerical solutions and formalize what they learn once they fully understand the problem and solution.

As an example, students often struggle with the precise definition of the limit in Calculus I. This precise definition is stated as follows: Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that "the limit of $f(x)$ as x approaches a is L, and we write $\lim_{x \to a} f(x) = L$ if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that if $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$" (Stewart et al., 2021, p. 106).

Students can get caught up in the symbols and the algebraic manipulation associated with absolute value, and the conceptual understanding is overshadowed by the mechanics. Even those students who are successful on tests may just be mimicking procedures and not really understand the concept.

It is important for students to be able to visualize the relationship between output tolerance limits and input allowed values to maintain those tolerances. However, many students are intimidated by the notation and lose sight of the actual practical applications of precise limits. By posing problems to the students initially that use contexts grounded in real-world examples (Adiredja, 2021), combined with allowing them to build visualization tools to define and explore those limits, we can equip our students with the problem-solving tools they need.

In this paper, we address this issue by describing how we teach this content. Our goal is to tie this abstract concept to real-world applications and problem-solving before moving into any symbolic manipulation. In addition, we customize our applications to the students' majors, which means that many of our applications stem from the field of engineering. By connecting the ideas to real-world applications first, we help our students see how the content is relevant to their respective fields.

This paper describes a series of activities that we developed to use with our Calculus 1 students. The lesson sequence uses an "Experience First, Formalize Later" (EFFL) pedagogical structure. EFFL lessons are structured so that the "students are working collaboratively to think, to discuss, and to construct their own understanding of new content before the teacher helps students to arrive at formal definitions and formulas" (Wilcox, 2023).

Introductory Activity 1.0

We begin the lesson sequence with a task for the students to explore in groups. No prior instruction about delta or epsilon is given to the students. The goal is for the students to experience a context that utilizes the ideas of the precise definition of the limit. The prompt given to the groups is shown below:

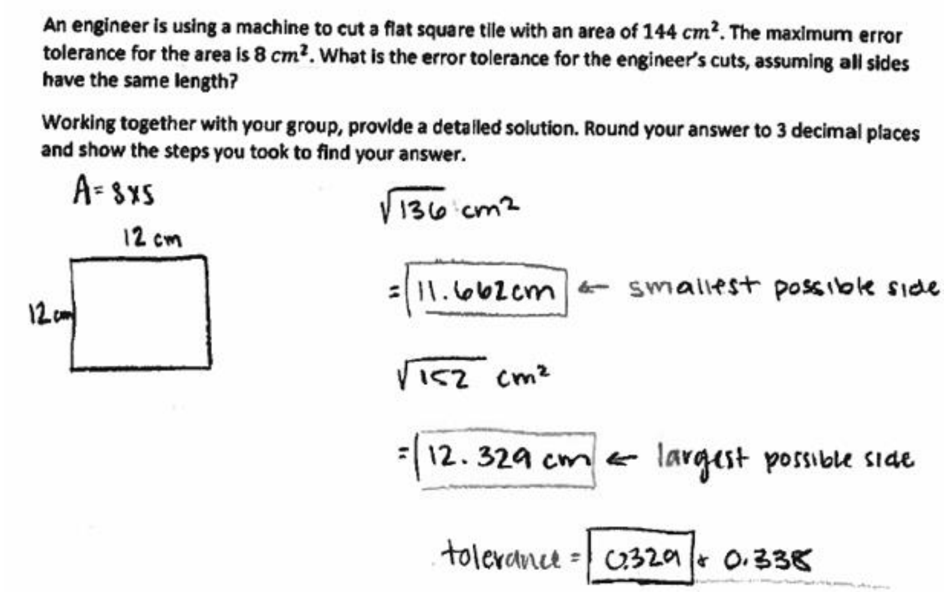
> *An engineer is using a machine to cut a flat square tile with an area of* $144\ cm^2$. *The maximum error tolerance for the area is* $8\ cm^2$. *What is the error tolerance for the engineer's cuts, assuming all sides have the same length?*

Working together with your group, provide a detailed solution. Round your answer to 3 decimal places and show the steps you took to find your answer.

