FIBONACCI GEMS TO FOSTER ENGAGEMENT IN A
PRE-SERVICE TEACHER COURSE

Jay L. Schiffman
Retired, Rowan University
604-36 South Washington Square # 815, Philadelphia, PA 19106
schiffmanjay911@gmail.com

Abstract: This paper features several surprising outcomes associated with an amazing sequence that are hopefully somewhat novel and include connections to geometry, number theory, modeling, and algebra. In the spirit of wonder, magic and mystery that personifies mathematics, Fibonacci permeates the landscape. The basic goal is to become further engaged and inspired about the discipline we love. A possible plan to achieve our desired outcomes is to have participants work in small groups and solve rich problems related to the Fibonacci sequence to discover neat mathematical patterns and explore connections including a Fibonacci-Pythagorean connection. One can then analyze patterns leading to conjectures based on the analysis of these patterns.

1. Introductory Ideas

Many pre-service teachers are familiar with the following Fibonacci number trick: Select any ten consecutive Fibonacci numbers and divide the sum by eleven. The quotient is always the seventh term in your sequence. Hence if I were given 13 as the first of ten consecutive terms of my sequence, I know 233 is my final answer. The question is how? This paper will resolve a series of Fibonacci number tricks that are very palatable and easily amenable to anyone who possesses a knowledge of basic algebra. We finally establish a Fibonacci-Pythagorean connection and an additional Fibonacci-Lucas connection and form several conjectures. Participants can view appealing, engaging, accessible and fun mathematics. To elicit clarity, recall that the Fibonacci sequence is recursively defined as $F_1 = F_2 = 1,\ F_n = F_{n-2} + F_{n-1};\ n \geq 3,$ while the companion Lucas sequence is recursively defined as $L_1 = 1,\ L_2 = 3,\ L_n = L_{n-2} + L_{n-1};\ n \geq 3$ and the Reverse Lucas sequence later discussed is recursively defined as $RL_1 = 3,\ RL_2 = 1,\ RL_n = RL_{n-2} + RL_{n-1};\ n \geq 3.$

The initial forty terms of the Fibonacci sequence are furnished in the following set below:
{1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765, 10946, 17711, 28657, 46368, 75025, 121393, 196418, 317811, 514229, 832040, 1346269, 2178309, 3524578, 5702887, 9227465, 14930352, 24157817, 39088169, 63245986, 102334155}.

Similarly, the initial forty terms of the Lucas sequence is provided in the accompanying set:
{1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843, 1364, 2207, 3571, 5778, 9349, 15127, 24476, 39603, 64079, 103682, 167761, 271443, 439204, 710647, 1149851, 1860498, 3010349, 4870847, 7881196, 12752043, 20633239, 33385282, 54018521, 87403803, 141422324, 228826127}.

We finally enumerate the initial forty terms of the Reverse Lucas sequence is provided in the accompanying set:

{3, 1, 4, 5, 9, 14, 23, 37, 60, 97, 157, 254, 411, 665, 1076, 1741, 2817, 4558, 7375, 11933, 19308, 31241, 50549, 81790, 132339, 214129, 346468, 560597, 907065, 1467662, 2374727, 3842389, 6217116, 10059505, 16276621, 26336126, 42612747, 68948873, 111561620, 18051493}.

2. The TI Voyage 200 Handheld Enters the Picture

One can generate the Fibonacci sequence on the HOME SCREEN. First recall the famous *Fibonacci sequence* is recursively defined as follows:

Define $F_1 = F_2 = 1\, and\; F_n = F_{n-2} + F_{n-1}\;\; for\; n \geq 3.$ Here $F_n =$ the nth term of the Fibonacci sequence. See FIGURE 1. We utilize the VOYAGE 200 handheld by Texas Instruments to generate the initial forty outputs in the Fibonacci sequence.

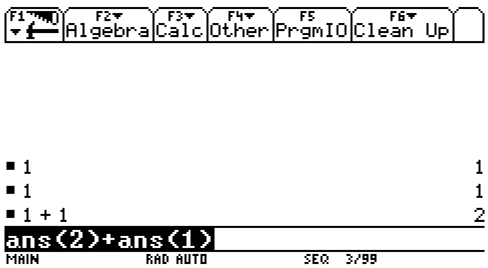

FIGURE 1: Entering the Fibonacci sequence.

.

In FIGURE 1, on The HOME SCREEN, we entered the initial two terms to start the recursion which are both 1 and then used the command $ans(2) + ans(1)$ followed by ENTER. This will furnish the sum of the next to the last answer on the HOME SCREEN followed by the last answer. Keep pressing ENTER to generate new terms of this sequence. See FIGURES 2-7:

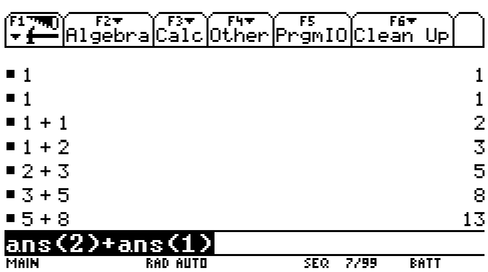

FIGURE 2: The first seven terms.

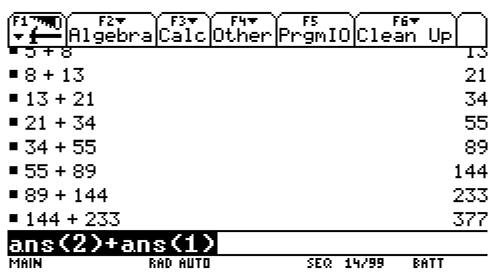

FIGURE 3: Terms eight to fourteen.

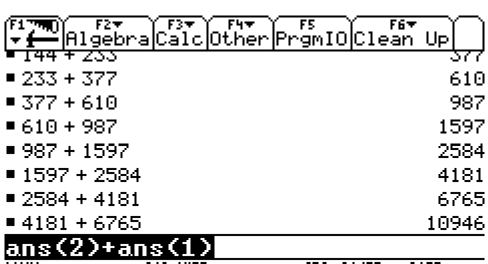

FIGURE 4: Terms 15-21.

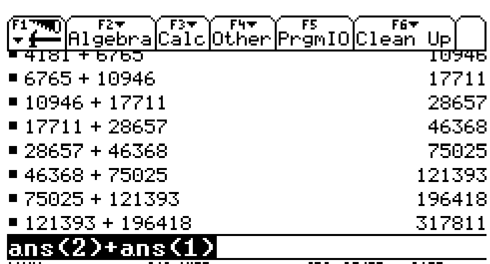

FIGURE 5: Terms 21-28.

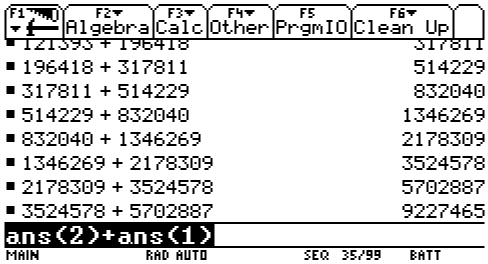

FIGURE 6: Terms 29-35.

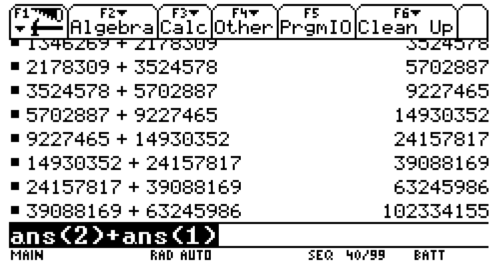

FIGURE 7: Terms 36-42.

If one reads this data, they see two numbers on the bottom right; for example, in FIGURE 5, one sees 28/99. The 28th term is the last answer in FIGURE 6 and is 317811. There are 99 possible answers retained by the calculator. One can adjust this last number. From the HOME SCREEN, use the keystrokes F1 9: Format (see FIGURE 8) and press ENTER. You will see what is called History Pairs and use the right arrow cursor to see the choices, which indicate the number of answers one can recover from the HOME SCREEN (see FIGURES 9-10). The factory setting for the History Pairs is 30.

