

EIGENVALUES AND EIGENVECTORS: A GRAPHICAL PEDAGOGICAL APPROACH USING GEOGEBRA

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ABSTRACT

Eigenvalues and eigenvectors are used extensively in mathematics, science, and engineering. We commonly are introduced to them in a linear algebra course. They are usually given the terse definition $L(\vec{u}) = \lambda\vec{u}$. In some cases, the teacher will say that along the eigenvector the linear transformation does not change the vector's direction but at most will stretch it by an amount that we call lambda. The student is then taught how to find the eigenvalues and eigenvectors and uses them to diagonalize a matrix but they are rarely given a visual representation of how the eigenvalues and eigenvectors of a matrix affect its linear transformation. Here we present a GeoGebra app that will remove this shortcoming.

INTRODUCTION

A good understanding of mathematical concepts consists of fluency with symbolic, numeric, and graphical representations. In particular, the connections between the symbolic and graphical representations play an important role in students' comprehension of mathematical ideas (Lipp, 1994; Presmeg, 2014). When a graphical representation of the mathematical concept demonstrates visually what is being studied symbolically, student comprehension is

strengthened. It is important for students to connect these multiple representations if they are to become robust problem solvers.

What is an eigenvector and eigenvalue? To answer this question, we need an n -dimensional linear operator $L(\vec{v})$. Here L is the linear operator n is a positive integer and \vec{v} is an n -dimensional vector. If we can find a special vector \vec{u} such that $L(\vec{u}) = \lambda\vec{u}$ then \vec{u} is an eigenvector and λ is its associated eigenvalue. \vec{u} is in some ways like a steady state $L(\vec{v}) = \vec{v}$ but it adds an extra component λ which either stretches or compresses the vector \vec{u} . Now if λ is 1 then \vec{u} actually is a steady state. (Note: L does not have to be a finite-dimensional operator. Eigenvector and eigenvalue also play a fundamental role in linear functional spaces, but we will stay in finite-dimensional spaces for this article.)

Traditionally, when learning about eigenvalues and eigenvectors, students are led through a symbolic procedure and encouraged to practice until memorized. It is not until later that applications of eigenvalues and eigenvectors, such as mass-spring systems, are explored with visualizations. Such applications are helpful to support student understanding by justifying why we need to calculate eigenvalues and eigenvectors. However, we suggest that dynamic visualizations of eigenvalues and eigenvectors can be used much earlier with students to support their understanding.

In Warren, Gresham, and Wyatt (2017, 2018), we discussed methods for utilizing dynamic 3D visualizations of polynomial and transcendental functions to strengthen students' understanding of non-real roots. In this paper, we present methods for utilizing dynamic visualizations of eigenvalues and eigenvectors to support students' learning of the traditionally abstract topic.

METHODS

To bring eigenvalues and eigenvectors to life let's look at linear operators as dynamical systems. What is a dynamical system? In a dynamical system you start somewhere, your initial condition. Then you operate on this initial condition using some function. You take the output of this operation as the input for the next iteration and continue this process. In our case the rule will be the linear operator L and v_0 will be our initial starting value. We have:

$$\vec{v}_{n+1} = L(\vec{v}_n)$$

Now remember that one of the main theorems in linear algebra is that any finite linear operator can be represented as a matrix. For visualization purposes let's let L be a 2×2 matrix and look at it as a linear operator. For this purpose we developed a 2×2 matrix dynamical systems viewer using C and OpenGL. In this app you select a 2×2 matrix and a screen will pop up with an x-y axis displayed. If you left-click on the screen a red dot will appear where the mouse is located. Then if you right-click the mouse the (x, y) location of the red dot will be translated into a vector and multiplied by the matrix you selected. The output of this operation will be placed back on the screen as a black dot. Subsequent right clicks of the mouse repeat this process using the new (x, y) position as the vector to be operated on. If you left click again a new sequence of points will be initiated. An example of this process is shown below in Figure 1.