Students could use any approach or tool that they wished. Many chose a numerical approach, and supported their solution processes with geometry diagrams. Some samples of student work are shown in Figures 1, 2, and 3 below.

Figure 1

Sample of Student Work from Introductory Activity 1.0



An engineer is using a machine to cut a flat square tile with an area of 144 cm^2. The maximum error tolerance for the area is 8 cm^2. What is the error tolerance for the engineer's cuts, assuming all sides have the same length?

Working together with your group, provide a detailed solution. Round your answer to 3 decimal places and show the steps you took to find your answer.

A= 8x5

12 cm

12 cm

$\sqrt{136}$ cm²

= $\boxed{11.662cm}$ ← smallest possible side

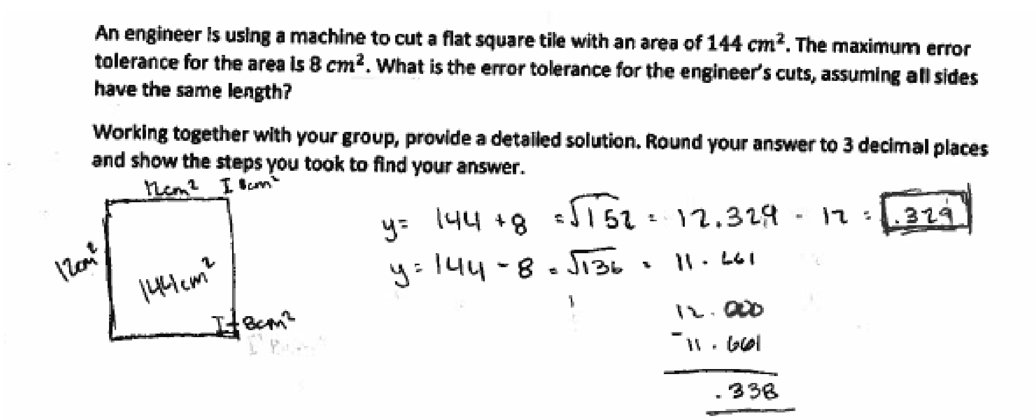
$\sqrt{152}$ cm²

= $\boxed{12.329 cm}$ ← largest possible side

tolerance = $\boxed{0.329}$ ← 0.338

Figure 2

Sample of Student Work from Introductory Activity 1.0



An engineer is using a machine to cut a flat square tile with an area of 144 cm^2. The maximum error tolerance for the area is 8 cm^2. What is the error tolerance for the engineer's cuts, assuming all sides have the same length?

Working together with your group, provide a detailed solution. Round your answer to 3 decimal places and show the steps you took to find your answer.

