EIGENVALUES AND EIGENVECTORS PART II: A GRAPHICAL APPROACH USING GEOGEBRA WITH COMPLEX EIGENVALUES

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ABSTRACT

In our presentation at ICTCM 2023 we focused on how to visualize non-repeating and real eigenvalues and their associated eigenvectors for 2x2 matrices. We now extend this work to include the realms of repeated real eigenvalues and complex non-real eigenvalues. Our aim again is to address the challenge in teaching eigenvalues and eigenvectors where they are typically presented as isolated procedures without context.

To rectify this, we introduce an interactive GeoGebra application that enables users to explore two-dimensional linear dynamical systems' behavior and classification based on eigenvalues and eigenvectors. This tool benefits educators and students seeking a deeper conceptual grasp. By bridging theory and application, we seek to transform the learning experience and facilitate a better understanding of these crucial linear algebra concepts.

INTRODUCTION

In our previous presentation we showed how we can use GeoGebra to illustrate eigenvalues and eigenvectors through repeated multiplication of a 2-dimensional vector V by a 2x2 matrix M. In this illustration the vector $\vec{v} = \langle 1,0 \rangle$ and the matrix $M = \begin{bmatrix} \frac{1}{2} & \frac{3}{4} \\ 1 & \frac{1}{4} \end{bmatrix}$.

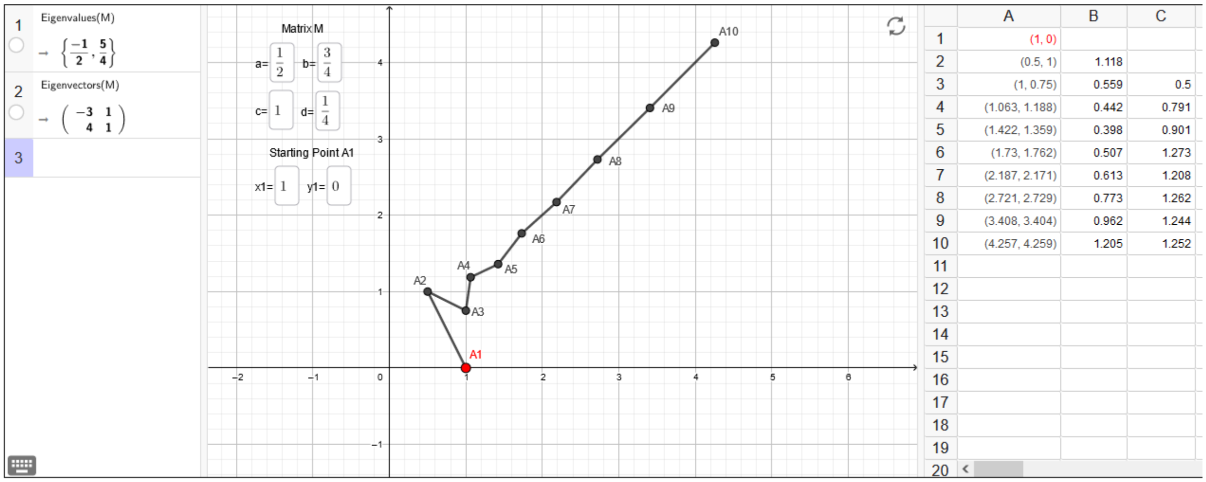

Figure 1 Start Vector = ⟨1,0⟩

This GeoGebra picture, whose link is https://www.geogebra.org/m/ywm5nvsk, shows the product vectors

$$\vec{v}_{n+1} = M\vec{v}_n$$

with coordinates given in column A of the spreadsheet window. The lengths of the product vectors are given in column B and the ratios of successive lengths is given in column C. These ratios approach the eigenvalue $\frac{5}{4}$ after repeated multiplications.

In the next illustration we have placed the starting point of the vector $\vec{v}_1$ at the eigenvector ⟨1,1⟩. Notice that all the product vectors line up and the ratios of successive lengths is 1.25.