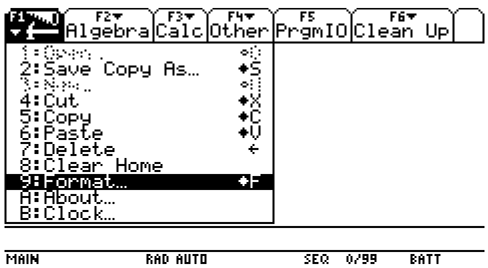

FIGURE 8: The Format option.

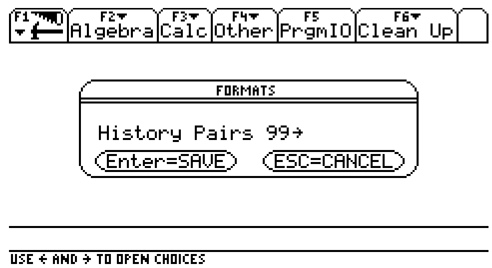

FIGURE 9: The History Pairs option.

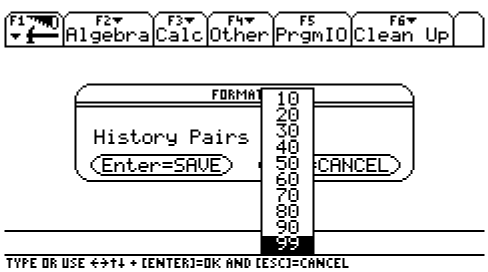

FIGURE 10: Setting the History Pairs option to view last 99 answers on the HOME SCREEN.

Based on the data in FIGURES 2-7, we conjecture that every fourth Fibonacci integer is divisible by three.

$$\left\{\begin{array}{l} F_4 = 3, F_8 = 21, F_{12} = 144, F_{16} = 987, F_{20} = 6765 \\ F_5 = 5, F_{10} = 55, F_{15} = 610, F_{20} = 6765, F_{25} = 75025 \\ F_8 = 21, F_{16} = 987, F_{24} = 46368, F_{32} = 2178309, F_{40} = 102334155 \end{array}\right\}.$$

Proceeding to SEQUENCE GRAPHING (use the keystrokes MODE followed by the right arrow cursor to option 4: SEQUENCE followed by ENTER), we see an SEQ at the bottom of the HOME SCREEN. See FIGURES 11-12:

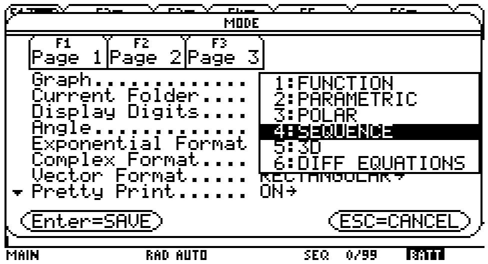

FIGURE 11: The Sequence MODE.

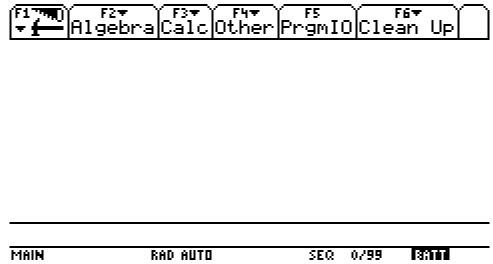

FIGURE 12: The SEQ MODE is revealed.

Next proceed to the Y= EDITOR and input the following as in FIGURE 13 with the Standard Viewing Window, Graph, Table Setup, and a portion of the TABLE in FIGURES 14-20:

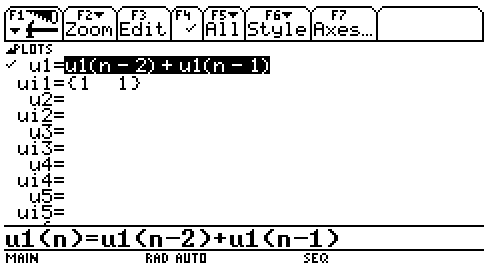

FIGURE 13: Entering the sequence.

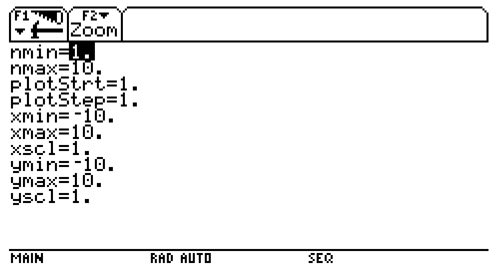

FIGURE 14: The Standard Window setting.

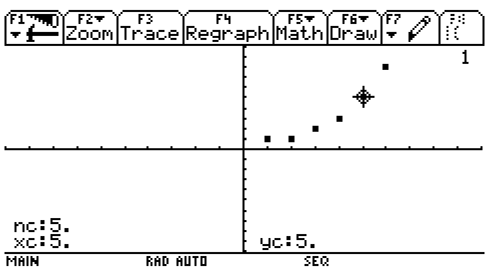

FIGURE 15: Graph and fifth term.

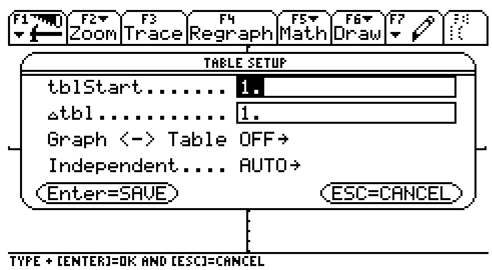

FIGURE 16: The Table Setup.

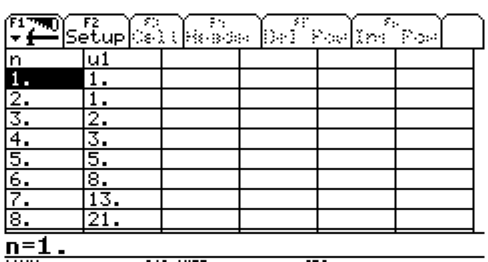

FIGURE 17: The table is revealed.

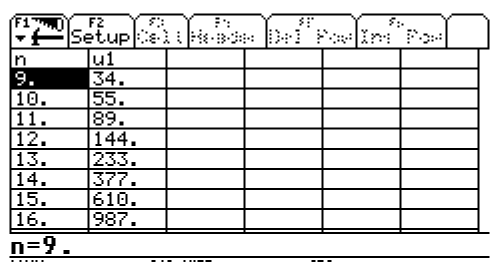

FIGURE 18: The table is revealed.

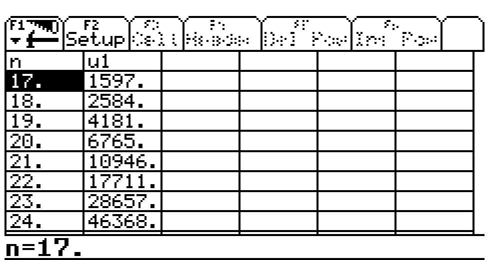

FIGURE 19: The table is revealed.

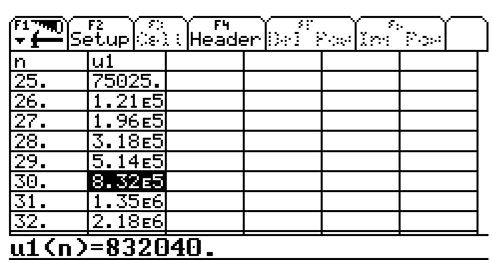

FIGURE 20: The table is revealed.

Some Comments on the above screen captures:

1. In FIGURE 13, note that the recursion rule is provided on the line headed by $u1$ while the line headed by $ui1$ records the initial two terms of the sequence, the second followed by the first. There is no comma between the two 1's in Pretty Print although one separates the two initial 1's with a comma on the entry line. See FIGURE 21:

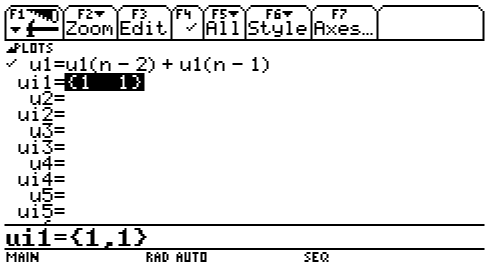

FIGURE 21: The recursion rule for the Fibonacci sequence.

