DESMOS PENNY CIRCLE ACTIVITY: ENHANCED ANALYSIS AND DATA POOLING

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Abstract

Two interactive Google Sheets spreadsheets were developed to expand the data analysis of the Desmos Penny Circle Activity. Via simple computations, students can examine 2D penny packing models, discover and perform an error analysis, and examine some statistical measures of variability via data pooling. This brings mathematical, statistical, and scientific thinking together around an elegant simple model in Desmos. Plus, students can validate the Desmos simulation!

Introduction

Measurement and its error are a critical part of the scientific method and need to be considered in the collection of quantitative data, especially in the age of big data. Students need to experience the melding of mathematical, statistical, and scientific thinking (Figure 1). Deducing the cause of an error can be a daunting task, especially for novice students. Sinex (2024) summarizes a variety of simple manipulative activities to introduce error and its treatment. Dealing with error is a mindset that must be nurtured and developed in students.

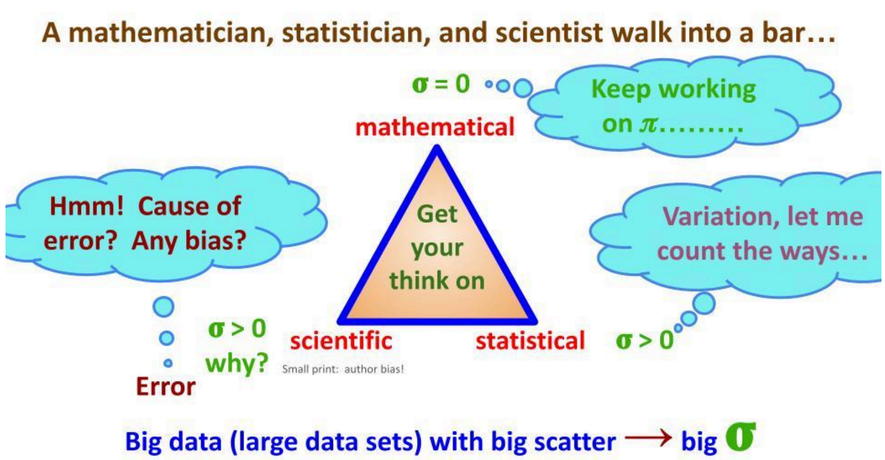


Figure 1 - Get your think on!

This article is an extension of the [Desmos Penny Circle Activity] (freely available online), via data analysis using interactive Google Sheets spreadsheets, the number one data analysis choice of 330 teachers (Rosenberg et al. 2022)! The Desmos activity investigates f(circle diameter) → maximum number of pennies to fill; however, here this is flipped to f(no.of pennies) → area occupied by pennies and then compares it to the area based on the circle diameter. The difference, A-pennies minus A-diameter, is the space not filled by pennies, which is the error.

The objectives of this activity are given below:

1. Demonstrate the Desmos Penny Circle Activity to obtain data and discover a quadratic relationship.
2. Use the original data to transform it to compare the area of a circle and the area occupied by pennies in that circle plotted against the number of pennies. Discover a nice linear relationship that illustrates a negative proportional systematic error (the empty pore space between pennies). The error is obvious and very easy for students to discover!
3. Examine a comparison plot, two methods compared to y = x line on plot (another method to determine error).
4. Discover the types of 2D packing of pennies and test the model data (2 above).
5. Offer a test to validate the Desmos activity, a black-box simulation.
6. Via data pooling in a shared prebuilt GSheets spreadsheet, examine a variety of ways to assess variation (adds statistical thinking).

The Desmos Penny Circle Activity

In the original analysis as outlined by Meyer (2014), students would collect data by setting the circle diameter and then adding pennies until no more could be added. The number of pennies would then be considered the area of the circle. Students would then graph the data, number of pennies as a function of circle diameter. Students would mathematically model the data to decide if a linear, quadratic, or exponential fit was best.

Students would then open [Desmos Penny Circle Activity: Enhanced Analysis] spreadsheet and examine the data on the "Desmos data" tab (number of pennies depends on diameter of the circle) to see if they had analyzed their data correctly (Figure 2). Here the linear, quadratic, or exponential regressions are accompanied by r-squared values to judge goodness-of-fit. Students can add a (0, 0) point in the data set to decide how that influences the results. The data collected was by the author and for each trial the pennies were rattled to ensure maximum fit of pennies! Students could compare their results to see how well they packed the circle with pennies!