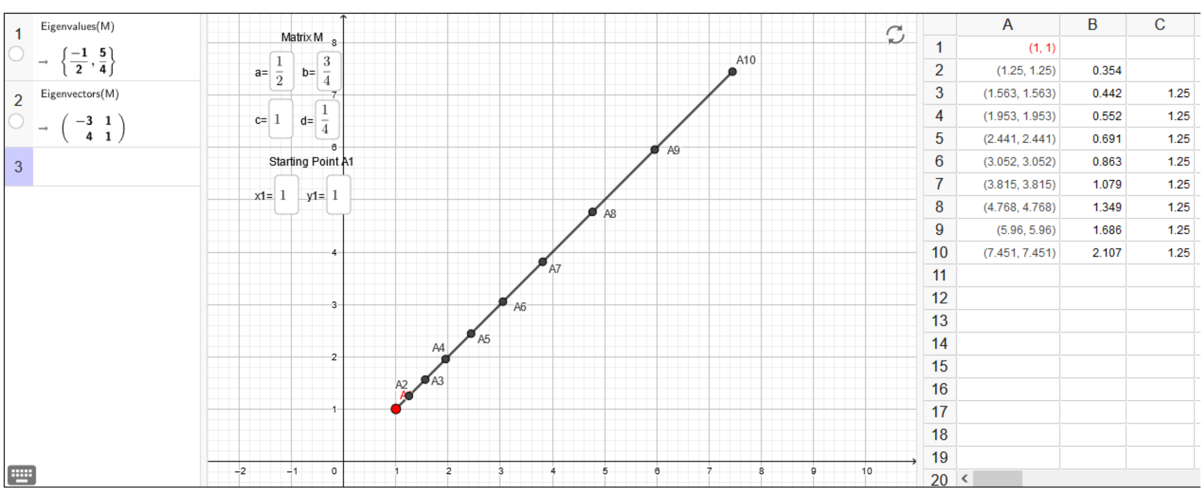

Figure 2 Start Vector = ⟨1,1⟩

Finally, we will move the starting vector point to the eigenvector $\langle -3,4 \rangle$ and we see that the length ratios are ½. Since the eigenvalue is negative, the product vectors bounce back and forth across the origin.

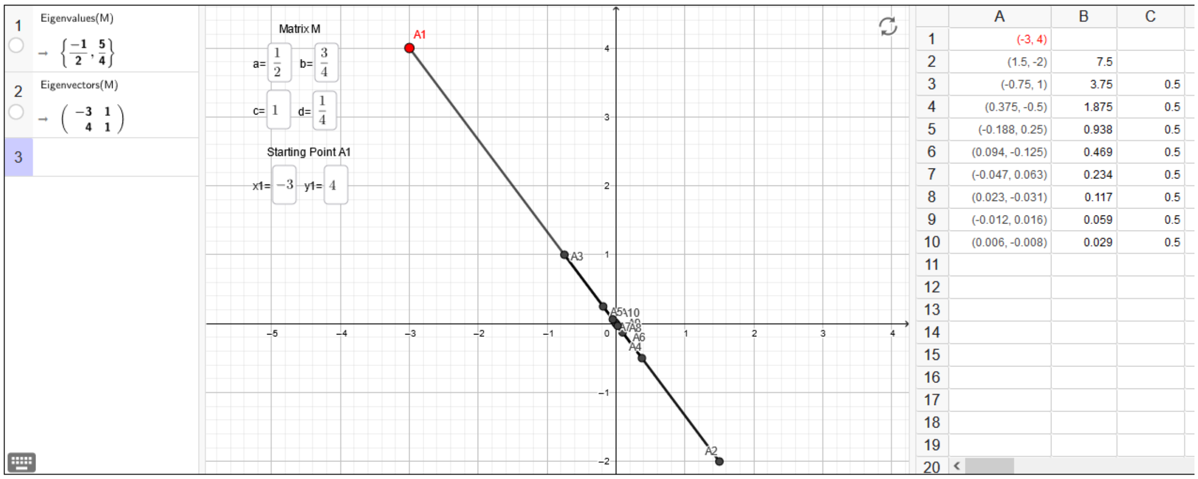

Figure 3 Start Vector = $\langle -3,4 \rangle$

ILLUSTRATIONS FOR MATRICES WITH COMPLEX EIGENVALUES

In this paper we will consider what happens when we use a matrix M with complex number eigenvalues. We will see that the absolute value of a complex eigenvalue (here named ev) influences the pattern of the product vectors:

1. If $|ev| < 1$ then the product vectors spiral in toward the origin.
2. If $|ev| > 1$ then the product vectors spiral in outward away from the origin.
3. If $|ev| = 1$ then the product vectors remain roughly equidistant from the origin.