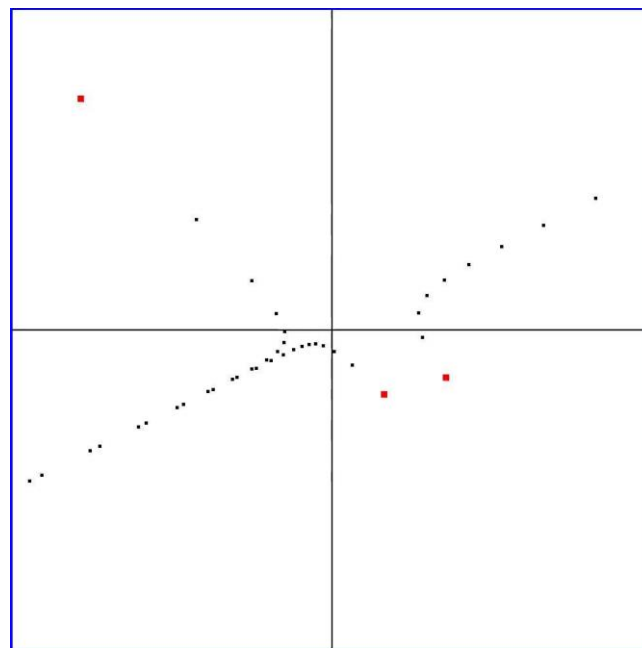


Figure 1 Eigenvalues 1.25, 0.5

By looking at this screen students can see two vectors emerge: $\langle 2, 1 \rangle$ and $\langle 1, -1 \rangle$. With closer inspection, they can see that dot spacings along the $\langle 2, 1 \rangle$ vector increase with each new interaction and the dot spacings decrease along the vector $\langle -1, 1 \rangle$. With even closer inspection they can see that the increase along the vector $\langle 2, 1 \rangle$ is about 1.25 and the decrease along the vector $\langle -1, 1 \rangle$ is about 0.5. If you have them then calculate the eigenvectors and eigenvalues for this system, they will find that these are indeed the correct values. The matrix that created this dynamical system was A ($A_{1,1} = 1$, $A_{1,2} = 1/2$, $A_{2,1} = 1/4$, $A_{2,2} = 3/4$).

To get an intuitive sense as to what is actually happening let's look at 1D linear operators. What is a 1D linear operator? $L(x) = cx$ where c is a constant and x is an input value. Do 1D linear operators have eigenvectors and eigenvalues? They are by definition eigenvectors and c is its eigenvalue. They just happen to operate in one dimension; let's call it the x -axis.

Consider the 1D linear operator $L(x) = 2x$ as a dynamical system. If you start with an initial condition greater than zero you will walk to the right with step sizes that increase by an order of 2. If you start with an initial condition less than zero, you will walk off to the left with step sizes that increase by an order of 2. If you start at zero then you will not move. Zero is a steady state.

$$L(x) = 2x$$

If $x_0 = 1$ then we have 1, 2, 4, 8, 16 ...

If $x_0 = -1$ then we have -1, -2, -4, -8, -16 ...

If $x_0 = 0$ then we have 0, 0, 0, 0, 0, ... a fixed point

Consider the 1D linear operator $L(x) = -2x$ as a dynamical system. If you start at an initial condition not equal to zero you will jump back and forth across zero as you bounce off to infinity in both directions. If you start at zero you will not move. Zero is a steady state.

$$L(x) = -2x$$

If $x_0 = 1$ then we have 1, -2, 4, -8, 16 ...

If $x_0 = -1$ then we have -1, 2, -4, 8, -16 ...

If $x_0 = 0$ then we have 0, 0, 0, 0, 0, ... a fixed point

Consider the 1D linear operator $L(x) = \frac{1}{2}x$ as a dynamical system. If you start with an initial condition greater than zero you will walk to zero from the right with step sizes that decrease by a factor of 1/2. If you start with an initial condition less than zero you will walk to zero from the left with step sizes that decrease by a factor of 1/2. If you start at zero you will not move. Zero is a steady state.

$$L(x) = \frac{1}{2}x$$

If $x_0 = 1$ then we have $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots$

If $x_0 = -1$ then we have $-1, -\frac{1}{2}, -\frac{1}{4}, -\frac{1}{8}, \dots$

If $x_0 = 0$ then we have 0, 0, 0, 0, 0, ... a fixed point

Consider the 1D linear operator $L(x) = -\frac{1}{2}x$ as a dynamical system. If you start at an initial condition not equal to zero you will jump back and forth across zero as you bounce down to zero. If you start at zero you will not move. Zero is a steady state.

$$L(x) = -\frac{1}{2}x$$

If $x_0 = 1$ then we have $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \dots$

If $x_0 = -1$ then we have $-1, \frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, \dots$

If $x_0 = 0$ then we have $0, 0, 0, 0, 0, \dots$ a fixed point

Consider the 1D linear operator $L(x) = 1x$ as a dynamical system. If you start anywhere, you just stay there because every point is a steady state.

$$L(x) = 1x$$

If $x_0 = 1$ then we have $1, 1, 1, 1, \dots$

If $x_0 = -1$ then we have $-1, -1, -1, -1, \dots$

If $x_0 = 0$ then we have $0, 0, 0, 0, 0, \dots$ a fixed point

Consider the 1D linear operator $L(x) = -1x$ as a dynamical system. If you start anywhere, you just jump back and forth across zero and go nowhere.

$$L(x) = -1x$$

If $x_0 = 1$ then we have $-1, 1, -1, 1, \dots$

If $x_0 = -1$ then we have $1, -1, 1, -1, \dots$

If $x_0 = 0$ then we have $0, 0, 0, 0, 0, \dots$ a fixed point

Now to understand 2D operators, just put two 1D linear operators together. You project your starting point down onto each eigenvector and let your eigenvalue adjust this value at each iteration, see Figure 2. The result is just a linear combination of these two linear operators in action.