Desmos generated data

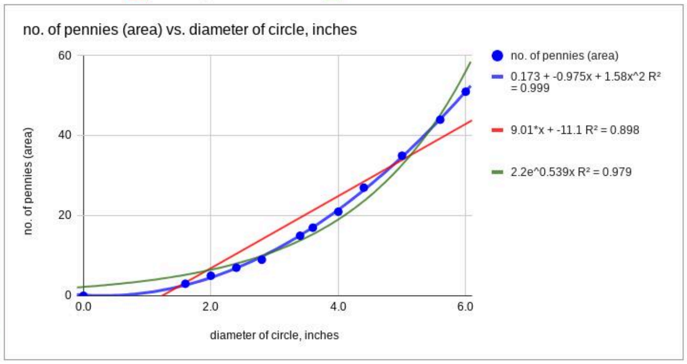
add (0,0) point

How does the diameter of the circle influence the maximum number of pennies that the circle will contain or hold?

linear fit | exponential fit

diameter of circle, inches	no. of pennies (area)
0.0	0
1.6	3
2.0	5
2.4	7
2.8	9
3.4	15
3.6	17
4.0	21
4.4	27
5.0	35
5.6	44
6.0	51

Does the number of pennies depend on the circle diameter?



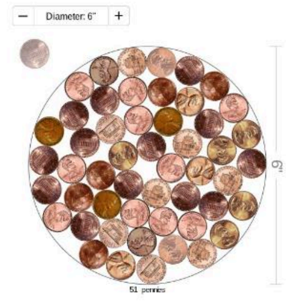


Figure 2 - Original Desmos penny circle activity data

The Extended Analysis

Now students are ready to examine the “Desmos data - alternate plot” tab (Figure 3). Here the number of pennies to fill the circle depends on the diameter of the circle (reversed from original activity). Now a number of results must be computed: area occupied by pennies, area of circle, difference (area occupied by pennies - area of circle, absolute error), and percent error including the sign of the error (%error = (area of pennies - area of circle)*100/ area of circle, relative error). For a discussion and graphical illustration of types of errors, see Sinex (2005) and see the Ruler and Measurement Error Excelet to explore the types of errors in a variety of ways.

The graph in Figure 3 shows the area occupied by the pennies (blue line) and the area of the circle (red line), showing the area occupied by the pennies underestimated the total area. The

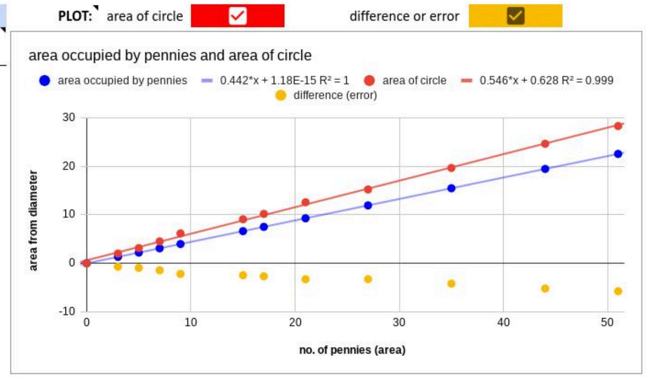
Desmos generated data - another view

diameter of penny = 0.750 in

How do you calculate the area of a circle?

PLOT: area of circle | difference or error

data - measured		results - calculated			
no. of pennies (area)	diameter of circle, inches	area occupied by pennies	area of circle	difference (error)	%error
0	0.0	0.0	0.0		
3	1.6	1.3	2.0	-0.7	-34.1
5	2.0	2.2	3.1	-0.9	-29.7
7	2.4	3.1	4.5	-1.4	-31.6
9	2.8	4.0	6.2	-2.2	-35.4
15	3.4	6.6	9.1	-2.5	-27.0
17	3.6	7.5	10.2	-2.7	-26.2
21	4.0	9.3	12.6	-3.3	-26.2
27	4.4	11.9	15.2	-3.3	-21.6
35	5.0	15.5	19.6	-4.2	-21.3
44	5.6	19.4	24.6	-5.2	-21.1
51	6.0	22.5	28.3	-5.7	-20.3
		0.0	0.0	0.0	



What is the cause of the error? See the "penny packing" tab

Figure 3 - Extended analysis of penny circle activity

difference or area of pennies minus the area of the circle is the error. The negative error grows larger as the number of pennies increases; hence, this is a negative proportional systematic error (decreased the slope for the area occupied by pennies). The slope of the area occupied by the pennies (0.442 in^2/penny) is the area of a penny.