In these examples our starting vectors $\vec{v}_1 = \langle 1,0 \rangle$.

Example 1: $M = \begin{bmatrix} \frac{1}{2} & -1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$. The eigenvalues are $ev = \frac{1}{2} \pm \frac{\sqrt{2}}{2} i$ with magnitude $\frac{\sqrt{3}}{2} < 1$

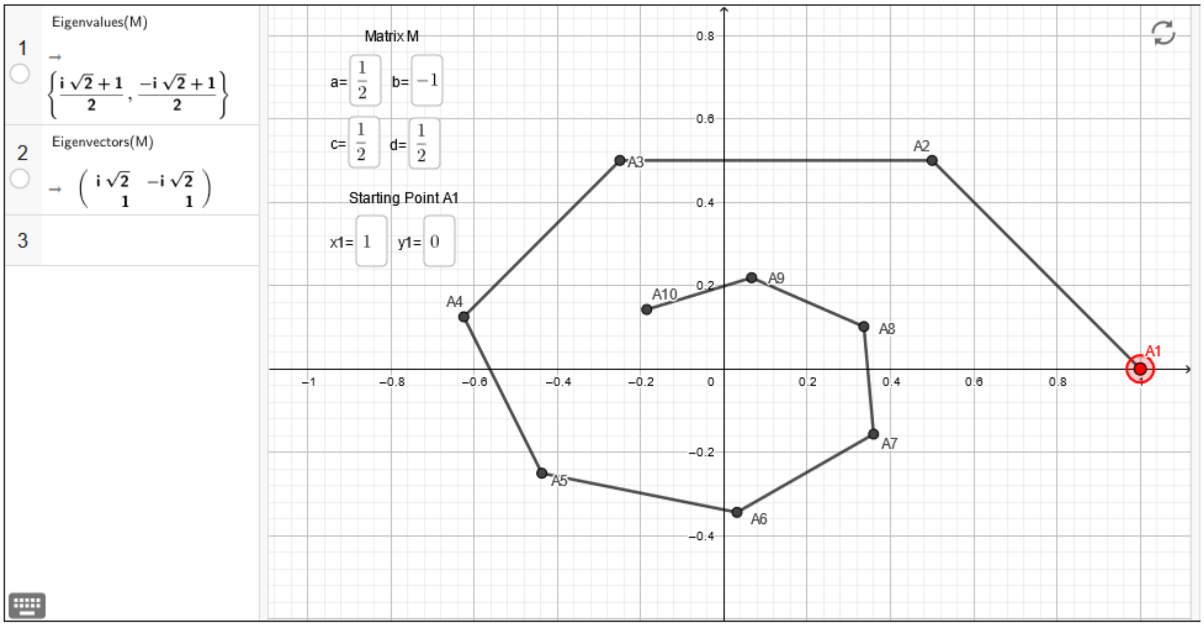

Figure 4 Complex Eigenvalue with Absolute Value Less Than 1

Example 2: $M = \begin{bmatrix} \frac{3}{5} & -\frac{6}{5} \\ \frac{3}{5} & \frac{3}{5} \end{bmatrix}$.

The eigenvalues are $ev = \frac{3}{5}\left(1 \pm \sqrt{2}i\right)$ with magnitude $\frac{3\sqrt{3}}{5} > 1$