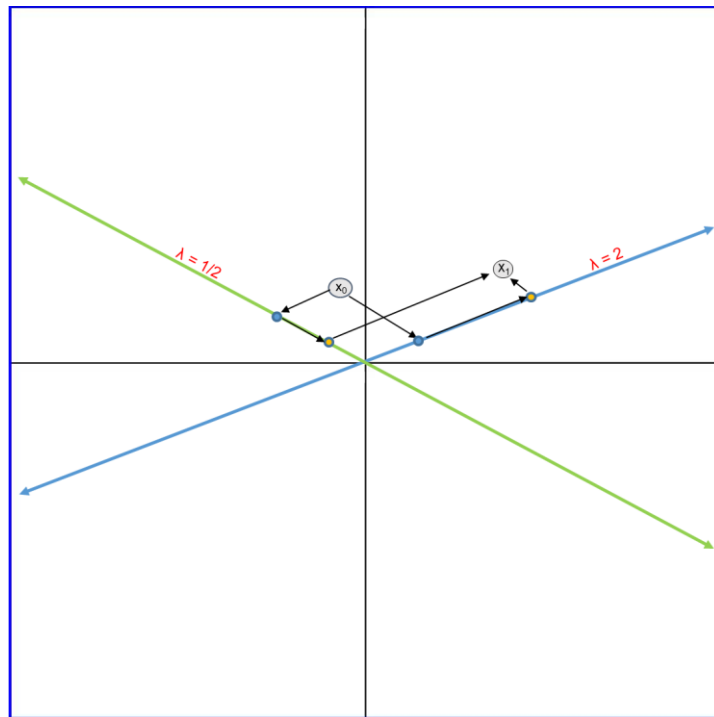


Figure 2 Two-dimensional Linear Operator

Look at Figure 3 and see if you can guess the eigenvectors and eigenvalues. You can see some interesting behavior along both the 45-degree lines in both runs. Hence, you might conjecture that the eigenvectors are $\langle 1, 1 \rangle$ and $\langle 1, -1 \rangle$. Then you will notice that the values tend to increase along $\langle 1, 1 \rangle$ so its eigenvalue must be greater than 1. Then you will notice that the values along $\langle 1, -1 \rangle$ tend to bounce back and forth across $(0, 0)$ and decrease. Hence, its eigenvalue must be negative and smaller than 1 in absolute value. The matrix A ($A_{1,1} = 1/2$, $A_{1,2} = 3/4$, $A_{2,1} = 1$, $A_{2,2} = 1/4$), created the sequence of dots in Figure 3. Calculate its eigenvectors and eigenvalues and see if they match what your eyes see.

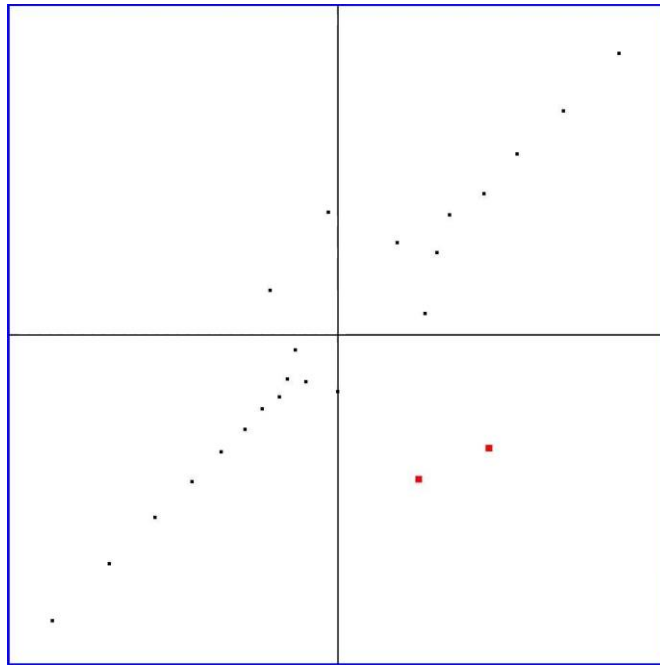


Figure 3 Eigenvalues $\frac{5}{4}$, $-\frac{1}{2}$

Look at Figure 4. It is a little harder to see what is going on here but once you know the eigenvectors and eigenvalues it becomes perfectly clear. The pairs $(\langle 1,0 \rangle, 1.1)$ and the pair $(\langle 0,1 \rangle, 1.2)$ are the two eigenvector and eigenvalue pairs. Wherever you start you increase down each eigenvector, but you move a little faster along $\langle 0,1 \rangle$ so a half parabolic type shapes emerge. Note there are 5 independent runs in this figure that were created by the matrix A ($A_{1,1} = 1.1, A_{1,2} = 0, A_{2,1} = 0, A_{2,2} = 1.2$).