12cm² I 8cm²

12cm²

144cm²

I 8cm²

y= 144 + 8 = $\sqrt{152}$ = 12.329 - 12 = $\boxed{.329}$

y = 144 - 8 = $\sqrt{136}$ = 11.661

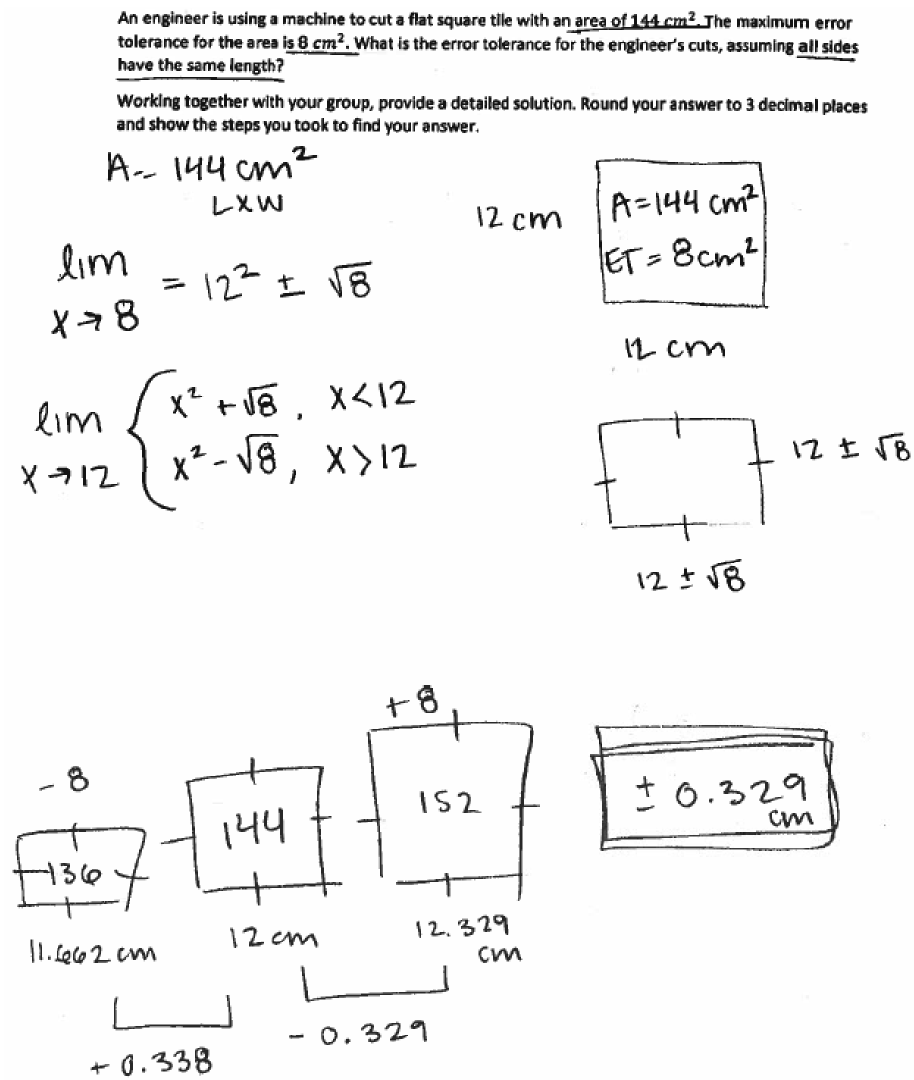
12.000

- 11.661

.338

Figure 3

Sample of Student Work from Introductory Activity 1.0



A copy of the Introductory Activity 1.0 is provided in Appendix A.

Introductory Activity 2.0

After the initial exploration in Introductory Activity 1.0, the students were given Introductory Activity 2.0 to explore in groups. Again, no formal instruction about delta or epsilon was given to the students, nor was any formal symbolic notation given. The prompt for this second introductory activity is as follows:

An engineer is creating a cube-shaped container to ship a variety of small objects and has been given the specifications that the container must have a length of 3 ft per side to minimize shipping costs. His shipper has allowed a tolerance of plus or minus 0.2 ft per side. For this scenario, the allowable side length depends on the volume of the item(s) being shipped. $(f(v) = \sqrt[3]{v})$

How much can the volume of 27 ft^3 vary in order to keep the side length of the container within acceptable limits?

Working together with your group, provide a detailed solution. Round your answer to 3 decimal places and show the steps you took to find your answer.

Students worked together to investigate the context and familiarize themselves with the parameters of the prompt. At this point in the lesson sequence, the students were using their graphing calculators as a tool for their calculations, but few students had used a graphical representation in the Introductory Activity 1.0. The scaffolding of providing a function representation within the prompt for the Introductory Activity 2.0 was intentional since we knew we would be guiding the students to use visualization tools in Desmos in the next phase of the lesson sequence to connect the context to a graphical representation.

A copy of the Introductory Activity 2.0 is provided in Appendix B.

Exploring with Technology

At this point, the students were becoming familiar with the idea of a "distance from the target input on the left" and a "distance from the target input on the right," and they had made numerical calculations of these quantities. We wanted our students to discover that, if these distances were different, they should choose the minimum of the two distances in order to stay within the allowed tolerance on the output value. So, we moved the exploration to Desmos in order to leverage the power of visualization tools and technology.