2. Note from FIGURE 15 that 5 is the fifth term of the Fibonacci sequence.
3. Since a sequence is a function whose domain is the set of positive integers, the Table Start begins at 1 in FIGURE 16.
4. Only five figures are possible in any cell. Thus, all terms of the Fibonacci sequence after the twenty-fifth are expressed in scientific notation. If one places their cursor on the output value, however, the exact value is determined as in FIGURE 20 where the thirtieth term is given exactly as 832040.

The Lucas sequence is like the Fibonacci sequence and is defined as follows:
$L_1 = 1, L_2 = 3, \text{ and } L_n = L_{n-2} + L_{n-1} \text{ for } n \geq 3.$ Here $L_n =$ the n^{th} term in the Lucas sequence.
Let us compute $L_2, L_6, L_{10}, L_{14}, \text{ and } L_{18}.$ We determine a conjecture that can be formed related to divisibility based on our observation, generalize, and justify.

Consider the initial fourteen terms in the Lucas sequence. We proceed to the HOME SCREEN. After the first two terms 1 and 3 are entered, use the command $ans(2) + ans(1)$ and keep pressing ENTER. See FIGURES 22-23:

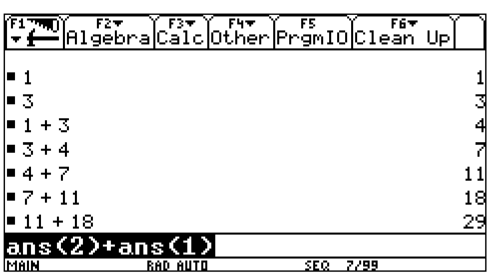

FIGURE 22: The initial seven terms.

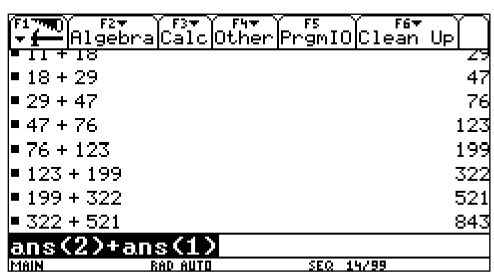

FIGURE 23: Terms 8-14 in the Lucas sequence.

Observe that $L_2 = 3, L_6 = 18, L_{10} = 123, L_{14} = 843.$ Each of these integers is divisible by three. Our conjecture is that every fourth integer from the second term onward is evenly divisible by three. This claim is proven via mathematical induction.

Conjecture: $3 \mid L_{4 \cdot n-2} \ \forall n \in \mathbb{N}$.

We justify our conjecture via The Principle of Mathematical Induction:

Step 1: The statement is true for $n=1$: $3 \mid L_{4 \cdot 1-2} \leftrightarrow 3 \mid L_2 \leftrightarrow 3 \mid 3$ since $3 = 3 \cdot 1$. (1)
Step 2: Assume that the statement is true for $n=k$:
We assume that $3 \mid L_{4 \cdot k-2}$. (2)
Step 3: We prove the statement true for $n=k+1$ given that the statement is assumed true for $n=k$: (3)
We wish to demonstrate that $3 \mid L_{4 \cdot (k+1)-2} \leftrightarrow 3 \mid L_{4 \cdot k+2}$ given that $3 \mid L_{4 \cdot k-2}$. (3)
To achieve (3), we utilize the recursion relation in the Lucas sequence as well as properties of divisibility. Note that

$$L_{4 \cdot (k+1)-2} = L_{4 \cdot k+2} = L_{4 \cdot k} + L_{4 \cdot k+1} = L_{4 \cdot k-2} + L_{4 \cdot k-1} + L_{4 \cdot k-1} + L_{4 \cdot k} = L_{4 \cdot k-2} + 2 \cdot L_{4 \cdot k-1} + L_{4 \cdot k} =$$
$$L_{4 \cdot k-2} + 2 \cdot L_{4 \cdot k-1} + L_{4 \cdot k-2} + L_{4 \cdot k-1} = 2 \cdot L_{4 \cdot k-2} + 3 \cdot L_{4 \cdot k-1}.$$

By The Induction Hypothesis, $3 \mid L_{4 \cdot k-2} \rightarrow 3 \mid 2 \cdot L_{4 \cdot k-2}$. Clearly $3 \mid 3 \cdot L_{4 \cdot k-1}$. Hence $3 \mid \left[2 \cdot L_{4 \cdot k-2} + 3 \cdot L_{4 \cdot k-1} \right] \leftrightarrow 3 \mid L_{4 \cdot k+2} \leftrightarrow 3 \mid L_{4 \cdot (k+1)-2}$. This establishes (3) and proves that $P(k) \rightarrow P(k+1)$.

Step 4: By the Principle of Mathematical Induction, since the statement is true for $n=1$, it must likewise be true for $n = 1+1 = 2, n = 2+1 = 3, \ldots$ *i.e.* $\forall n \in \mathbb{N}$. □

In FIGURES 24-31, we demonstrate the graphical phenomena for the Lucas sequence in SEQUENCE MODE:

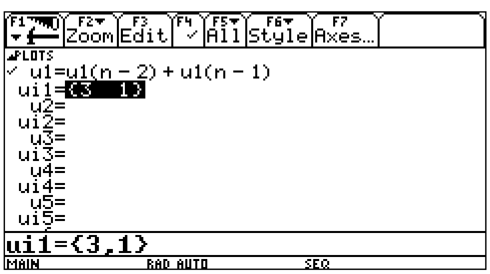

FIGURE 24: The Lucas sequence.

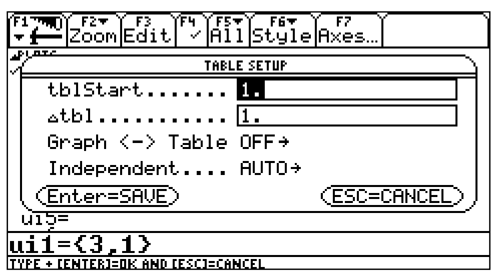

FIGURE 25: The table setup.

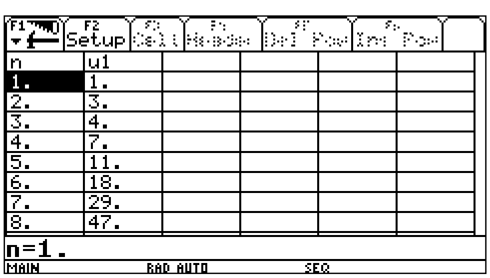

FIGURE 26: The initial eight terms.

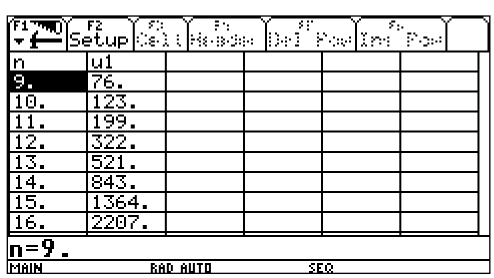

FIGURE 27: Terms 9-16 are revealed.

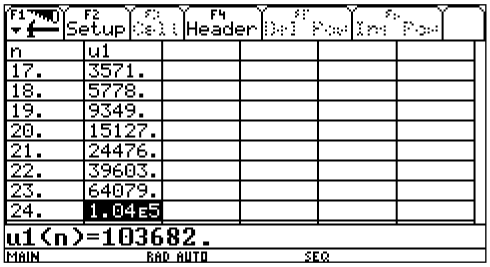

FIGURE 28: Terms 17-24 are revealed.