Meyer (2014) offers an alternative method for estimating the number of pennies to occupy a circle of known radius, where number of pennies = $\pi r^2/A_{penny}$ and r is the radius of the circle. This produces a positive proportional systematic error and is illustrated on this tab, just scroll down.

Another approach ("another comparison plot" tab) used to compare two methods is a y = x line plot as illustrated in Figure 4. If the two methods agree, the data will plot on the y = x line (slope = 1, y-intercept = 0). Systematic errors, which can be positive or negative, show up as proportional (slope ≠ 1) and constant (y-intercept ≠ 0). Random error increases scatter, which lowers R-square, and randomly influences the slope and y-intercept. The y = x line is added on the plot by repeating the x-axis data in the third column.

method 1 - more trusted	method 2 - new method	repeat method 1
area from diameter	area from no. of pennies	y = x line comparison
0.0	0.0	0.0
2.0	1.3	2.0
3.1	2.2	3.1
4.5	3.1	4.5
6.2	4.0	6.2
9.1	6.6	9.1
10.2	7.5	10.2
12.6	9.3	12.6
15.2	11.9	15.2
19.6	15.5	19.6
24.6	19.4	24.6
28.3	22.5	28.3

Compare two different analysts measuring the same quantity such as number of pennies for the same circle diameter!

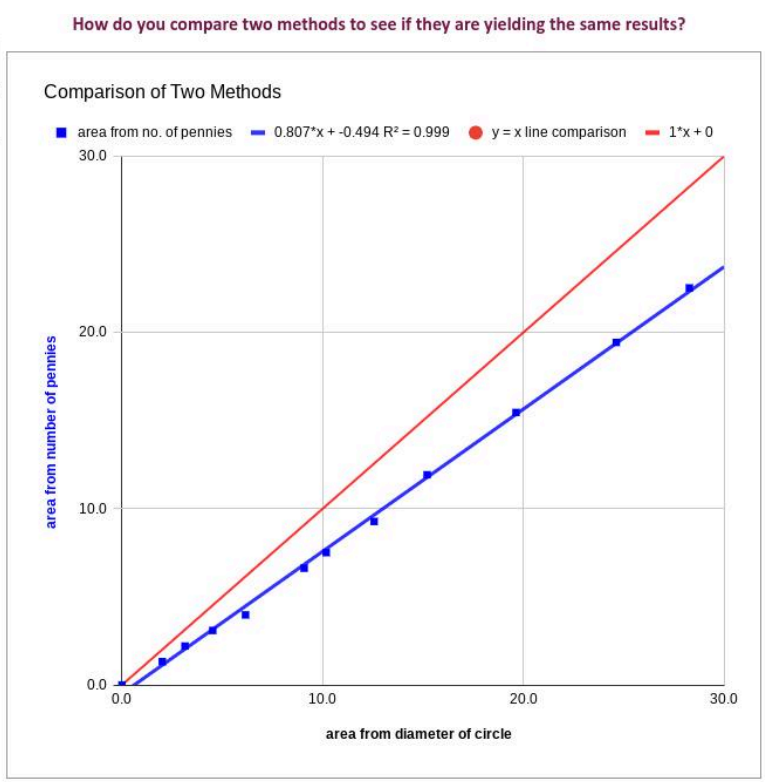


Figure 4 - Comparison plot for two methods

Two-dimensional Packing of Pennies

The packing of regular shaped objects has numerous applications in mathematics and the sciences. Starting with two-dimensional packing is a great lead-in for three-dimensional packing. So let's pack pennies (small circles of constant size) into a larger circle. Packing arrangements fall into three categories: square packing, hexagonal packing, and random packing. The random packing leaves additional empty space and is not as efficient.

A student activity on [2D packing of pennies] is provided that includes the geometric formula to compute the volume of the circle compared to the square or hexagonal arrangement as seen in Figure 5 with examples labeled of the three categories. Students can then complete Table 1.