At our institution, Calculus I, II, and III each have a lab component tied to the course. Most labs are done in our computer lab during a scheduled lab time so that each student has access to a computer while the teacher is present to help set the stage for the task and then monitor student progress and assist where needed. The Desmos lab happened on the class day following the Introductory Activity 1.0 and Introductory Activity 2.0.

Students were given a copy of the Desmos lab instructions. A copy of the instructions for the lab is provided in Appendix C. While each student worked at his or her own computer, students were encouraged to work together with those sitting near them to discuss what they observed at each stage of the lab. Students followed the instructions on the Desmos lab to build a graphical representation, find the tolerances on the output value (ε), and shade the tolerances for the input value to find the desired distance (δ). The students responded to scaffolded questions as they progressed through the lab. Several Desmos technology skills were strengthened during the lab such as how to create and use a slider and how to shade areas on the graph using inequalities. In addition, the lab guided them to use several formatting features of Desmos such as changing the graph of a line from a solid line to a dashed line and creating variable names that use subscripts.

The sequence of figures below illustrates the progression of concepts for which the students created models using Desmos as a visualization tool.

In Figure 4, the graph of the function is shown, along with horizontal lines representing the tolerance boundaries for the output. The vertical dotted lines in the graph are constructed by graphing the vertical line at each abscissa value for the intersection points of the function and the horizontal lines.

Figure 4

Initial Graphical Representation for Relationship Between Delta and Epsilon

After creating the initial graphical representation, students were prompted to create some shading on their graph. Students calculated the tolerance on the right, and shaded that area on either side of the target input value, as shown in Figure 5. Then, the students calculated the tolerance on the left, and shaded that area on either side of the target input value, as shown in Figure 6.

Figure 5

Graphical Representation for Relationship Between Delta and Epsilon – Delta on the Right

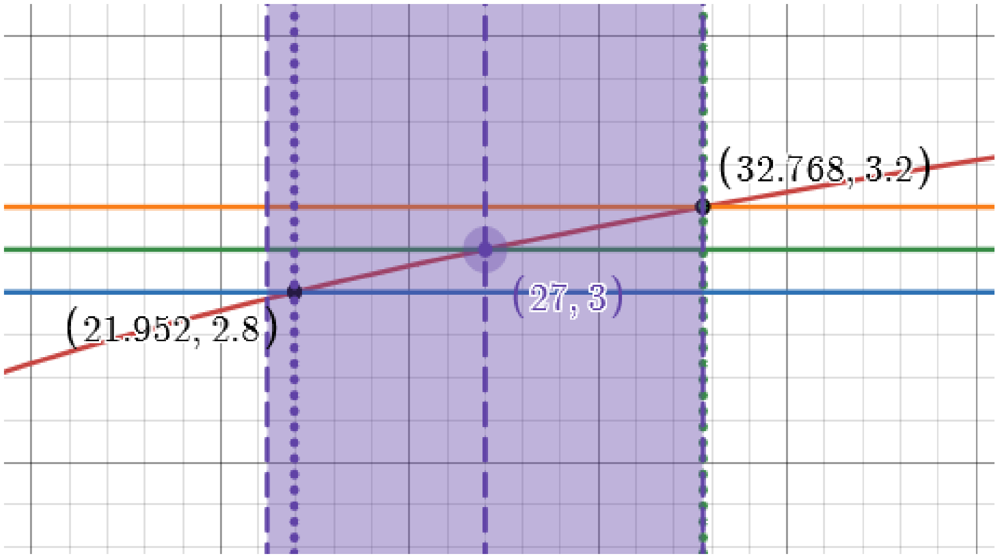
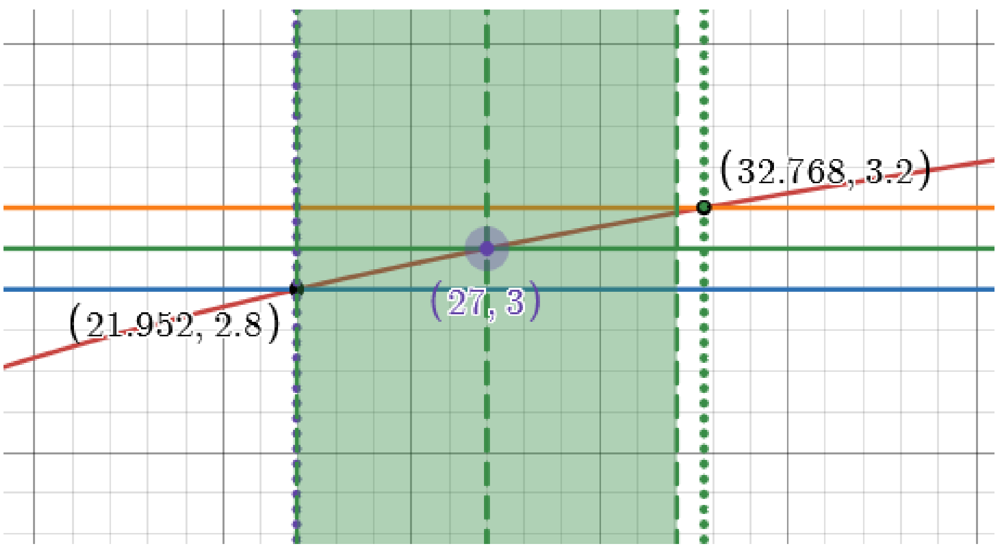