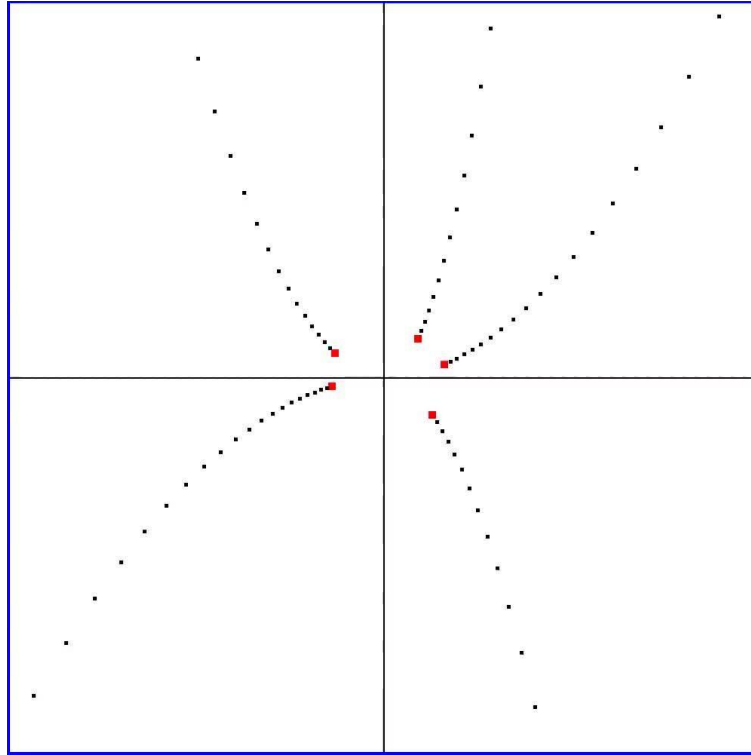


Figure 4 Eigenvalues 1.1, 1.2

Look at Figure 5, again like in Figure 4 the eigenvectors are the standard bases but all 5 of the runs go either straight up or straight down. The $\langle 1,0 \rangle$ values do not move hence its eigenvalue must be 1. The values along the $\langle 0,1 \rangle$ vector grow so its eigenvalue must be positive and greater than 1. Note that there are 5 independent runs in this figure that were created by the matrix A ($A_{1,1} = 1, A_{1,2} = 0, A_{2,1} = 0, A_{2,2} = 1.2$).

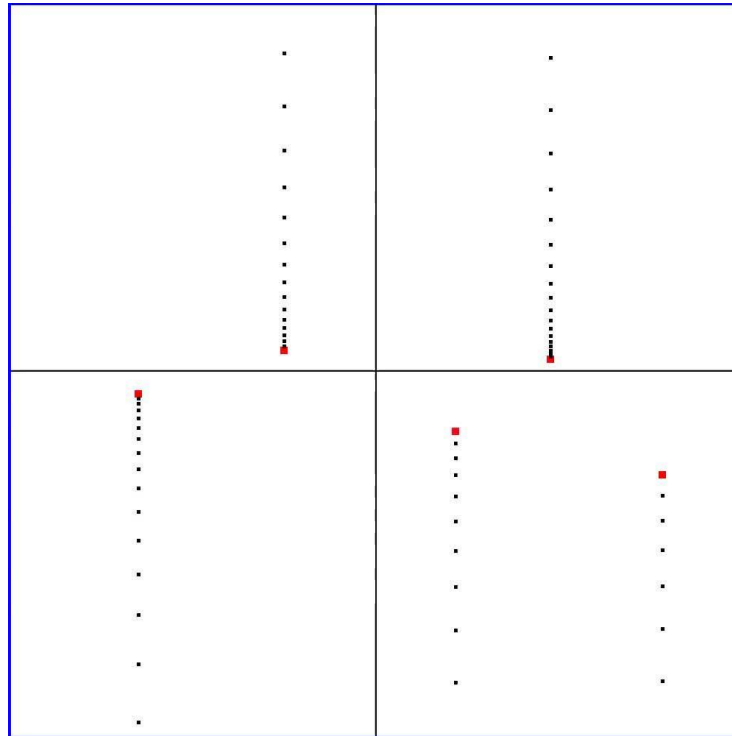


Figure 5 Eigenvalues 1, 1.2

Look at Figure 6. It is a little harder to see the eigenvectors in this figure but you can deduce that nothing is decreasing so both eigenvalues must be greater than zero in absolute value and in both runs the points jump around a great deal so at least one of the eigenvalues must be negative. Matrices of this type prompted us to build an interactive GeoGebra app to help students get a deeper understanding of eigenvectors and eigenvalues even when the dynamical system is not as obvious as in the previous cases. The original code was written in C and the graphics were done in OpenGL; a link to its GitHub repository is given in the appendix. The GeoGebra App is much more user-friendly and can be run in the cloud; a link to it is also found in the appendix. Note there are 2 separate runs in this figure that were created by the matrix A ($A_{1,1} = 1/2$, $A_{1,2} = 2$, $A_{2,1} = 1/2$, $A_{2,2} = -1/2$).

