EXCURSIONS WITH RECURSION

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Introduction

Mathematics faculty at St. John Fisher are looking to integrate more technology into the curriculum - both at the program and course levels. At Fisher, Discrete Mathematics is an introduction to proof course with an audience comprised of students from mathematics, computer science, statistics, and cyber security. With an emphasis on proof and proof techniques, the course can benefit from a technological infusion, and, rife with potential for this audience, is recursion and recursive relationships.

Many people are familiar with the Fibonacci sequence, defined by the sequence where terms are generated by adding the two previous terms, beginning with 1 and 1. Fibonacci numbers are interesting for a whole host of reasons ranging from their relationship to nature, their aesthetic appeal, and relationships with other areas of mathematics like Pascal's triangle. Less well known are the Lucas numbers. The Lucas numbers share the same recursive relationship with the Fibonacci sequence, but instead start with 1 and 3.

In this paper, we discuss Fibonacci numbers, Lucas numbers, potentially surprising results, extensions, and a writing assignment in Discrete Mathematics in which students used the technology of their choice to explore the ratios of successive terms in sequences with Fibonacci growth to generate data to make conjectures. Technological trials, tribulations, and trifles are tendered.

The Inspiration and Problem Context

The inspiration for this assignment was a dearth of technology in an Introduction to Proof course in the mathematics major. This course is also an elective for computer science, statistics, and cybersecurity majors. The course is only required for mathematics majors and is often students' first encounter with proof. As a bridge course to upper level proving courses, the course emphasizes proof over computation. Historically, students have used Excel or other spreadsheet software to explore recursively defined sequences, and specifically, the Fibonacci numbers. Students would collect data and make conjectures about relationships involving Fibonacci numbers, often with the intent to generalize and prove their conjectures. Eventually, it became hit-or-miss as to whether students had experience with spreadsheet software. Many students were unfamiliar with how to use both closed and recursive formulas to generate data. Unfortunately, the semester often runs short of time to discuss recursive relationships (and their proofs) with much depth, so there is rarely extra class time to dedicate to learning new technology. As a result, over time, students' explorations of Fibonacci sequences have become an optional activity.

There has always been insufficient technology in this course, even for the mathematics majors, but especially so for the majors for whom the course is an elective. However, the choice of technology was not obvious. Each of these major cohorts use quite different technologies in their disciplines. Statistics students used R and historically computer science students started their foray into programming with Java. Until recently, mathematics students did not have to have any formal experience in computer programming but rather would learn algorithmic thinking to create mini-programs or subroutines in Maple or other computer algebra systems as part of specific classes that emphasize technology.

The mathematics department is prioritizing the integration of technology more holistically into the curriculum and the program itself and is now making use of Python in the sophomore seminar class. Computer science also made the switch to Python as their introductory programming language. All students take a data analysis course, and in doing so, gain experience using spreadsheet software. While it has been an on-going problem to decide how to support the different technologies without committing much class time to said support, the recent convergence to fewer likely technologies makes for a more manageable assignment going forward.

First, we start with some background on the problem context. Many readers are likely familiar with Fibonacci numbers, or the numbers generated by the Fibonacci sequence, defined by the recursive relationship, $F_1 = 1$, $F_2 = 1$, and $F_{n+1} = F_n + F_{n-1}$, for natural numbers n, $n \geq 2$.

The Fibonacci numbers are a popular sequence to explore for the mathematically curious or for the curious at heart, as not all Fibonacci phenomena are mathematically specific. For example, Fibonacci numbers are commonly found in nature, when looking at the number of petals in certain flowers or the number of spirals (clockwise and counterclockwise) in pinecones, pineapples, sunflowers, and other objects found in nature. There are numerous resources detailing Fibonacci numbers in nature like those provided by Rod Pierce (2023) via the Math is Fun website and Lamb and Shields (2023) on the How Stuff Works webpage available online at https://www.mathsisfun.com/numbers/fibonacci-sequence.html, and https://science.howstuffworks.com/math-concepts/fibonacci-nature.htm, respectively.

Those interested in more information about Fibonacci numbers can find plenty of activities and lesson plans on the internet, like those provided by Mensa for kids (2024) available online at https://www.mensaforkids.org/teach/lesson-plans/fabulous-fibonacci/.
Fibonacci aficionados can celebrate Fibonacci day on November 23 or publish in the Fibonacci Quarterly (2023), the principal publication of the Fibonacci Association with web presence at https://www.fq.math.ca/. The editors invite manuscripts involving new results, novel proofs of known relationships, research proposals, or interesting problems.