Figure 6

Graphical Representation for Relationship Between Delta and Epsilon – Delta on the Left



At our institution, we use Canvas as our Learning Management System. For this lab assignment, students recorded their answers in a Canvas quiz for the scaffolded questions within the Desmos Lab. The last question in the Canvas quiz that accompanied this Desmos lab asked students for a copy of their Desmos file. Students saved their completed files in

Desmos, then uploaded a link to the Desmos file to a Canvas assignment using the "Share Graph" icon within Desmos.

Student Observations

Questioning strategies were intentional in the Desmos lab to help students solidify their logic for how to make a decision for delta when the distances on the right and the left were different. We chose to collect these responses through a Canvas quiz that the students used as they worked through the lab. A sample of questions and selected student responses is included below.

After students created the graph shown in Figure 5, students were asked if a tolerance of this size (δ_{right}) for the input of volume would keep them within the tolerance for the output of side length. Also, students were asked how the shading on the graph helped them decide. A sample of student responses is shown below.

- *It goes outside of the boundaries set on (21.952, 2.8).*
- *The shaded part goes past the left side, helping me visually see that it doesn't stay within the tolerance.*
- *Shading helped uncover the rejects that wouldn't fit.*
- *d_{right} goes past the tolerance on the left.*

Next, students created the graph shown in Figure 6 and were asked if a tolerance of this size (δ_{left}) for the input of volume would keep them within the tolerance for the output of side length. In addition, the students were asked to articulate how the shading on the graph informed their decision. Selected student responses include the following.

- *The shading stays within the points set. On (21.952, 2.8) it lands right on the line, while it does not reach the other boundary set at (32.768, 3.2).*
- *The shading helps me see that d_{left} stays inside of the parameters on both sides.*
- *Shading helped show that the values all fit within acceptable range.*
- *d_{left} stays within the tolerance on both sides.*

To conclude the lab, students were asked the following question: Based on your exploration, if the values of d_{left} and d_{right} for the distance from the target input volume are different, which value should you choose in order to guarantee that you remain within the tolerance parameters on the output of side length? Selected student responses are shown below.

- *To guarantee that I remain within the tolerance parameters, I must choose d_{left}.*
- *I would choose the 5.048 value since that is the one associated with d_{left} because I know that would keep me within parameters.*
- *You should choose d_{left} because the values stay within the range of output values.*

- *You should choose the value that stays between the lines and doesn't go outside of them. In the case of the lab, it would be* d_{left}.

The students were successful with discovering this conclusion based on the visualization tools that they constructed during the lab.