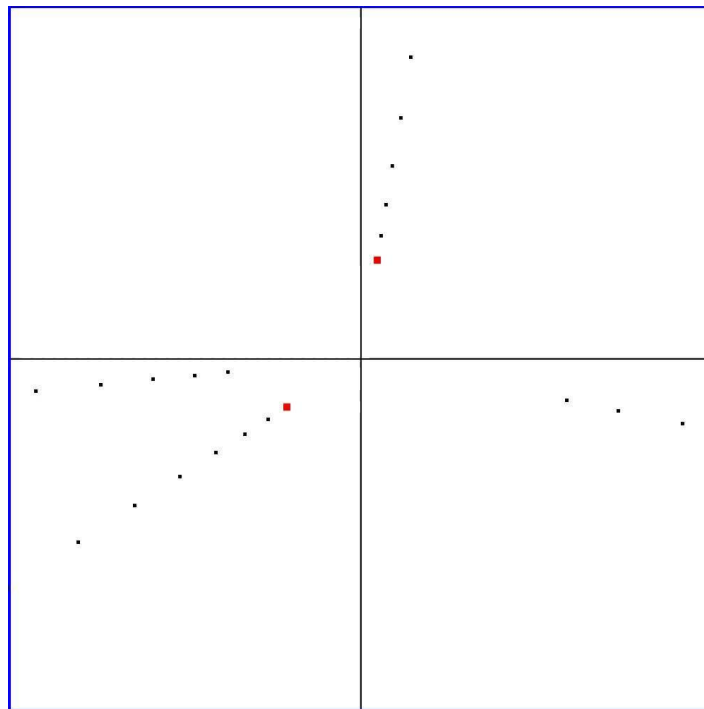


Figure 6 Eigenvalues $\frac{\sqrt{5}}{2}$, $-\frac{\sqrt{5}}{2}$

Figures 7-9 on the next page are snapshots from the GeoGebra app operating on the same matrix that created Figure 3. The red dot is the initial condition followed by 9 interactions of the dynamical system. The user can drag the red dot around to generate different scenarios. The far right column lists the successive ratios of distances between points. If the student drags the red point around until it creates a line, then they know they are on an eigenvector because the projection from a point on an eigenvector to the other eigenvector will be zero. We saw this in the 1D examples. Zero is always a fixed point. Also, they will see that the right column will display the eigenvalue for the eigenvector that you are on.