Fibonacci Numbers, in addition to their connections to nature and other areas of mathematics, have an interesting property when looking at the limiting behavior of the sequence. Consider the limit of the ratio of successive terms of the Fibonacci sequence as n approaches infinity, or $\lim_{n \to \infty} \frac{F_{n+1}}{F_n}$. The ratios settle down to a real number, so that the limit exists, and is equal to the irrational number $\frac{1+\sqrt{5}}{2}$. We call this number the Golden Ratio and it is denoted by the Greek letter ϕ, so that $\lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \phi = \frac{1+\sqrt{5}}{2}$.

The Lucas numbers are another number sequence with similar properties. The Lucas numbers are also generated by the recurrence relation, $L_{n+1} = L_n + L_{n-1}$, for natural numbers n, $n \geq 2$. Instead, the initial values are given by $L_1 = 1$ and $L_2 = 3$. Given the way in which additional terms of the Lucas number sequence are generated, it is natural to wonder if the ratio of successive Lucas numbers approaches a limit as n approaches infinity. Using technology, it is easy to conjecture that the same result as the Fibonacci numbers is true for the Lucas numbers, or that for Lucas numbers, L_n , $\lim_{n \to \infty} \frac{L_{n+1}}{L_n} = \frac{1+\sqrt{5}}{2}$.

This could be a surprising result, inviting more questions. Is this just a curious coincidence that the ratio of successive Fibonacci and Lucas numbers both approach the Golden ratio, or is there something deeper going on? What about other Fibonacci-adjacent sequences (sequences with "Fibonacci growth"), or those sequences defined recursively by $F_1 = a$, $F_2 = b$, and $F_{n+1} = F_n + F_{n-1}$, for natural numbers n, $n \geq 2$ and real numbers a and b. Are there any "trouble seeds", or real number initial values, a and b, for which the limit is not the Golden ratio? Do the starter values matter at all? Do all roads (seeds) lead to $\lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2}$?

It was with these goals and questions in mind that the "excursions with recursion" was conceived, an assignment in which students used their technology of choice to explore sequences with Fibonacci growth to determine if the ratio of successive terms always approached the Golden ratio or if there were any problematic initial values to be avoided.

The Technology "Problem" and the Assignment

In constructing this assignment, the guiding principle was that it was a writing assignment in which students can use any technology other than AI-based tools, to make their explorations. It was expected that students would complete this work using either spreadsheet software like Excel or Google sheets or would use a programming language like Python or another language familiar to the student. Even with a narrowed technological scope, there were still many questions to be resolved. How does one accommodate ANY technology? What content should students turn in? What form should their work take? How should student work be assessed?

In finalizing this assignment, multiple colleagues were solicited for constructive suggestions on how students might submit their work. Our preferences are strong, and the responses were consistently "have them use Excel" in response to the query. To be fair, that was my technology of choice as well, but the guiding principle to this assignment was that students would be able to use their technology of choice. Given that many students had experience using both Python and spreadsheet software like Excel or Google sheets, there was also interest in finding out if students had any insight into the relative advantages (or disadvantages) of the potential technologies. Mathematics faculty had strong opinions. Perhaps the students did as well.

Not everyone was familiar with the Fibonacci numbers, so we started the class conversation by constructing the sequence from scratch and discussing the recursively defined way to generate subsequent terms of the sequence. To that end, not all the students were familiar with the Golden ratio, so there was a fair amount of set up and discussion to make sure everyone understood the problem context and assignment itself.

The final product was to be a Google document that was shared with the instructor electronically, consisting of an exposition of the student's explorations of the problem context, their results, discussion and analysis. Students were asked to include supporting screenshots of their "coding work" and resulting data. The assignment was posted in Brightspace as a link to an editable Google document that was discussed together as a group in class. The intent was so that we would be able to add clarification to our shared document if needed. While the document was not updated, questions for clarification were asked and answered in class.

The Results, including Hard to Find Trouble Seeds and Assessment

At St. John Fisher, faculty members take research involving human subjects seriously, and all research is required to go through the Institutional Review Board (IRB) process which starts with CITI Program training, available at https://about.citiprogram.org/, and is followed by a rigorous collections of tasks, including application, designed to protect research subjects, outlined on the SJF Institutional Review Board guidelines for IRB page, available online at https://www.sjf.edu/services/institutional-review-board/guidelines-for-irb-application/. The intent of this project was more "preliminary report" than "proper research" as the design was to test drive a student assignment and see what worked well, what worked less well, and what could be improved. Fortunately, we can share these sorts of general findings and observations without going through the IRB process.