EFFL – Connections in the *Formalize Later* Component

Since we were using the EFFL lesson strategy, the next phase of the lesson sequence was to help our students formalize these concepts and tie in some notation. We used the handout provided in Appendix D as an artifact for our students to turn in as a homework grade. This formalization component allowed us to connect the Greek letters used in the precise definition of the limit in our textbook to the attributes and quantities that our students had experienced in the Desmos lab. Even though Desmos did not allow us to name variables with Greek letters, we tried to streamline the connections where possible. For example, the lab guided the students to use d in Desmos for the quantity that would later connect to δ when we formalized the notation at the end of the lesson sequence.

After experiencing the lesson and formalizing the notation, the students completed an online homework assignment through WebAssign in Canvas where they had opportunity to continue to apply the concepts they had learned.

Additional Resources

There are numerous resources containing pre-built visualization models that are readily available for students to use. While many of these resources are certainly high in quality, we wanted our students to experience the lesson sequence described in this paper prior to using these pre-built models so that the pre-built models could reinforce the conceptual understanding. Posting links to these resources in your Learning Management System will allow students the flexibility to interact with the content in a reinforcement phase as needed. Here are some possibilities for pre-built visualization models that might be helpful for your students.

The first resources utilize GeoGebra. Mulholland (2013) has an activity that is a type of Epsilon-Delta game. The student chooses epsilon to be a number, then attempts to find a value for delta by dragging a slider so that each *x*-value in the delta interval is sent to the epsilon interval. Next, Gresham (2016) shares a resource where students select a function and then a target input and output can be examined using sliders for delta and epsilon such that the limit definition is satisfied. Third, Navetta (2016) provides a resource where values for the target input, a, on a curve determine the target output, L. Sliders are provided for delta and for epsilon so that results can be examined to determine the correct delta value to remain within the tolerance on the output. Fourth, Jackson (2020) presents an interactive exploration with questions included. Students are able to change the function, the target input and target output with sliders for delta and epsilon to adjust and find tolerances. Finally, Quinn and Wysocki (2016) provide an exploration with shading to reinforce allowable values of delta given a function and a target input. Sliders are present for the input, delta, and epsilon. The shading makes it apparent when delta values exceed the tolerance on the limit.

Moretti (2009) provides a resource using *Mathematica*. The model shows a visual representation of the delta and epsilon ranges. Values can be entered for the target input, a, and for the target output, L, and sliders for delta and epsilon can be used to explore tolerances.

Concluding Remarks and Next Actions

While the precise definition of a limit in Calculus 1 can be a challenging concept for students, the lesson sequence described in this paper can help your students construct their understanding effectively. We acknowledge that these activities focus primarily on the concepts through a numerical and graphical representation lens, and our next goal is to expand this lesson sequence to flow into writing delta-epsilon proofs. With this expansion, we can help our students strengthen their conceptual understanding by connecting the numerical, graphical, and symbolic representations.

References

Adiredja, A.P. (2021). The pancake story and the epsilon-delta definition. *PRIMUS*, *31*(6), 662-677. https://doi.org/10.1080/10511970.2019.1669231

Gresham, J. (2016). *Epsilon-delta illustrator for definition of limit.* GeoGebra. https://www.geogebra.org/m/Cary5q6p

Jackson, J. L., II. (2020). *Formal epsilon-delta definition of a limit.* GeoGebra. https://www.geogebra.org/m/nnb5auut

Moretti, C. (2009). *Limits using epsilons and deltas.* Southeastern Oklahoma State University. https://www.se.edu/cmoretti/main/mathematicalibrarymain/mathematica-notebooks-calculus-1/

Mulholland, J. (2013). *Epsilon-delta game (Formal definition of a limit).* GeoGebra. https://www.geogebra.org/m/tCnmrWg2

Navetta, A. (2016). *Limit definition.* GeoGebra. https://www.geogebra.org/m/QUdWvJys

Quinn, M. & Wysocki, J. (2016). *Delta epsilon limit definition.* GeoGebra. https://www.geogebra.org/m/cx4RweDd

Stewart, J., Clegg, D., & Watson, S. (2021). *Calculus: Early transcendentals* (9th ed.). Cengage.