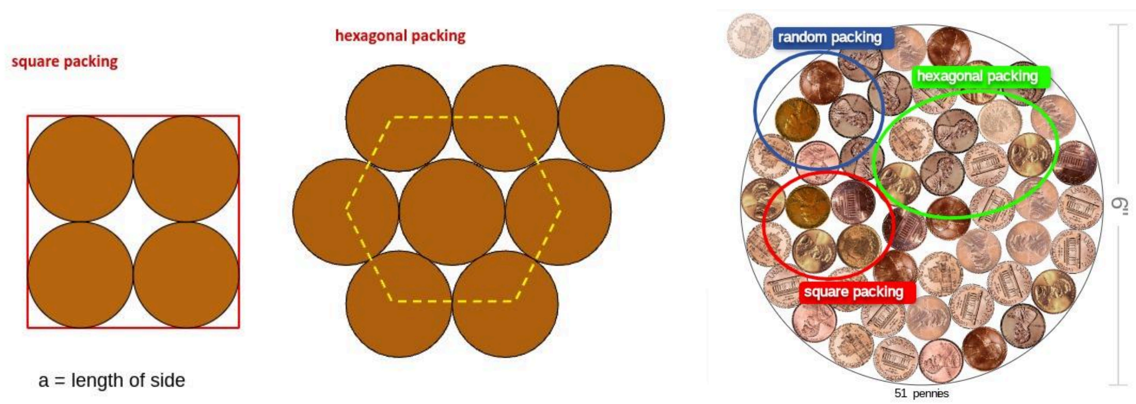


Figure 5 - Square and hexagonal 2D packing (left) and Pennies packed in circle (right)

Table 1 - Determination of 2D Packing Efficiency for Square and Hexagonal Packing

Determine the following	For the square	For the hexagon
How many pennies are contained?	4	1 +6 x (⅓) = 3
Calculate area occupied by pennies	4 x 0.442	3 x 0.442
Packing efficiency = area of pennies x 100/area packed	4 x 0.442 x100/2.25 = 78.6%	3 x 0.442 x 100/1.46 = 90.2%

Students then go to the "penny packing" tab to compute the bulk packing efficiency for the data collected earlier as illustrated in Figure 6. The data has an overall average of 73% with a range of 65 to 80%.

Calculate the 2D Packing Efficiency using the Desmos Data - Alternate Plot

data - measured		results - calculated		
no. of pennies (area)	diameter of circle, inches	area occupied by pennies	area of circle	2D packing efficiency
0	0.0	0.0	0.0	--
3	1.6	1.3	2.0	66
5	2.0	2.2	3.1	70
7	2.4	3.1	4.5	68
9	2.8	4.0	6.2	65
15	3.4	6.6	9.1	73
17	3.6	7.5	10.2	74
21	4.0	9.3	12.6	74
27	4.4	11.9	15.2	78
35	5.0	15.5	19.6	79
44	5.6	19.4	24.6	79
51	6.0	22.5	28.3	80

TASK

How do the 2D packing efficiencies calculated using the Desmos data compare to the theoretical calculation in the square and hexagon packing models?

Is the Desmos penny circle a fair comparison to the square and/or hexagon? Explain your reasoning!

Figure 6 - Computation of 2D packing efficiency

The 2D penny packing of circles is an excellent transition to the 3D packing of spheres, which is a great lead-in to 3D crystal structures. Many aspects of the cubic crystal structures of metals can be introduced and illustrated by examining the 2D position of circles in the faces of 3D structures. For the diehards, see Exploring the Crystal Structure of Metals I: Cubic Structures, an interactive Excel spreadsheet.

Validation of the Desmos Simulation

The Desmos Penny Circle Activity is a black-box simulation (calculations/programming not visible to the user). How do you know if this simulation is realistic? Validation of a model is an aspect of the mathematical modeling process that is usually not an easy step to assess. Here via a little simple manipulative experimentation using pennies, flat lids and/or embroidery hoops of various diameters, and a measuring device as illustrated in Figure 7.