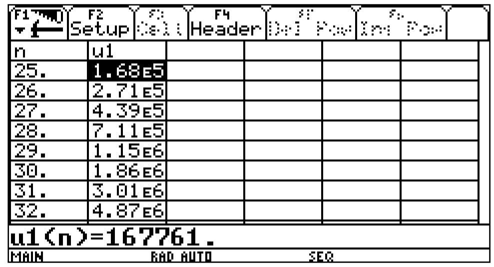

FIGURE 29: Terms 25-32 are revealed.

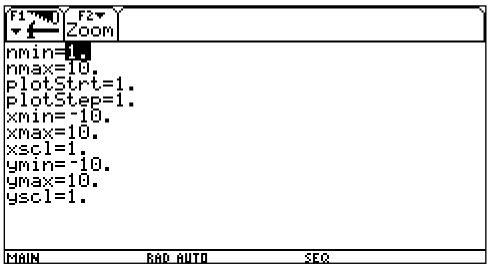

FIGURE 30: The standard window.

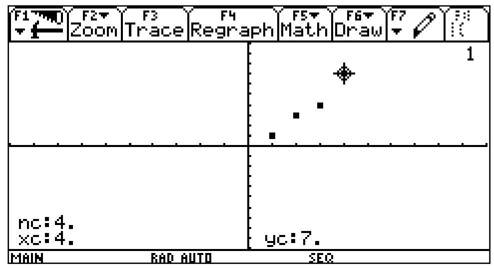

FIGURE 31: The graph and the fourth term.

In FIGURE 26, our hand-held correctly asserts that the fourth term in the Lucas sequence is 7.

For the Reverse Lucas Sequence, see FIGURES 32-35:

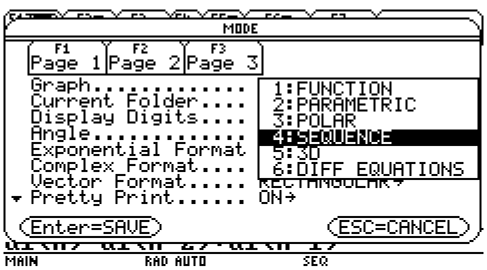

FIGURE 32: Sequence Mode.

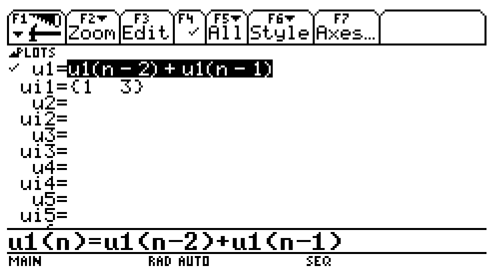

FIGURE 33: Entering the Reverse Lucas sequence.

FIGURE 33: Entering the sequence.

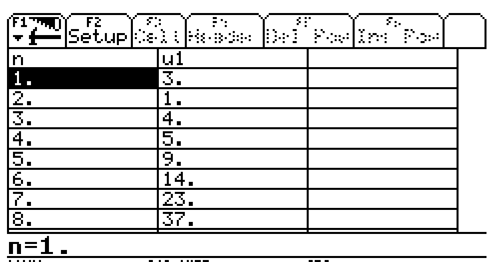

FIGURE 35: The table for the initial eight terms.

We note that the first ten terms in the table are 3, 1, 4, 5, 9, 14, 23, 37, 60 and 97. The calculator reads the second term followed by the first terms when we enter 1, 3. Hence the expected Lucas sequence is not generated. Instead, the sequence we obtain will heretofore be classified as the Reverse Lucas sequence.

3. Sums of consecutive Fibonacci numbers - an activity ripe for exploration

In Table 1, we consider sums of consecutive Fibonacci numbers from two to forty. We form three conjectures based on the analysis of Table 1. Observe the coefficients of the variables x *and* y when one considers the sums of $4\cdot n-2$ *and* $4\cdot n\ (n\in\mathbb{N})$ terms respectively in the quotient column are Fibonacci and Lucas numbers respectively. The cumulative sum for any term can be secured by a diagonal procedure. Note that $(x+2\cdot y)+(2\cdot x+2\cdot y)=3\cdot x+4\cdot y$.

Table 1: Generalized Fibonacci Table

Term:	Term Value:	The Cumulative Sum (C.S.) and the value the C.S. is divisible by in ( ):	Quotient and Comments: Note: We are dividing the Cumulative Sum by the GCD of the coefficients of x and y.
1	x	x (1)	
2	y	$x+y$ (1)	
3	$x+y$	$2\cdot x+2\cdot y$ (2)	$x+y$ (3rd term)
4	$x+2\cdot y$	$3\cdot x+4\cdot y$ (1)	$3\cdot x+4\cdot y$ (3rd + 5th terms)
5	$2\cdot x+3\cdot y$	$5\cdot x+7\cdot y$ (1)	
6	$3\cdot x+5\cdot y$	$8\cdot x+12\cdot y$ (4)	$2\cdot x+3\cdot y$ (5th term)
7	$5\cdot x+8\cdot y$	$13\cdot x+20\cdot y$ (1)	
8	$8\cdot x+13\cdot y$	$21\cdot x+33\cdot y$ (3)	$7\cdot x+11\cdot y$ (5th + 7th terms)
9	$13\cdot x+21\cdot y$	$34\cdot x+54\cdot y$ (2)	$17\cdot x+27\cdot y$ (3rd + 6th + 9th terms)
10	$21\cdot x+34\cdot y$	$55\cdot x+88\cdot y$ (11)	$5\cdot x+8\cdot y$ (7th term)
11	$34\cdot x+55\cdot y$	$89\cdot x+143\cdot y$ (1)	
12	$55\cdot x+89\cdot y$	$144\cdot x+232\cdot y$ (8)	$18\cdot x+29\cdot y$ (7th + 9th terms)
13	$89\cdot x+144\cdot y$	$233\cdot x+376\cdot y$ (1)	
14	$144\cdot x+233\cdot y$	$377\cdot x+609\cdot y$ (29)	$13\cdot x+21\cdot y$ (9th term)
15	$233\cdot x+377\cdot y$	$610\cdot x+986\cdot y$ (2)	$305\cdot x+493\cdot y$ (3rd + 6th + 9th + 12th + 15th terms)
16	$377\cdot x+610\cdot y$	$987\cdot x+1596\cdot y$ (21)	$47\cdot x+76\cdot y$ (9th + 11th terms)
17	$610\cdot x+987\cdot y$	$1597\cdot x+2583\cdot y$ (1)	
18	$987\cdot x+1597\cdot y$	$2584\cdot x+4180\cdot y$ (76)	$34\cdot x+55\cdot y$ (11th term)
19	$1597\cdot x+2584\cdot y$	$4181\cdot x+6764\cdot y$ (1)	