Primary technologies used by the student included Excel (or Google sheets) and Python (using Google Colab or GitHub). One student wrote their work on paper, and another did not seem to fully understand the assignment, as incomplete code, with no results, was typed into the student's submission. Mathematics, computer science, and cybersecurity students all have experience with Python either through the first-year programming course or through the sophomore seminar class required for mathematics majors. All students get some experience with spreadsheet software in a data analysis course, so students who had no computer science experience or had not yet taken the sophomore seminar class used Excel or Google sheets.

Trifles were promised, so perhaps surprising to no one, students used the technology they were comfortable with. One noteworthy result was that when given an alternative to Google sheets or Excel, like Python, students took it. The data analysis course that students take is part of the new Core curriculum and varies by instructor. It was striking that their experience in these course(s) did not translate into being able to generate data using a basic recursive relationship. Related is that the "coding" piece seemed difficult in some instances, so why would students code when they have a recursive friendly spreadsheet software that requires little to no coding? It turns out that many (most) sections of the data analysis course focus on built-in statistical functions and simply do not elucidate the recursive richness or mathematical power of spreadsheet software.

There were no particularly insightful perspectives on the relative advantages (or disadvantages) of the different potential technologies. Perhaps that was too much to hope for, but it remains a good question and worth asking. Most students were able to construct a conjecture about the limit of ratio of successive terms in Fibonacci-adjacent sequences, or those generated by the Fibonacci recursive growth pattern. However, none of the students were able to identify the seeds to avoid so that their conjecture was true. Some students spoke of trouble seeds involving zeros but less obvious are the trouble seeds for which the limit of ratio of successive terms in these sequences is not the Golden Ratio. Regarding assessment, it was always the intent to give full credit if students met the basic guidelines and criteria of the assignment, whether they were able to determine the trouble seeds or not. The seeds to avoid are irrational, and thus are very unlikely to be found by trial and error. Ultimately, full credit was given generously, if students had tried positive, negative, and fractional starting values, the supporting work was included, and the exposition was reasonable, then full credit was earned.

It turns out that the hard-to-find trouble seeds are related to the other root of the quadratic equation generated by the Fibonacci recurrence relation, $F_{n+1} = F_n + F_{n-1}$. Dividing both sides of the Fibonacci recurrence relation by the n^{th} Fibonacci number, F_n, it follows that $\frac{F_{n+1}}{F_n} = \frac{F_n + F_{n-1}}{F_n} = 1 + \frac{1}{\frac{F_n}{F_{n-1}}}$.

Thus, if we take the limit of both sides of the equation, we have that

$$\lim_{n\to\infty}\left(\frac{F_{n+1}}{F_n}\right) = \lim_{n\to\infty}\left(\frac{F_n + F_{n-1}}{F_n}\right) = \lim_{n\to\infty}\left(1 + \frac{1}{\frac{F_n}{F_{n-1}}}\right).$$

We also have that $\lim_{n\to\infty}\frac{F_n}{F_{n-1}} = \lim_{n\to\infty}\frac{F_{n+1}}{F_n}$ and we will call this limit L.

By substitution, we have that $L = 1 + \frac{1}{L}$.

Now multiplying both sides of this equation by L and rearranging so that the right-hand side is equal to 0, yields $L^2 - L - 1 = 0$. Using the quadratic formula to solve for L, we get that $L = \frac{1 \pm \sqrt{5}}{2}$, meaning that $L = \frac{1+\sqrt{5}}{2}$ or $L = \frac{1-\sqrt{5}}{2}$.

The first value is the golden ratio, $\phi = \frac{1+\sqrt{5}}{2}$ and the second value is going to generate the hard to find (by trial-and-error) trouble seeds. What these roots indicate is that for the recursively defined sequence, $F_1 = a,\ F_2 = b$, and $F_{n+1} = F_n + F_{n-1}$, for natural numbers

n, $n \geq 2$ and real numbers a and b, initial values $a = 1$ and $b = \frac{1-\sqrt{5}}{2}$ will generate a limit of $\lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \frac{1-\sqrt{5}}{2}$. And more generally, $a = c, c \neq 0$ and $b = \frac{c(1-\sqrt{5})}{2}$, will also generate a limit of $\lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \frac{1-\sqrt{5}}{2}$.