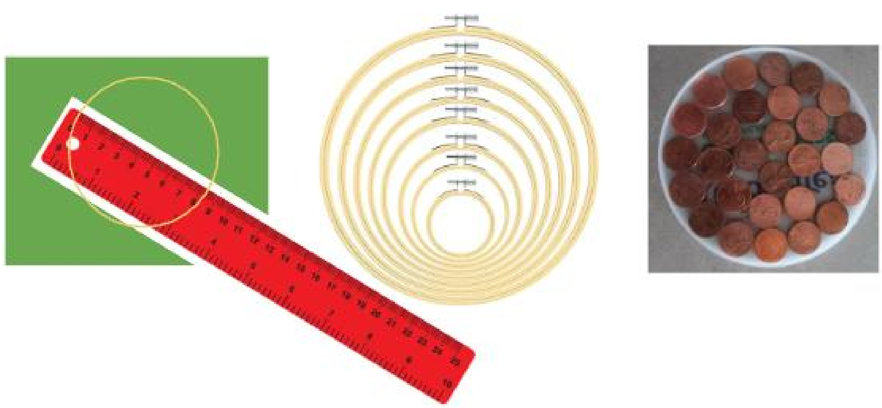


Figure 7 - Required materials for validation

Using the ruler, measure the diameter, being sure to get the maximum distance across, of a variety of different size flat lids and/or embroidery hoops. Then pack pennies in them and record the number of pennies for each diameter. Ideally, generating a y = x comparison plot would be the choice. However, this would require matching the x-variable for both data sets by matching diameters, so that the number of pennies would in theory be the same. Since the number of pennies do not match for our reality versus virtual comparison, a plot of area of the circles and area occupied by the pennies for both methods as a function of the number of pennies will be plotted as illustrated in Figure 8. How does the virtual compare to reality?

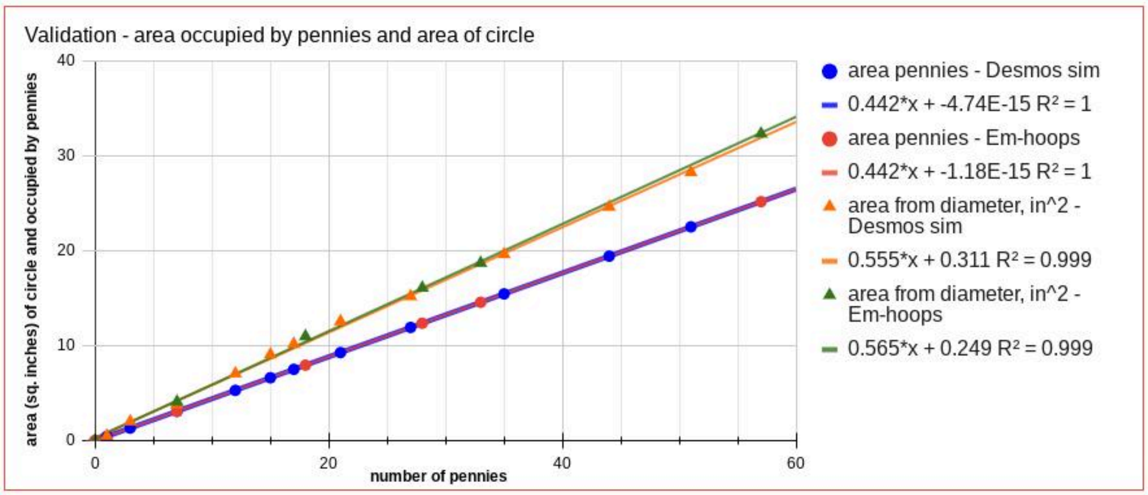


Figure 8 - Validation of Desmos simulation

For the area occupied by the pennies, both Desmos and real measurements yield the identical equations with R-squared equal to one: $A_{pennies} = 0.442n_{pennies}$ where the slope is the area of a penny in square inches using the US Mint penny diameter of 0.750 inch (appeared earlier in Figure 3) and the y-intercept is zero. Now the area of the circle from the diameter shows the two results are very close, with the slopes differing by about 2% and the y-intercepts off in the direction that could be due to slight increase in slope. The measurement of the hoop diameters with a ruler could surely be the difference. In any case, the Desmos virtual penny measurement is as close to reality as needed.

Data Pooling to Examine Variability

Pooling class data from groups (or individuals) is a great way to generate a class mean and examine variation for the class. Plus groups get to compare their results to other groups, which can enhance competitiveness. To accomplish this with this activity, *groups must all use the same choice of prescribed circle diameters* (see spreadsheet for diameters).

Two different pooling tabs are provided on [Desmos Penny Circle Activity II: Data Pooling] with a tab for mean and range (plots max and min values) and a tab for mean and standard deviation (plots +/- one std. dev.). This is as close to a one standard deviation error bar for each data point as you can get. The mean and standard deviation tab is shown in Figure 9 with no student data added. Spreadsheets are glass-box simulations, as all computations are visible. The quadratic regression and R-square will recalculate as data is added.