Wilcox, L. (2023, December 4). *Increase equity with EFFL*. Math Medic Foundation. https://mathmedicfoundation.org/our-work/increase-equity-with-effl

Appendix A

Introductory Activity 1.0

An engineer is using a machine to cut a flat square tile with an area of $144\ cm^2$. The maximum error tolerance for the area is $8\ cm^2$. What is the error tolerance for the engineer's cuts, assuming all sides have the same length?

Working together with your group, provide a detailed solution. Round your answer to 3 decimal places and show the steps you took to find your answer.

Appendix B

Introductory Activity 2.0

Getting to the Root of the Cube!

An engineer is creating a cube-shaped container to ship a variety of small objects and has been given the specifications that the container must have a length of 3 ft per side to minimize shipping costs. His shipper has allowed a tolerance of plus or minus 0.2 ft per side. For this scenario, the allowable side length depends on the volume of the item(s) being shipped. ($f(v) = \sqrt[3]{v}$)

How much can the volume of 27 ft^3 vary in order to keep the side length of the container within acceptable limits?

Working together with your group, provide a detailed solution. Round your answer to 3 decimal places and show the steps you took to find your answer.

Appendix C

Desmos Lab – Getting to the Root of the Cube!

During class, we have been exploring the ideas of limits. In today's lab, we will look more closely at what we mean when we take the limit of a function.

Let's use the following scenario to guide our exploration:

> An engineer is creating a cube-shaped container to ship a variety of objects and has been given the specifications that the container must have a length of 3 ft. per side to minimize shipping costs. The shipper has allowed a tolerance of plus or minus 0.2 ft. per side. For this scenario, the allowable side length depends on the volume of the item(s) being shipped.
>
> - How much can the volume of 27 ft^3 vary and still keep the side length of the cube-shaped container within the parameters set by the shipper?

The following function allows us to determine the side length of the cube-shaped container, given the volume that the box needs to hold.

$$f(x) = \sqrt[3]{x}$$

What is the target input/output pair for the scenario described above?

Target input (Volume): ___________________ ft^3

Target output (Side Length): ________________ ft

How far above and below the "target output" must you consider for the side length? _____________

How much can the volume of 27 vary and still keep the side length of the cube-shaped container within the parameters set by the shipper? ***Let's use Desmos to build an illustration to assist us in exploring this problem.*** 😊

Use this recording form to help you fill in the answers to the Canvas Quiz Assignment. This form is for your use. You will submit your quiz and your Desmos graph for your Lab 2 Grade.

Name your Desmos file: ***First & Last Name – Math 2413-### – Lab 2***.
For Example: Nancy Summer – Math 2413-020 – Lab 2.

Completed?	Your task… 😊
☐	Open a new Desmos graphing calculator page. (Be sure to save your graph using the naming convention shown above.)
☐	Graph the function $f(x) = \sqrt[3]{x}$.
☐	Define a variable and create a slider for a, and set its value to 27.
☐	Define a variable and create a slider for L, and set its value to 3.
☐	Plot the point (a, L). Be sure to check the box in Desmos to show the label for the point on the graph.
☐	Graph a horizontal line that passes through the point (a, L). You can do this by defining a function $j(x) = L$.
☐	Define a variable and create a slider for p, and set its value to 0.2.
☐	Graph two horizontal lines to show how far you can deviate above and below the target output of L. You can do this by graphing the following: $g(x) = L + p$ $h(x) = L - p$
☐	At this point, you may want to adjust your viewing window to better see the points and lines of interest. You can do this by selecting the wrench icon in the upper right corner of the graph. Pick values that seem reasonable for your inputs (volume) and outputs (side length of container). You do not need to pick a “Step” value but reasonable upper and lower limits for your input and output variables will allow you to view the “action!” You can play around with your values until you get a view that you like. 😊