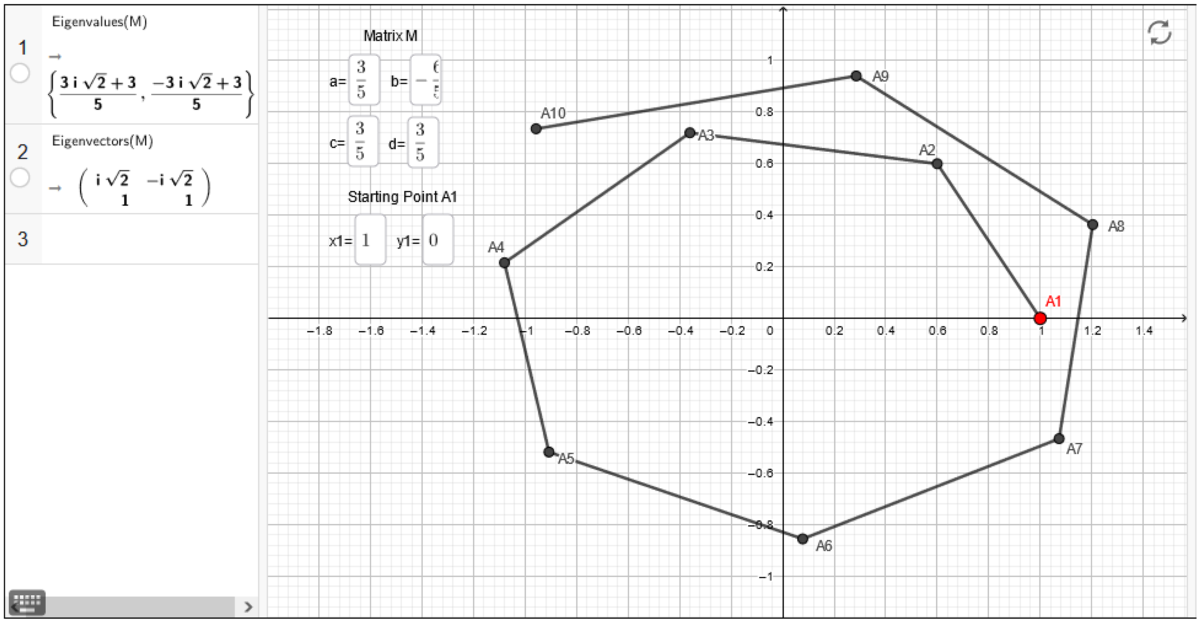

Figure 5 Complex Eigenvalue with Absolute Value Greater Than 1

In the following examples we will consider matrices with a complex eigenvalue whose absolute value is 1. First, we will look at a rotation matrix.

Example 3: Rotation Matrix $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ with $\theta = \frac{\pi}{3}$. $M = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$.

$$ev = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

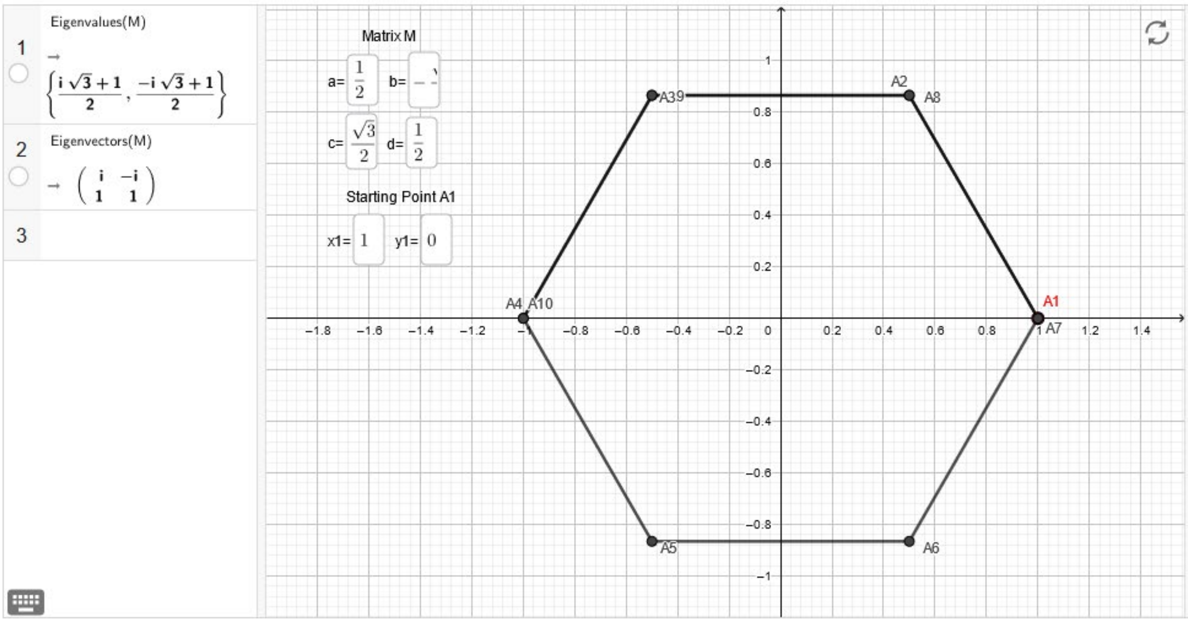

Figure 6 Complex Eigenvalue with Absolute Value Equal to 1

Using rotation matrices with $\theta = \frac{\pi}{2}$ will give a square.