Pooling the Data for the Pennies in a Circle Activity

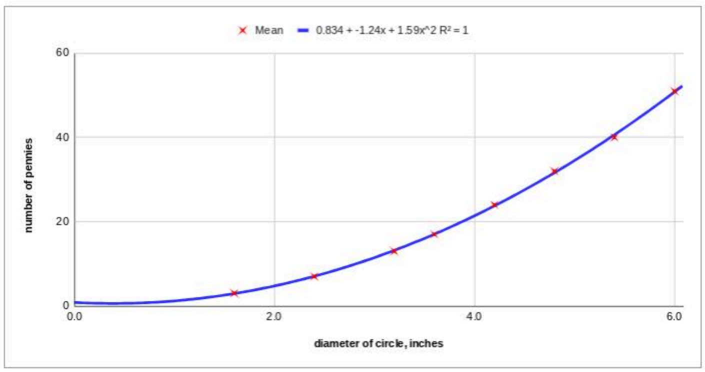
Add the group data in the yellow cells below. They will plot and the class data columns and regression will adjust.

Protection is on except for the check box and the yellow cells below, where you enter your data!

Check box to

☐

insert a (0,0) point



Using mean +/- std, deviation error points for each mean

This is the standard error bar!

The graph starts with the instructor points only and no error points; however, they will appear as student is added!

The blue cells will recalculate as data is added.

	author*	G1	G2	G3	G4	G5	G6	G7	G8	Class Values - mean with +/- error points		
diameter	no. of pennies	no. of pennies	no. of pennies	no. of pennies	no. of pennies	no. of pennies	no. of pennies	no. of pennies	no. of pennies	Mean	Mean + SD	Mean - SD
1.6	3									3	#DIV/0!	#DIV/0!
2.4	7									7	#DIV/0!	#DIV/0!
3.2	13									13	#DIV/0!	#DIV/0!
3.6	17									17	#DIV/0!	#DIV/0!
4.2	24									24	#DIV/0!	#DIV/0!
4.8	32									32	#DIV/0!	#DIV/0!
5.4	40									40	#DIV/0!	#DIV/0!
6.0	51									51	#DIV/0!	#DIV/0!

Figure 9 - Class variability via data pooling

To use this spreadsheet with students, go to File > Make a copy… to save in your Google account. Then go to Share (button in upper right) where you should be the owner now. Change the General access to Anyone with the link, change Viewer to Editor, and supply the Copy link to your students or post on webpage or shared document. Lastly, press Done!

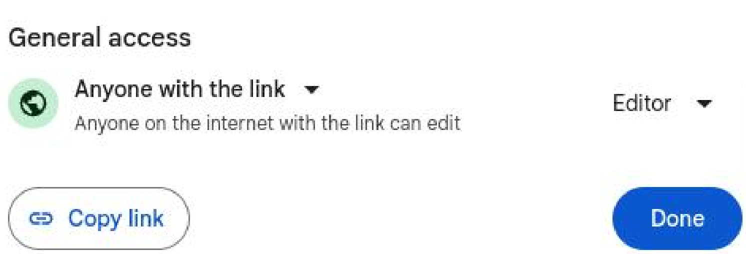


Figure 10 - Sharing the spreadsheet for data collection

For those thinking about considering online collaboration using Google Drive, see Sinex and others (2016a, 2016b) for a variety of ideas and tips. For a serious mathematical activity measuring pi, see Sinex and Chambers (2018).

Some Final Thoughts

The modified Desmos Penny Circle Activity allows students to easily discover an error that is visually apparent (empty space between pennies) and examine its behavior by investigating a simple clear system. Students get to validate the Desmos virtual application using tactile manipulatives, real pennies in embroidery hoops, to see it matches reality. This is a great introduction to dealing with errors!

Getting students into error analysis is not an easy task. Over the years, this author has developed a variety of simple experimental tasks to get students into mathematical modeling of data collected by students and investigating errors (Sinex 2013, Sinex 2017, & Sinex 2018). Hunter-Thomson (2022) has suggested a number of ways to introduce examining variability into the middle-school realm. This is a worthy idea as it sets up considering what causes the variability, which is error. Error analysis is a mindset that needs iteration to develop.

Acknowledgements

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Classroom Resources (accessed June 2024)

Ruler and Measurement Error Excelet to explore all the possible ways to mess up with a ruler! Examines relative and absolute errors, illustrates the errors, and shows how the y = x comparison behaves due to errors.

Desmos Penny Circle Activity (student computer-based activity from Desmos)

Student activity on 2D packing of pennies (key)

Desmos Penny Circle Activity: Enhanced Analysis spreadsheet

Desmos Penny Circle Activity II: Data Pooling spreadsheet

Data Pooling and On-line Collaboration (author's site with "how to" instructions and examples) https://sites.google.com/site/datapoolinthecloud

Dealing with Scientific Data in Google Sheets: Data > Model > Simulation
https://sites.google.com/view/ssinex/home/dealing-with-data-in-gsheets