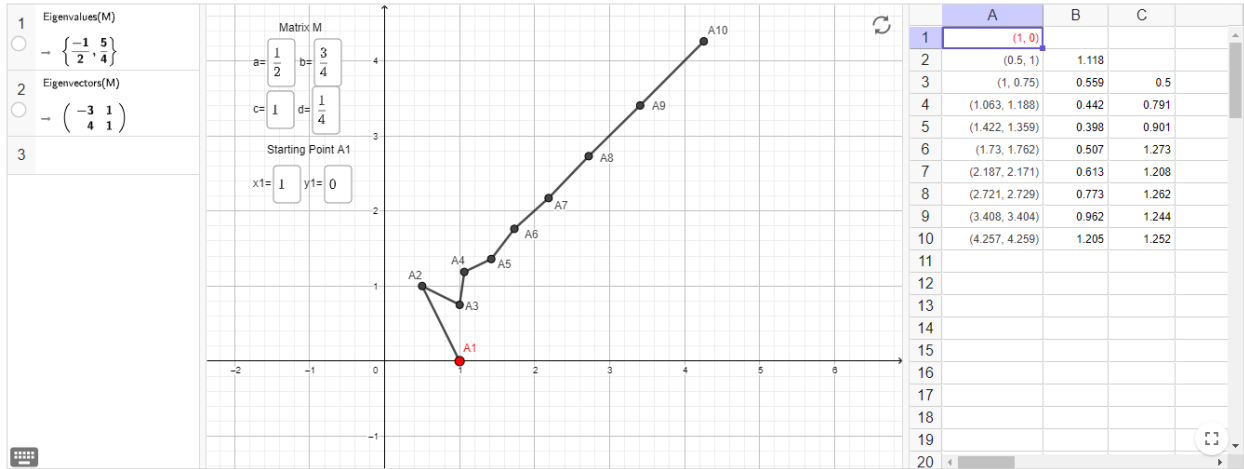


Figure 7 Eigenvalues $\frac{5}{4}, -\frac{1}{2}$

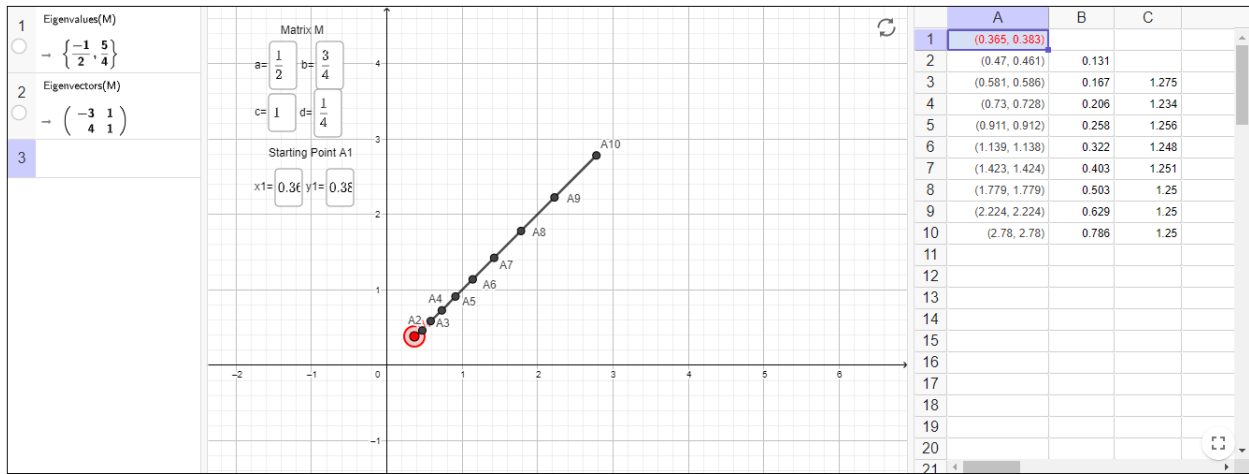


Figure 8 Eigenvalues $\frac{5}{4}, -\frac{1}{2}$ starting on an eigenvector.

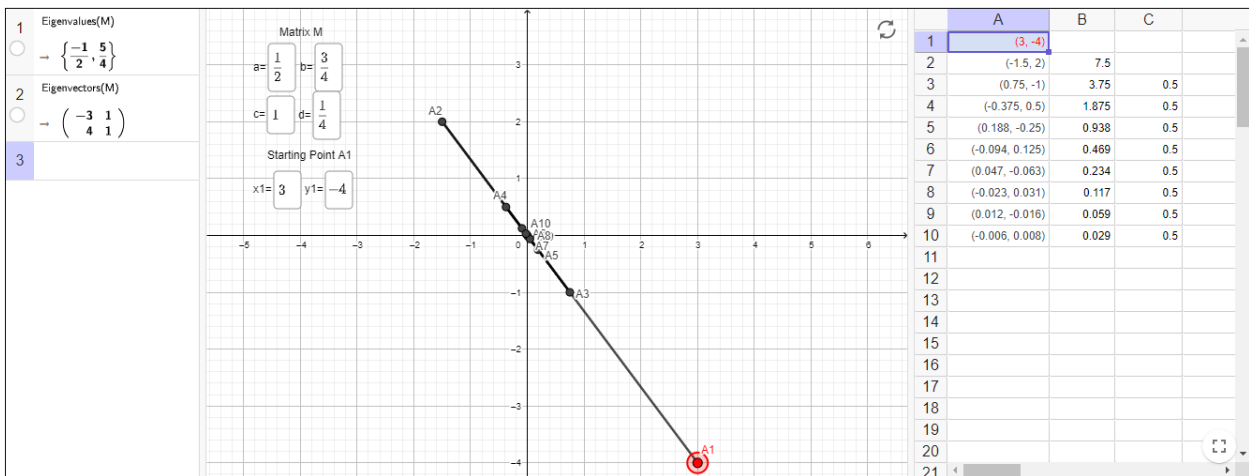


Figure 9 Eigenvalues $-\frac{5}{4}, -\frac{1}{2}$ starting on an eigenvector.

Figures 10 - 13 are snapshots from the GeoGebra app operating on the same matrix that created Figure 6. Figure 10 starts at the initial condition that produced the sequence of points in quadrants 1 and 4. Figure 11 starts at the initial condition that produced the sequence of points in quadrants 2 and 3. Figures 12 and 13 are screenshots where the initial condition was moved to find the two eigenvectors and the columns to the right show approximations to the associated eigenvalues.