Example 4: A non-rotation matrix with complex eigenvalues whose absolute value is 1. In these examples we consider a multiplication matrix with form

$$M = \begin{bmatrix} 0 & -1 \\ 1 & d \end{bmatrix}$$

The eigenvalues have the form

$$\frac{d \pm \sqrt{d^2 - 4}}{2}$$

When $|d| < 2$, the eigenvalues are complex numbers whose absolute value is 1. We have created a GeoGebra illustrator for this situation in which there is a slider for d which can be used to experiment with different values for d.

The link to this illustrator is https://www.geogebra.org/m/wgvej9fb

In this example we let d=1: $M = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$.

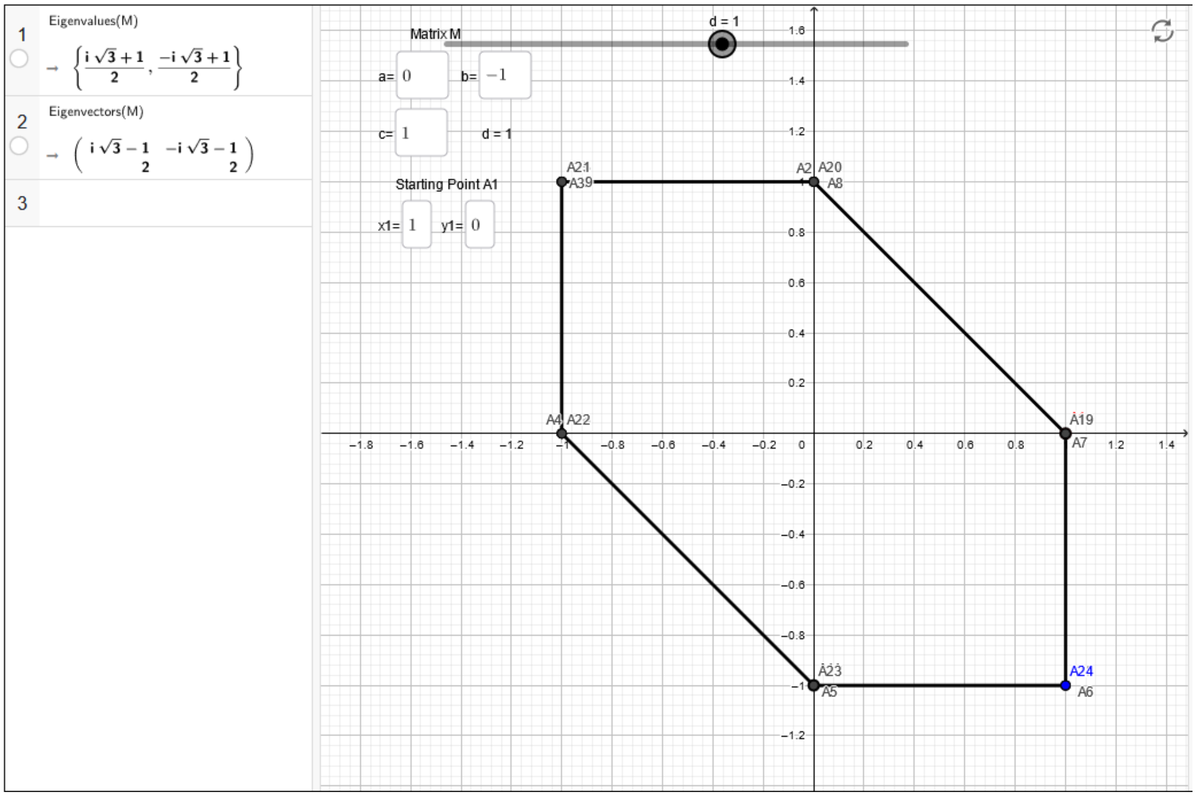

Figure 7: d=1 with a 6-cycle of points

Notice that the points A are in a cycle of 6: A1=A7, A2=A8, etc.

Example 5: In this example we let d = -1: $M = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$.