20	$2584 \cdot x + 4181 \cdot y$	$6765 \cdot x + 10945 \cdot y$ (55)	$123 \cdot x + 199 \cdot y$ (11^{th} + 13^{th} terms)
21	$4181 \cdot x + 6765 \cdot y$	$10946 \cdot x + 17710 \cdot y$ (2)	$5473 \cdot x + 8855 \cdot y$ (3^{rd} + 6^{th} + 9^{th} + 12^{th} + 15^{th} + 18^{th} + 21^{st} terms)
22	$6765 \cdot x + 10946 \cdot y$	$17711 \cdot x + 28656 \cdot y$ (199)	$89 \cdot x + 144 \cdot y$ (13^{th} term)
23	$10946 \cdot x + 17711 \cdot y$	$28657 \cdot x + 46367 \cdot y$ (1)	
24	$17711 \cdot x + 28657 \cdot y$	$46368 \cdot x + 75024 \cdot y$ (144)	$322 \cdot x + 521 \cdot y$ (13^{th} + 15^{th} terms)
25	$28657 \cdot x + 46368 \cdot y$	$75025 \cdot x + 121392 \cdot y$ (1)	
26	$46368 \cdot x + 75025 \cdot y$	$121393 \cdot x + 196417 \cdot y$ (521)	$233 \cdot x + 377 \cdot y$ (15^{th} term)
27	$75025 \cdot x + 121393 \cdot y$	$196418 \cdot x + 317810 \cdot y$ (2)	$98209 \cdot x + 158905 \cdot y$ (3^{rd} + 6^{th} + 9^{th} + 12^{th} + 15^{th} + 18^{th} + 21^{st} + 24^{th} + 27^{th} terms)
28	$121393 \cdot x + 196418 \cdot y$	$317811 \cdot x + 514228 \cdot y$ (377)	$843 \cdot x + 1364 \cdot y$ (15^{th} + 17^{th} terms)
29	$196418 \cdot x + 317811 \cdot y$	$514229 \cdot x + 832039 \cdot y$ (1)	
30	$317811 \cdot x + 514229 \cdot y$	$832040 \cdot x + 1346268 \cdot y$ (1364)	$610 \cdot x + 987 \cdot y$ (17^{th} term)
31	$514229 \cdot x + 832040 \cdot y$	$1346269 \cdot x + 2178308 \cdot y$ (1)	
32	$832040 \cdot x + 1346269 \cdot y$	$2178309 \cdot x + 3524577 \cdot y$ (987)	$2207 \cdot x + 3571 \cdot y$ (17^{th} + 19^{th} terms)
33	$1346269 \cdot x + 2178309 \cdot y$	$3524578 \cdot x + 5702886 \cdot y$ (2)	$1762289 \cdot x + 2851443 \cdot y$ (3^{rd} + 6^{th} + 9^{th} + 12^{th} + 15^{th} + 18^{th} + 21^{st} + 24^{th} + 27^{th} + 30^{th} + 33^{rd} terms)
34	$2178309 \cdot x + 3524578 \cdot y$	$5702887 \cdot x + 9227464 \cdot y$ (3571)	$1597 \cdot x + 2584 \cdot y$ (19^{th} term)
35	$3524578 \cdot x + 5702887 \cdot y$	$9227465 \cdot x + 14930351 \cdot y$ (1)	
36	$5702887 \cdot x + 9227465 \cdot y$	$14930352 \cdot x + 24157816 \cdot y$ (2584)	$5778 \cdot x + 9349 \cdot y$ (19^{th} + 21^{st} terms)
37	$9227465 \cdot x + 14930352 \cdot y$	$24157817 \cdot x + 39088168 \cdot y$ (1)	
38	$14930352 \cdot x + 24157817 \cdot y$	$39088169 \cdot x + 63245985 \cdot y$ (9349)	$4181 \cdot x + 6765 \cdot y$ (21^{st} term)
39	$24157817 \cdot x + 39088169 \cdot y$	$63245986 \cdot x + 102334154 \cdot y$ (2)	$31622993 \cdot x + 51167077 \cdot y$ (3^{rd} + 6^{th} + 9^{th} + 12^{th} + 15^{th} + 18^{th} + 21^{st} + 24^{th} + 27^{th} + 30^{th} + 33^{rd} + 36^{th} + 39^{th} terms)
40	$39088169 \cdot x + 63245986 \cdot y$	$102334155 \cdot x + 165580140 \cdot y$ (6765)	$15127 \cdot x + 24476 \cdot y$ (21^{st} + 23^{rd} terms)

We now consider the Fibonacci sequence or any Fibonacci-like sequence (a sequence whose first two terms can be anything one pleases, but each term thereafter follows the Fibonacci recursion rule), form the sum of any six consecutive terms and divide this sum by four. We do this for three separate data sets and form a conjecture. The results are tabulated in the following Table:

Sum of six consecutive Fibonacci numbers:	Sum of the members in the set:	Quotient when the sum is divided by four:
$\{2,3,5,8,13,21\}$	52	13…fifth term
$\{1,1,2,3,5,8\}$	20	5… fifth term
$\{55,89,144,233,377,610\}$	1508	377… fifth term

<u>CONJECTURE:</u> The sum of any six consecutive Fibonacci numbers is divisible by four and the quotient will always be the fifth term in the sequence.

<u>Proof:</u> Consider the initial two terms of the Fibonacci sequence to be $x \text{ and } y$. The six consecutive terms of the sequence are as follows: $\{x, y, x+y, x+2\cdot y, 2\cdot x+3\cdot y, 3\cdot x+5\cdot y\}$.
We employ the symbolic capability of the VOYAGE 200 to form the sum and divide the resulting sum by four. See FIGURE 36:

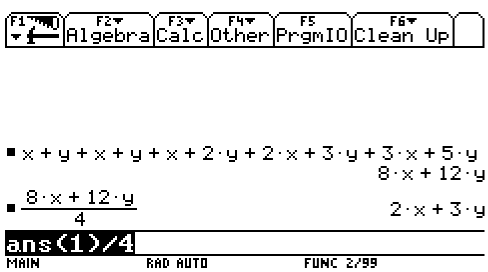

FIGURE 36: The proof is revealed.

Let us next form the sum of any nine consecutive Fibonacci integers and divide this sum by two. View this for three separate data sets and form a conjecture. The results are tabulated below:

Sum of nine consecutive Fibonacci numbers:	Sum of the members in the set:	Quotient when the sum is divided by two:
$\{2,3,5,8,13,21,34,55,89\}$	230	115. $115=5+21+89.$ Sum of the third, sixth and ninth terms.
$\{1,1,2,3,5,8,13,21,34\}$	88	44. $44=2+8+34.$ Sum of the third, sixth and ninth terms.
$\{55,89,144,233,377,610,987,1597,2584\}$	6676	3338. $3338=144+610+2584.$ Sum of the third, sixth and ninth terms.

CONJECTURE: The sum of any nine consecutive Fibonacci numbers is divisible by two and the quotient will always be the sum of the third, sixth and ninth terms in the sequence.
Proof: Consider the initial two terms of the Fibonacci sequence to be $x \text{ and } y.$ The nine consecutive terms of the sequence are as follows:

$\{x, y, x+y, x+2\cdot y, 2\cdot x+3\cdot y, 3\cdot x+5\cdot y, 5\cdot x+8\cdot y, 8\cdot x+13\cdot y, 13\cdot x+21\cdot y\}.$ We employ the TI-89 to form the sum and divide the resulting sum by two. See FIGURES 37-39:

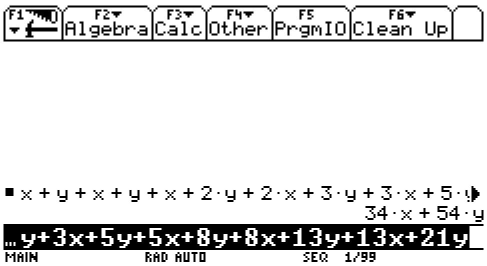

FIGURE 37: The proof is revealed.

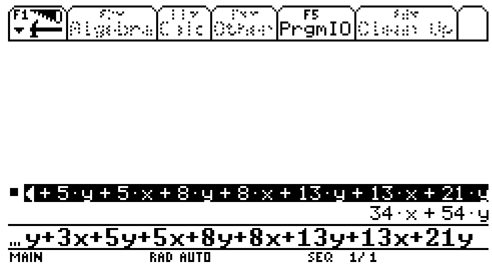

FIGURE 38: The proof is revealed.

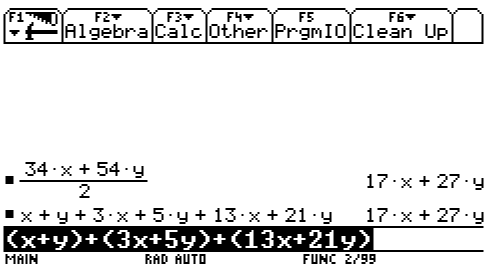

FIGURE 39: The proof is revealed.

Notice our sum is $34\cdot x+54\cdot y.$ When we divide our sum by two, our quotient is $17\cdot x+27\cdot y.$ The quotient represents the sum of the third, sixth and ninth terms in the sequence. Observe that $(x+y)+(3\cdot x+5\cdot y)+(13\cdot x+21\cdot y)=17\cdot x+21\cdot y.$ This is a neat Fibonacci number trick.