The roots to the quadratic equation, $L^2 - L - 1 = 0$, are the only possible values for a limiting value for the ratio of successive Fibonacci-adjacent terms as n approaches infinity. If we start with trouble seeds that form a ratio of $\frac{1-\sqrt{5}}{2}$, the limit will never move away from $\frac{1-\sqrt{5}}{2}$. If trouble seeds are avoided, the limit of ratio of successive Fibonacci-adjacent terms will be the Golden ratio $\phi = \frac{1+\sqrt{5}}{2}$. For those with interest, kindly look for a related manuscript (in preparation) that speaks to these excursions in this problem context with a focus on the mathematical details of the hard-to-find trouble seeds.

In Figure 1 on the next page, we see examples of what happens when we start with the trouble seeds (as approximated by Excel) in both scenarios outlined above. Given are three sequences, each with their corresponding ratio sequence, R_n . In the first sequence, T_n , the initial values are $a = 1$ and $b = \frac{1-\sqrt{5}}{2}$ and in the second sequence, G_n , the initial values are $a = 3$ and $b = \frac{3(1-\sqrt{5})}{2}$. In both situations the ratio of successive Fibonacci values starts with an approximation to $\frac{1-\sqrt{5}}{2}$ and doesn't vary beyond round off error.

In the third sequence, D_n , a hand-entered decimal approximation was used for $\frac{1-\sqrt{5}}{2}$. Specifically, $a = 3$ and $b = 3(-.61803398874)$. What is of significant is that the ratio sequence looks like it may behave the same way as the first two sequences, but upon examination of additional terms, by the 40th term, we see that this sequence will, in fact, converge to the Golden ratio, $\frac{1+\sqrt{5}}{2}$. This result was a surprising result because Excel approximates irrational numbers. A natural question is how accurate an approximation must be so that the ratio sequence converges to a particular root of the quadratic equation? Surely related to the program's accuracy, Excel's approximation to $\frac{1-\sqrt{5}}{2}$ was sufficient to ensure that the sequence of Fibonacci ratios would converge to a decimal approximation for $\frac{1-\sqrt{5}}{2}$. In contrast, when we approximated $\frac{1-\sqrt{5}}{2}$ by $-.61803398874$, the ratios of successive terms converged to $\frac{1+\sqrt{5}}{2}$.

Figure 1: Three Interesting Sequences of Fibonacci Ratios

n	Tn	Rn	n	Gn	Rn	n	Dn	Rn
1	3		1	3		1	3	
2	-1.854101966	-0.618033988	2	-1.854101966	-0.6180339887	2	-1.854101966	-0.6180339887
3	1.145898034	-0.618033988	3	1.145898034	-0.6180339887	3	1.145898034	-0.6180339888
4	-0.7082039325	-0.618033988	4	-0.708203932	-0.6180339887	4	-0.7082039324	-0.6180339887
5	0.4376941013	-0.618033988	5	0.437694101	-0.6180339887	5	0.4376941013	-0.6180339889
6	-0.2705098312	-0.618033988	6	-0.270509831	-0.6180339887	6	-0.2705098311	-0.6180339883
7	0.16718427	-0.618033988	7	0.16718427	-0.6180339887	7	0.1671842702	-0.61803399
8	-0.1033255612	-0.618033988	8	-0.103325561	-0.6180339887	8	-0.1033255609	-0.6180339856
9	0.06385870876	-0.618033988	9	0.063858708	-0.6180339887	9	0.06385870938	-0.6180339971
10	-0.03946685249	-0.618033988	10	-0.039466852	-0.6180339887	10	-0.03946685148	-0.6180339669
11	0.02439185627	-0.618033988	11	0.024391856	-0.6180339888	11	0.0243918579	-0.6180340459
12	-0.01507499622	-0.618033988	12	-0.015074996	-0.6180339887	12	-0.01507499358	-0.6180338391
13	0.009316860045	-0.618033988	13	0.009316860	-0.6180339888	13	0.00931686432	-0.6180343806
14	-0.005758136177	-0.618033988	14	-0.005758136	-0.6180339887	14	-0.00575812926	-0.6180329628
15	0.003558723869	-0.618033988	15	0.003558723	-0.6180339888	15	0.00355873506	-0.6180366746
16	-0.002199412308	-0.618033988	16	-0.002199412	-0.6180339887	16	-0.0021993942	-0.618026957
17	0.001359311561	-0.618033988	17	0.001359311	-0.6180339888	17	0.00135934086	-0.6180523982
18	0.000840100746	-0.618033989	18	-0.000840100	-0.6180339887	18	-0.00084005334	-0.6179857935
19	0.0005192108147	-0.618033988	19	0.000519210	-0.618033989	19	0.00051928752	-0.6181601754
20	0.000320889931	-0.618033990	20	-0.000320889	-0.6180339881	20	-0.00032076582	-0.6177036953
21	0.000198320883	-0.618033984	21	0.000198320	-0.6180339905	21	0.000198521699	-0.6188991705
22	0.000122569048	-0.618034001	22	-0.000122569	-0.6180339842	22	-0.00012224412	-0.61577208
23	.000075751834	-0.618033956	23	0.000075751	-0.6180340005	23	0.000076277579	-0.6239774951
						24	-0.00004596654	-0.6026219019
						25	0.000030311039	-0.6594152932
						26	-0.00001565550	-0.5164950076
						27	0.000014655538	-0.9361271364
						28	-0.00000099996	-0.0682309711
						29	0.000013655577	-13.65610094
						30	0.000012655615	0.9267726561
						31	0.000026311192	2.079013276
						32	0.000038966808	1.480997409
						33	0.000065278001	1.675220627
						34	0.000104244810	1.596936298
						35	0.000169522811	1.626199054
						36	0.000273767622	1.61493087
						37	0.000443290434	1.619221552
						38	0.000717058056	1.617580713
						39	0.00116034849	1.618207173
						40	0.001877406547	1.617967845
						41	0.003037755037	1.618059254
						42	0.004915161583	1.618024338
						43	0.00795291662	1.618037675
						44	0.0128680782	1.618032581