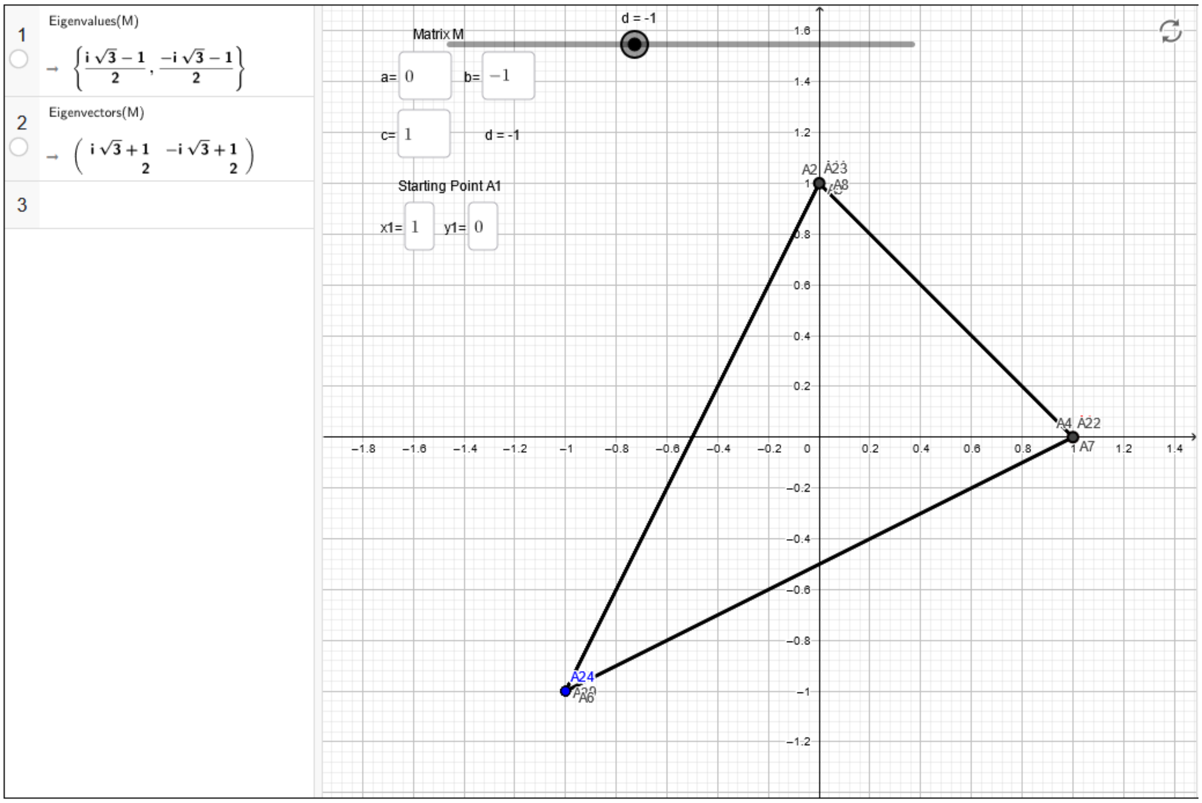

Figure 8: d = -1 with a 3-cycle of points

As we manipulate the slider for d we can search for values of d that produce integer cycles of points. For instance, if we let $d = 1.618$, the golden ratio, we get a 10-cycle.

Example 6: In this example we let d = 1.618: $M = \begin{bmatrix} 0 & -1 \\ 1 & 1.618 \end{bmatrix}$.