Let us next form the sum of any twelve consecutive integers and divide this sum by eight for three separate data sets and form a conjecture. The results are tabulated below:

Sum of twelve consecutive Fibonacci numbers:	Sum of the members in the set:	Quotient when the sum is divided by eight:
$\{2,3,5,8,13,21,34,55,89,144, 233, 377\}$	984	123. $123=34+89.$ Sum of the seventh and ninth terms.
$\{1,1,2,3,5,8,13,21,34,55,89,144\}$	376	47. $47=13+34.$ Sum of the seventh and ninth terms.
$\{55,89,144,233,377,610,987,1597,2584,4181,6765,10946\}$	3571	3571. $987+2584=3571.$ Sum of the seventh and ninth terms.

CONJECTURE: The sum of any twelve consecutive Fibonacci numbers is divisible by eight and the quotient will always be the sum of the seventh and ninth terms in the sequence.
Proof: Consider the initial two terms of the Fibonacci sequence to be x *and* y. The twelve consecutive terms of the sequence are as follows:

$$\left\{\begin{matrix} x, y, x+y, x+2\cdot y, 2\cdot x+3\cdot y, 3\cdot x+5\cdot y, 5\cdot x+8\cdot y, 8\cdot x+13\cdot y, 13\cdot x+21\cdot y, 21\cdot x+34\cdot y, \\ 34\cdot x+55\cdot y, 55\cdot x+89\cdot y \end{matrix}\right\}.$$

The VOYAGE 200 is used to form the sum and divide the total by eight. See FIGURES 40-43:

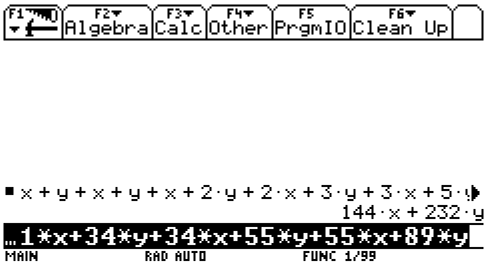

FIGURE 40: The proof is revealed.

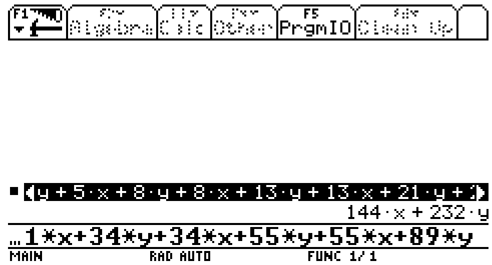

FIGURE 41: The proof is revealed.

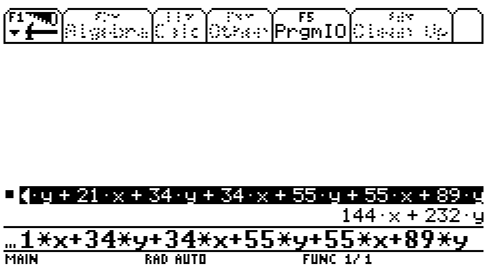

FIGURE 42: The proof is revealed.

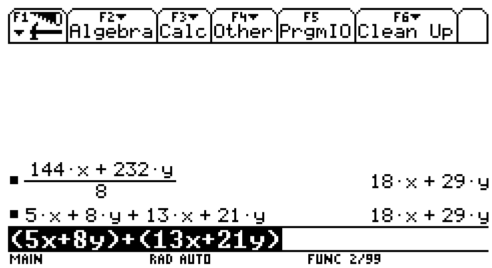

FIGURE 43: The proof is revealed.

Observe that the sum of the twelve consecutive terms is $144\cdot x+232\cdot y$. When this sum is divided by eight, our quotient is $18\cdot x+29\cdot y$. Observe $18\cdot x+29\cdot y=(5\cdot x+8\cdot y)+(13\cdot x+21\cdot y)$ and represents the sum of the seventh and ninth terms in the sequence.

We next consider ten consecutive terms in the Fibonacci sequence of our choosing.
The results for three separate data sets are tabulated in the following Table:

Sum of ten consecutive Fibonacci numbers:	Sum of the members in the set:	Quotient when the sum is divided by eleven:
$\{2,3,5,8,13,21,34,55,89,144\}$	374	34…seventh term
$\{1,1,2,3,5,8,13,21,34,55\}$	143	13… seventh term
$\{55,89,144,233,377,610,987,1597,2584,4181\}$	10857	987… seventh term

CONJECTURE: The sum of any ten consecutive Fibonacci numbers is divisible by 11 and the quotient will always be the seventh term in the sequence.

Proof: Denote the initial two terms of the Fibonacci sequence by x *and* y. Then ten consecutive terms of the sequence are as follows:

$$\{x, y, x+y, x+2\cdot y, 2\cdot x+3\cdot y, 3\cdot x+5\cdot y, 5\cdot x+8\cdot y, 8\cdot x+13\cdot y, 13\cdot x+21\cdot y, 21\cdot x+34\cdot y\}$$

Employ our handheld to form the sum and divide 11. See FIGURES 44-47:

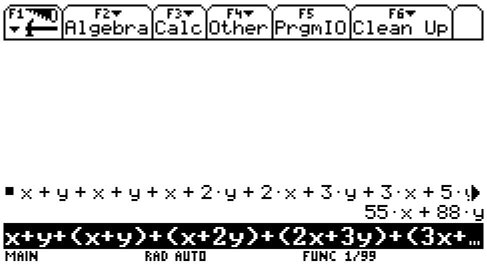

FIGURE 44: The proof is revealed.

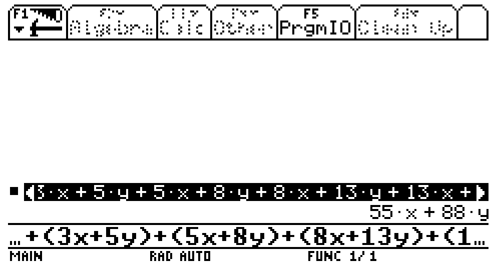

FIGURE 45: The proof is revealed.

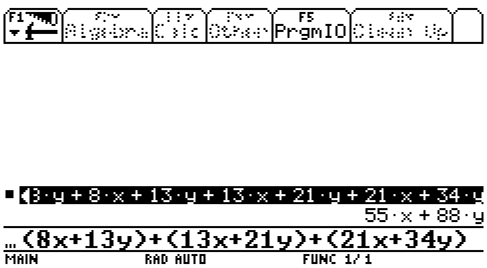

FIGURE 46: The proof is revealed.

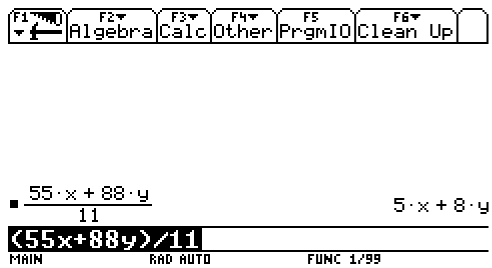

FIGURE 47: The proof is revealed.

Observe that $5\cdot x+8\cdot y$ is the seventh term in the sequence. This is a neat Fibonacci trick.

Let us next form the sum of any fourteen consecutive integers and divide this sum by 29 for three separate data sets and form a conjecture. The results are tabulated below:

Sum of fourteen consecutive Fibonacci numbers:	Sum of the members in the set:	Quotient when the sum is divided by twenty-nine:
$\{2,3,5,8,13,21,34,55,89,144, 233, 377, 610, 987\}$	2581	89… ninth term
$\{1,1,2,3,5,8,13,21,34,55, 89, 144, 233, 377\}$	986	34… ninth term
$\left\{\begin{matrix}55,89,144,233,377,610,987,1597,2584, \\ 4181,6765, 10946, 17711, 28657\end{matrix}\right\}$	74936	2584… ninth term

CONJECTURE: The sum of any fourteen consecutive Fibonacci numbers is divisible by 29 and the quotient will always be the ninth term in the sequence.