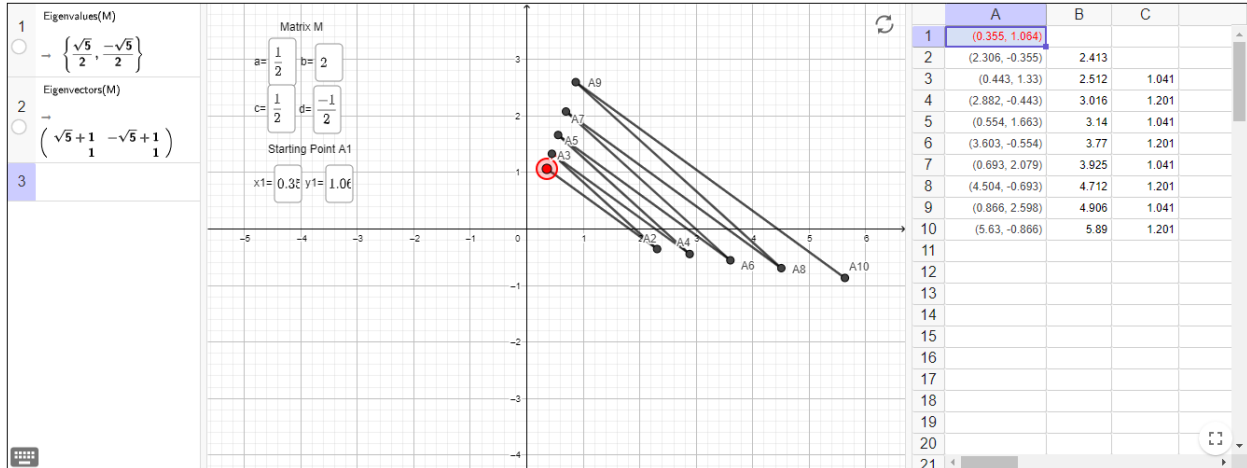


Figure 10 Eigenvalues $\frac{\sqrt{5}}{2}, -\frac{\sqrt{5}}{2}$

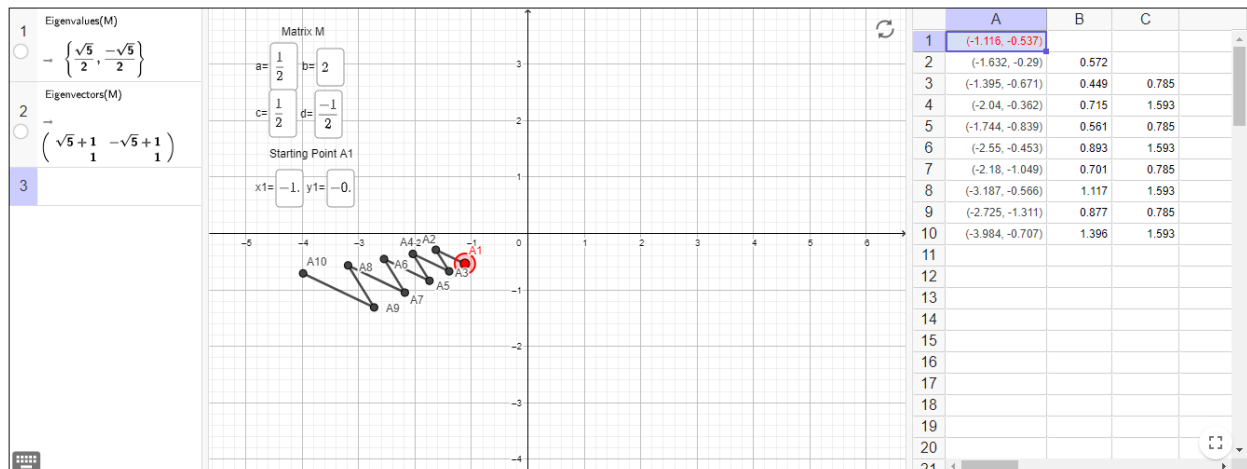


Figure 11 Eigenvalues $\frac{\sqrt{5}}{2}, -\frac{\sqrt{5}}{2}$

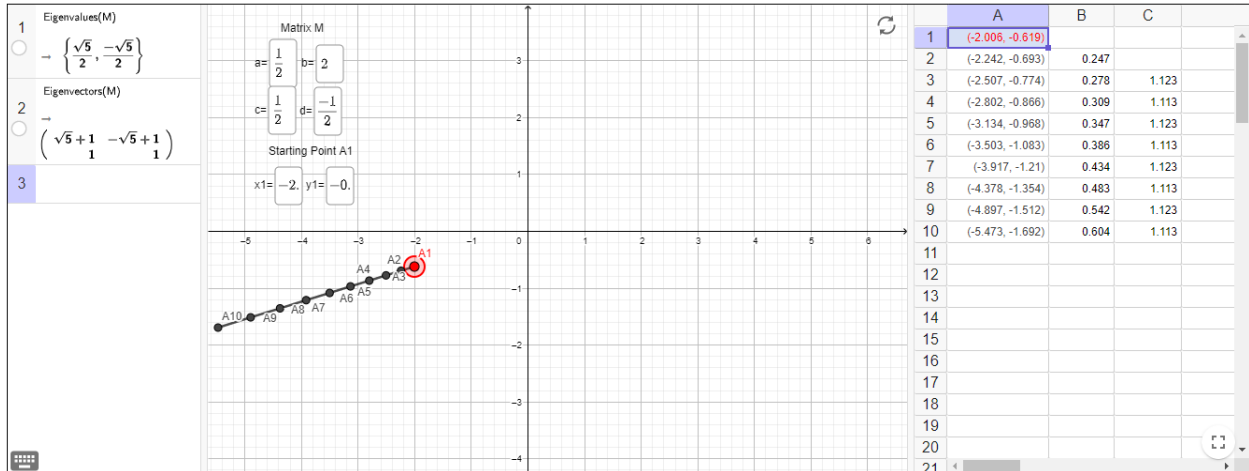


Figure 12 Eigenvalues $\frac{\sqrt{5}}{2}$, $-\frac{\sqrt{5}}{2}$ starting on an eigenvector.

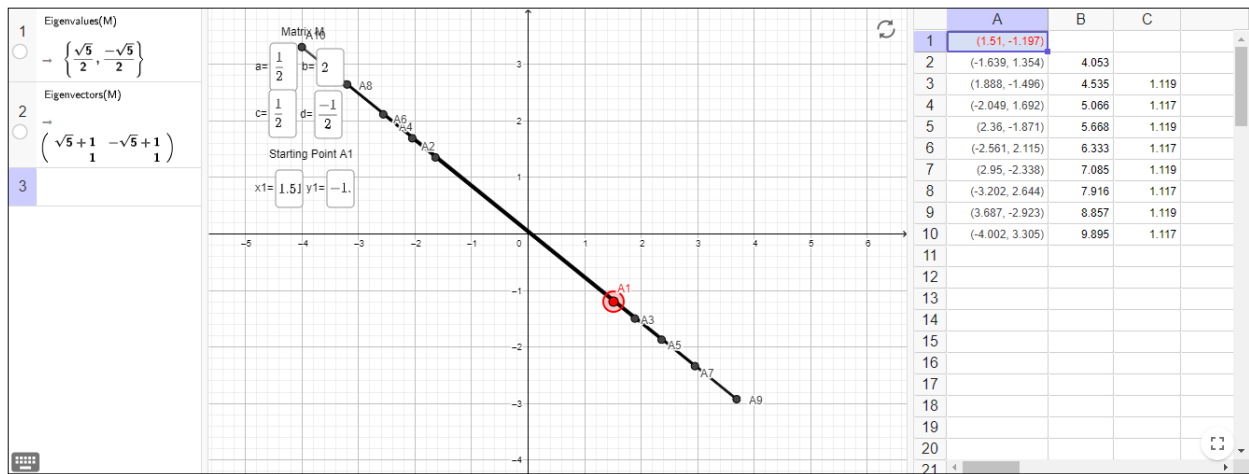


Figure 13 Eigenvalues $\frac{\sqrt{5}}{2}$, $-\frac{\sqrt{5}}{2}$ starting on an eigenvector.

In summary, we see that eigenvectors are like a set of straight roads that go through the origin and eigenvalues are like speed limits associated to its individual road. In this presentation, we did not talk about imaginary eigenvectors and eigenvalues or about repeated eigenvectors. We will save this for our next presentation.

CONCLUSION

Graphical representations are valuable when they illustrate the connections between what we know from symbolic mathematics and the underlying concepts. Students need to see the connections between the symbolic, numeric, and graphical representations of mathematical

concepts when possible. Traditional methods of calculating eigenvalues and eigenvectors emphasize abstract, symbolic manipulation. This application allows the teacher to give a deeper conceptual understanding of eigenvalues and eigenvectors in 2-dimensions which will allow the student to extend the 2D understanding into higher dimensions. We hope the reader will be able to use this graphical approach to help students better understand the calculation of eigenvalues and eigenvectors.

MEDIA LINKS

Link to GeoGebra file: <https://www.geogebra.org/m/ywm5nvsk>

Link to GitHub file:

<https://github.com/TSUParticleModelingGroup/eigenValuesViewer/blob/5d2afdee8800e13fb55f442494af529258c70602/EigenValueViewerICTCM.cu>

Link to Dr. Gresham's page:

https://www.tarleton.edu/faculty/jgresham/ictcm/ictcm_index.html

Tarleton's applied mathematics website: <https://www.tsucomputationalmathematics.com/>

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