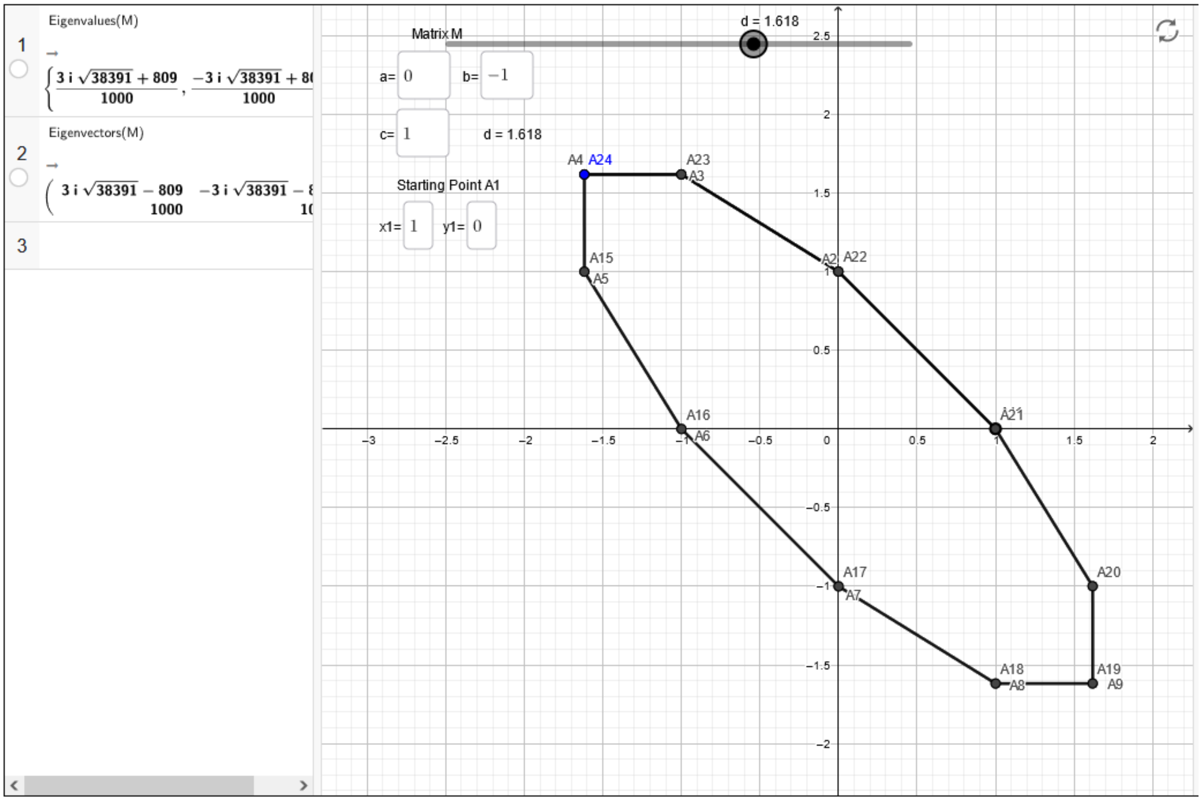

Figure 9: d = 1.618 with a 10-cycle of points

The slider d in the GeoGebra illustrator will allow you to try values of d that will produce cycles of all integer values. For instance, d = 1.247 will produce a 7-cycle. Other values we found by experimenting with the slider are

1 cycle, d = 2 and start on y = -x
2 cycle, d = -2 and start on y = x
3 cycle, d = -1
4 cycle, d = 0
5 cycle, d = golden ratio-1
6 cycle, d = 1
7 cycle, d ≈ 1.247????
8 cycle, d = sqrt(2)
9 cycle, d ≈ 1.532???
10 cycle, d = golden ratio
11 cycle, d ≈ 1.6825????
12 cycle, d = sqrt(3)

The question then arose as to what the relation is between these special numbers for d and the number of cycles. We appreciate Dr. Emmert's solution using powers of M to find that for an n-cycle we let

$$d = 2\cos(2\pi/n)$$

CONCLUSION

We have shown examples of dynamical systems with complex eigenvalues and how successive multiplications lead to geometric patterns in the 2-dimensional case.

We are interested in finding the relationship between the absolute value of complex eigenvalues and the ratios between lengths of successive multiplication vectors. We also are interested in developing a 3d GeoGebra illustrator using multiplication by a 3x3 matrix.

MEDIA LINKS

Link to GeoGebra file: https://www.geogebra.org/m/ywm5nvsk

Link to second GeoGebra file: https://www.geogebra.org/m/wgvej9fb

Link to GitHub file:
https://github.com/TSUParticleModelingGroup/eigenValuesViewer/blob/5d2afdee8800e13fb55f442494af529258c70602/EigenValueViewerICTCM.cu

Link to ICTCM 2023 paper:
https://www.pearson.com/content/dam/global-store/en-us/resources/EigenvaluesandEigenvectorsAGraphicalPedagogicalApproachUsingGeogebra.pdf

Link to Dr. Gresham's page: https://faculty.tarleton.edu/jgresham/ictcm/

Tarleton's applied mathematics website: https://www.tsucomputationalmathematics.com/

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