Proof: Consider the initial two terms of the Fibonacci sequence to be x *and* y. The fourteen consecutive terms of the sequence are as follows:

$$\left\{\begin{matrix}x, y, x+y, x+2\cdot y, 2\cdot x+3\cdot y, 3\cdot x+5\cdot y, 5\cdot x+8\cdot y, 8\cdot x+13\cdot y, 13\cdot x+21\cdot y, 21\cdot x+34\cdot y, \\ 34\cdot x+55\cdot y, 55\cdot x+89\cdot y, 89\cdot x+144\cdot y, 144\cdot x+233\cdot y\end{matrix}\right\}$$

The VOYAGE 200 is used to form the sum and divide the total by 29. See FIGURES 48-52:

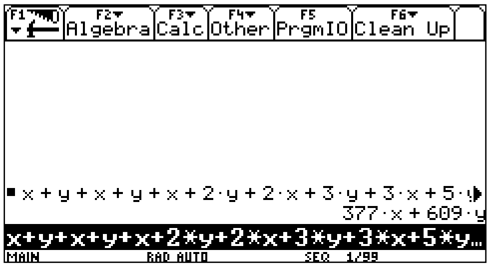

FIGURE 48: The proof is revealed.

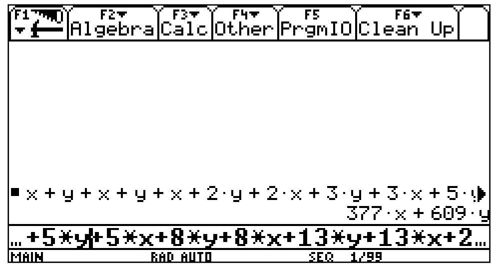

FIGURE 49: The proof is revealed.

:

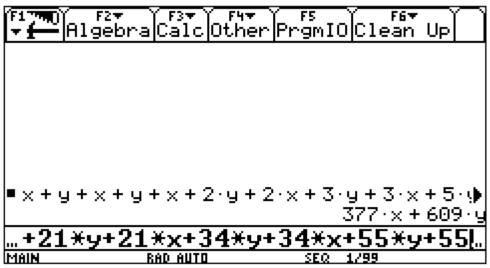

FIGURE 50: The proof is revealed.

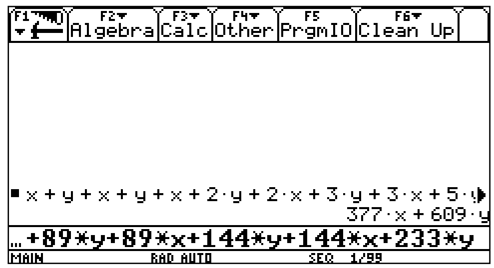

FIGURE 51: The proof is revealed.

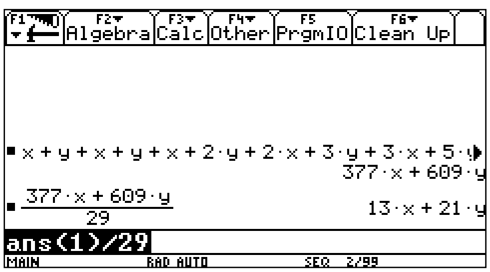

FIGURE 52: The proof is revealed.

We conclude this section by forming three conjectures based on Table 1:

Conjecture I: The sum of every $3\cdot n,\ n$ odd *and* $n\in\mathbb{N}$ consecutive terms in any Fibonacci-like sequence is divisible by two, and the quotient is the sum of all the terms that are multiples of three from the third term up to term $3\cdot n$.

Conjecture II: The sum of any $4\cdot n-2,\ n\in\mathbb{N}$ consecutive terms in a Fibonacci-like sequence is divisible by $L_{2\cdot n-1}$, $n\in\mathbb{N}$ and the quotient is the term of the form $2\cdot n+1,\ n\in\mathbb{N}$.
We note that $F_n=\{1,1,2,3,5,8,13,21,...\}$ *and* $L_n=\{1,3,4,7,11,18,29,...\}$. This is another Fibonacci-Lucas connection.

Conjecture III: The sum of any $4\cdot n,\ n\in\mathbb{N}$ consecutive terms in a Fibonacci-like sequence is divisible by $F_{2\cdot n}$, $n\in\mathbb{N}$, and the quotient is the sum of the terms $2\cdot n+1$ *and* $2\cdot n+3$.
We unpack an example associated with each of the conjectures articulated above.

For Conjecture I, one can show by algebra that the sum of twenty-seven consecutive terms in a Fibonacci-like sequence $(196418 \cdot x + 317810 \cdot y)$ is divisible by two and the quotient $(98209 \cdot x + 158905 \cdot y)$ is the sum of the third, sixth, ninth, twelfth, fifteenth, eighteenth, twenty-first, twenty-fourth and twenty-seventh terms in the sequence.

For Conjecture II, one can show by algebra that the sum of ten $(10 = 4 \cdot 3 - 2)$ consecutive terms is divisible by eleven $(L_{10/2} = L_5 = 11)$ and the quotient is the seventh term in the sequence $(2 \cdot 3 + 1 = 6 + 1 = 7)$.

For Conjecture III, one can show by algebra that the sum of twelve $(12 = 4 \cdot 3)$ consecutive terms in a Fibonacci-like sequence $(144 \cdot x + 232 \cdot y)$ is divisible by eight $(F_{2 \cdot 3} = F_6 = 8)$ and the quotient $(18 \cdot x + 29 \cdot y)$ is the sum of the seventh and ninth terms in the sequence. Note that $7 = 2 \cdot 3 + 1$ *and* $9 = 2 \cdot 4 + 1$.

4. Geometry and the Fibonacci Sequence.

In this activity, we next take any four consecutive Fibonacci numbers. Form the product of the first and fourth terms of the sequence. Next take twice the product of the second and terms. Finally take the sum of the squares of the second and third terms. Observe the relationship to the Pythagorean Theorem in plane geometry. We gather some empirical evidence via the following three examples:

Example 1: Consider the set of four consecutive Fibonacci numbers $\{3,5,8,13\}$. Observe the truth of the following with the aid of our hand-held. See FIGURE 53:

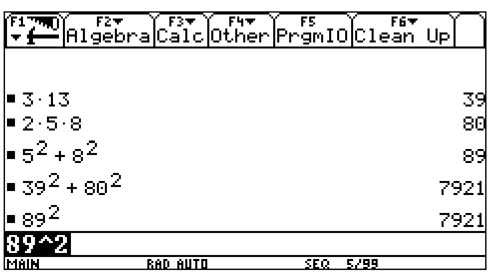

FIGURE 53: The Pythagorean triple is revealed.

Observe that the primitive Pythagorean Triple $(39, 80, 89)$ is formed.

Example 2: We next consider the sequence of four consecutive Fibonacci numbers $\{8,13,21,34\}$. We observe the truth of the following computations furnished by the VOYAGE 200. See FIGURE 54:

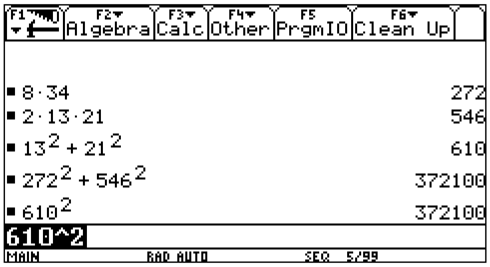

FIGURE 54: The Pythagorean triple is revealed.

The Pythagorean Triple $(272, 546, 610)$ (albeit not primitive; for 2 is a common factor among each of the components) is formed. The associated primitive Pythagorean Triple is $(136, 273, 305)$.

<u>Example 3:</u> Consider the sequence of four consecutive Fibonacci numbers $\{13, 21, 34, 55\}$. See FIGURE 55 for the relevant computations.

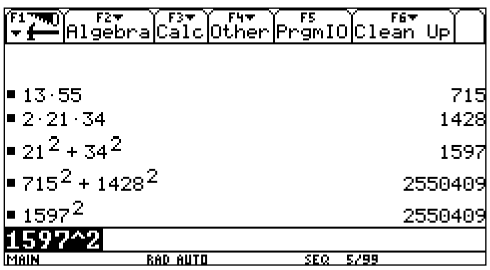

FIGURE 55: The Pythagorean triple is revealed.