The Aftermath and The Next Iteration
or How I Learned to Love Assignments from the Island of Misfit Assignments

This was the students' least favorite assignment. It does not seem like they hated it with every fiber of their being, but when asked what their favorite and least favorite aspects of the course were, the most common response to "least favorite" was this writing assignment. In their defense, it is not hard to see why. Because we only had time for a light treatment of recursion, the assignment was a bit disconnected and not as integrated into other course content for students to have gotten the maximal benefit. If it felt like an add-on to the professor, it is easy to see how it could feel like an add-on to the students. I am undeterred. If anything, this assignment invites even more questions and opportunities for interesting explorations than originally expected. It is a preliminary report, so there will be more attempts.

Several improvements include the following. First, I expect to separate the Fibonacci and Lucas Numbers set up and the assignment itself into two separate documents. With students' varying backgrounds, not everyone will need the set-up, but it will be available. Also, it sometimes seemed like too much information was offered and this change would also solve that problem, as the set-up info will be available on demand and with interest.

Another refinement to consider is a better way to direct students to look for irrational numbers as starter seeds and potential trouble spots. Some students had previously explored the Fibonacci sequence and Lucas numbers, and related questions in another class, so there was hope that some students might recognize a path to the troubling initial values. Unfortunately, this didn't happen, so next iteration I expect to require many examples of all different types of numbers for seeds – rational, irrational, large, small, positive, and negative. Even with requiring students to try irrational numbers, they need broader insight into the Fibonacci numbers. It certainly helps to have had the "right" Fibonacci number conversations to have even a chance at making an in-road. If anything, irrational numbers are harder to come by through trial-and-error than rational numbers! Another version could be to be forthright once students have a conjecture and let them know there are indeed trouble seeds, infinitely many, and that part of the assignment is to find them.

Mathematics faculty preference for spreadsheet software is two-fold. One, it is easy to use, and two, we can see the behavior of the ratio sequences. Since not all the students find Excel easy, it may not be a good fit for everyone. Most students who used Python gave the limit of a ratio of large Fibonacci number to the previous term, like F_{25} to F_{24} to estimate $\lim_{n\to\infty}\left(\frac{F_{n+1}}{F_n}\right)$, missing out on the overall behavior of the ratio sequence, $\frac{F_{n+1}}{F_n}$. As we saw in the third sequence, D_n, in Figure 1, the behavior of the sequence itself can be interesting. This is easily fixed by changing the assignment so that students construct ratio sequences using their technology of choice and use this data to make conjectures. Perhaps a follow-up question could be to compare the relative advantages (and disadvantages) to their solution method to the strategy of using a ratio of large Fibonacci number to the previous term, like, say F_{25} to F_{24} to estimate $\lim_{n\to\infty}\left(\frac{F_{n+1}}{F_n}\right)$.

Explaining is hard, and these modifications would also serve to provide more direction on what students should include as part of their discussion. It would also be nice if it was not students' least favorite assignment, so more effort will be made to create hype and have it feel like less of an "add on" to the course. Thank you for sharing in the journey of this "choose your own technology" assignment. It is lovely in its simplicity and its generalizability. Please kindly take what you can use and adapt for your own situations.

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