☐	Plot a point at the intersection of each of these two horizontal lines with the original function, $f(x)$. Be sure to check the box in Desmos to show the label for the points on the graph. You can plot these points with the following commands: Where the top horizontal line intersects the original function: $\left((g(x))^3, f((g(x))^3)\right)$ Where the bottom horizontal line intersects the original function: $\left((h(x))^3, f((h(x))^3)\right)$
☐	Let's capture the x-components of the intersection points, and graph vertical lines to mark these boundaries on the graph. You can plot these vertical lines with the following commands. $x = (h(x))^3$ $x = (g(x))^3$ Click and hold the colored circle to the left of the command line so that you can bring up the formatting options menu for the line. Change the format of the vertical lines from solid to dashed.
☐	Finally, let's write some Desmos commands to measure distances for us from the target input value (for volume). ***Note**: You can create a subscript in Desmos with "SHIFT_" then type the text that you wish to be in your subscript.* $d_{left} = abs((h(x))^3 - a)$ $d_{right} = abs((g(x))^3 - a)$ What values did you calculate (using Desmos) for d_{left} and for d_{right}? $d_{left} =$ ______________________________ (3 decimals) $d_{right} =$ ______________________________ (3 decimals)

☐	Finally… let's have Desmos do some shading for us to help us determine which of these distances from the target input volume would allow us to remain within the tolerance for the side length. First, let's see what happens if we use the distance that we calculated from the intersection to the ***RIGHT*** of the target volume. Use Desmos to shade that distance as the tolerance below and above the target volume using the following commands. $a < x < a + d_{right}$ $a - d_{right} < x < a$ Will a tolerance of this size for the input of volume keep you within your tolerance on the output of side length? ____ How does the shading and the graph help you decide? ____ ____
☐	Uncheck the icons in your left menu to remove the shading from the inequalities that we just graphed. We want a clean slate as we explore the effect of the other distance.
☐	Next, let's see what happens if we use the distance that we calculated from the intersection to the ***LEFT*** of the target volume. Use Desmos to shade that distance as the tolerance below and above the target volume. $a < x < a + d_{left}$ $a - d_{left} < x < a$ Will a tolerance of this size for the input of volume keep you within your tolerance on the output of side length? ____ How does the shading and the graph help you decide? ____ ____

☐	Based on your exploration, if the values of d_{left} and d_{right} for the distance from the target input volume are different, which value should you choose in order to guarantee that you remain within the tolerance parameters on the output of side length? ______________________________
☐	When you have completed this lab, save your file in Desmos. You will enter your answers into the Canvas lab, then upload a link to your Desmos file by selecting the “Share Graph” icon in the upper right corner of the Desmos webpage. Copy and paste the link for your graph in the space provided on the last question in the Canvas quiz. Great job! 😊

Appendix D

Homework - Getting to the Root of the Cube!

Revisiting our shipping container problem, let's see if we can apply the concepts that we learned in Section 2.4 to determine our tolerance on the volume!

Here's the original problem:

An engineer is creating a cube-shaped container to ship a variety of small objects and has been given the specifications that the container must have a length of 3 ft per side to minimize shipping costs.

His shipper has allowed a tolerance of plus or minus 0.2 ft per side. For this scenario, the allowable side length depends on the volume of the item(s) being shipped. ($f(v) = \sqrt[3]{v}$)

How much can the volume of 27 ft^3 vary in order to keep the side length of the container within acceptable limits?

1. First, we need a function:

$y =$ ____________________

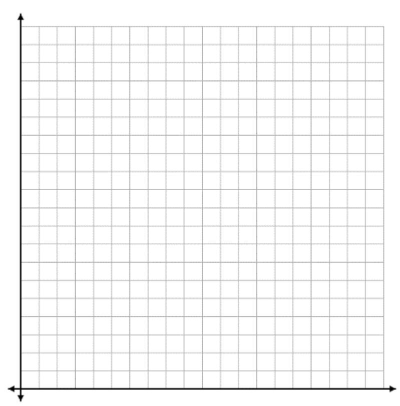
2. Next let's write a limit relating our input and output:

$$\lim_{x \to a} (\_\_\_\_\_\_\_\_\_\_) = \_\_\_\_\_\_\_\_$$

3. Let's identify everything we know:

L	ε	a	δ	$f(x)$

4. Let's graph $f(x)$ and identify $L, \varepsilon, a, and\ \delta$.



5. $\delta_L =$ ____________________ $\delta_R =$ ____________________

$\delta =$ ____________________