The primitive Pythagorean triple $(715, 1428, 1597)$ is formed.
Note that the hypotenuses of each of the right triangles formed are Fibonacci numbers $(89, 610, 1597)$. Based on the observations in the three examples, one suspects that a Pythagorean triple is always formed, and this is indeed the case. We justify our conjecture with the aid of the VOYAGE 200:

Suppose $\{x, y, x+y, x+2\cdot y\}$ represent any four consecutive terms of the Fibonacci (or Fibonacci-like sequence). We view our inputs and outputs in FIGURE 57 using the expand (command (See FIGURE 52) from the Algebra menu on the HOME SCREEN:

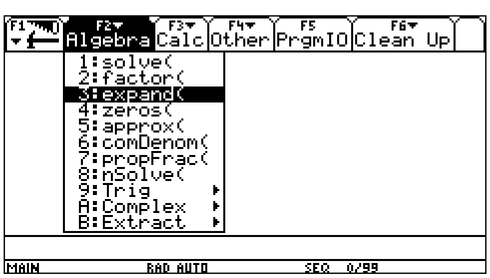

FIGURE 56: The expand command.

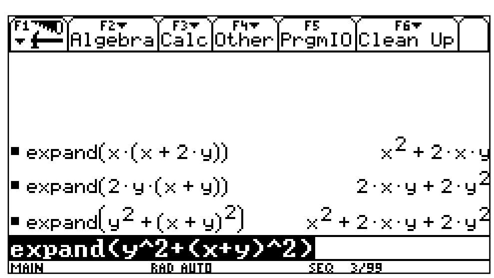

FIGURE 57: Using the expand command.

To show that $\left(x^2+2\cdot x\cdot y, 2\cdot x\cdot y+2\cdot y^2, x^2+2\cdot x\cdot y+2\cdot y^2\right)$ forms a Pythagorean Triple, see FIGURES 58-60 for our inputs and outputs:

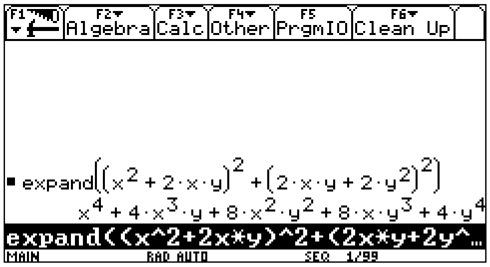

FIGURE 58: The proof is displayed.

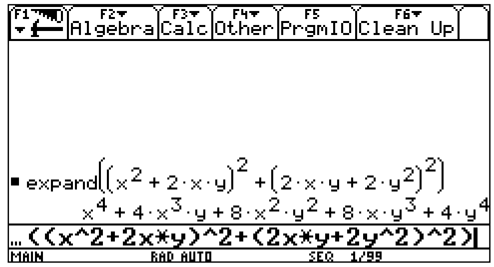

FIGURE 59: The proof is displayed.

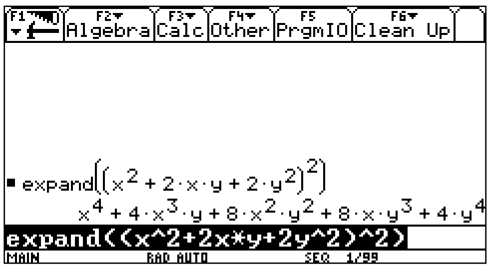

FIGURE 60: The proof is displayed.

It should be noted that this scheme works for any Fibonacci-like sequence in which the first two terms are arbitrary, but each term thereafter follows the Fibonacci recursion rule. In addition, not all PPT's are generated by this method. For example, the PPT $(15,8,17)$ is missed, but its related Pythagorean triple $(16,30,34)$ is generated when considering four consecutive terms in the Fibonacci sequence and is obtained by multiplying each of the sides of the PPT $(15,8,17)$ by two. Numerous other PPT's are missed including the PPT $(9,40,41)$. We generate the initial dozen Pythagorean Triples via this method for the Fibonacci sequence recursively defined by $F_1=F_2=1,\ F_n=F_{n-2}+F_{n-1};\ n\geq 3,$ the Lucas sequence defined by $L_1=1,\ L_2=3,\ L_n=L_{n-2}+L_{n-1};\ n\geq 3$ and the Reverse Lucas sequence recursively defined as follows: $RL_1=3,\ RL_2=1,\ RL_n=RL_{n-2}+RL_{n-1}; n\geq 3.$ See Tables 2, 3 and 4 respectively below:

Table 2: Four Consecutive Terms in the Fibonacci Sequence:

Four Consecutive Terms in the Fibonacci Sequence:	Pythagorean Triple Generated:
1, 1, 2, 3	(3, 4, 5)
1, 2, 3, 5	(5, 12, 13)
2, 3, 5, 8	(16, 30, 34)
3, 5, 8, 13	(39, 80, 89)
5, 8, 13, 21	(105, 208, 233)
8, 13, 21, 34	(272, 546, 610)
13, 21, 34, 55	(715, 1428, 1597)
21, 34, 55, 89	(1869, 3740, 4181)
34, 55, 89, 144	(4896, 9790, 10946)
55, 89, 144, 233	(12815, 25632, 28657)
89, 144, 233, 377	(33553, 67104, 75025)
144, 233, 377, 610	(87840, 176582, 196418)

Table 3: Four Consecutive Terms in the Lucas Sequence:

Four Consecutive Terms in the Lucas Sequence:	Pythagorean Triple Generated:
1, 3, 4, 7	(7, 24, 25)
3, 4, 7, 11	(33, 56, 65)
4, 7, 11, 18	(72, 154, 170)
7, 11, 18, 29	(203, 396, 445)
11, 18, 29, 47	(517, 1044, 1165)
18, 29, 47, 76	(1368, 2726, 3050)
29, 47, 76, 123	(3567, 7144, 7985)
47, 76, 123, 199	(9353, 18696, 20905)
76, 123, 199, 322	(24472, 48954, 54730)
123, 199, 322, 521	(64083, 128156, 143285)
199, 322, 521, 843	(167757, 1335524, 375125)
322, 541, 843, 1364	(439208, 878406, 982090)

Table 4: Four Consecutive Terms in the Reverse Lucas Sequence:

Four Consecutive Terms in the Reverse Lucas Sequence:	Pythagorean Triple Generated:
3, 1, 4, 5	(15, 8, 17)
1, 4, 5, 9	(9, 40, 41)
4, 5, 9, 14	(56, 90, 106)
5, 9, 14, 23	(115, 252, 277)
9, 14, 23, 37	(333, 644, 725)
14, 23, 37, 60	(840, 1702, 1898)
23, 37, 60, 97	(2231, 4440, 4969)
37, 60, 97, 157	(5809, 11640, 13009)
60, 97, 157, 254	(15240, 30458, 34058)
97, 157, 254, 411	(39867, 79756, 89165)
157, 254, 411, 665	(104405, 208788, 233437)
254, 411, 665, 1076	(273304, 546630, 611146)

We conclude the paper by stating two conjectures based on the hypotenuses of the right triangles formed in Tables 2 and 3 and attempt to verify them. The first asserts that each of the hypotenuses for the right triangles formed in Table 2 is a Fibonacci number. This is not an accident.

Thousands of Fibonacci identities exist. Some are easily proved by the Principle of Mathematical Induction illustrated earlier in the paper while others utilize algebra and/or the closed Binet formulas for the Fibonacci and Lucas sequences. At times, a combination of these approaches is required. Moreover, it appears that there is a gap of one Fibonacci number each time in the hypotenuses formed {5, 13, 34, 89, 233, 610, 1597, 4181, 10946, 28657, 75025, 196418} in Table 2 above. The respective Fibonacci numbers missed are 8, 21, 55, 144, 377, 987, 2584, 6765, 17711, 46368, and 121393. The Fibonacci identity associated with the phenomena in Table 2 is $F_{n-1}{}^2 + F_n{}^2 = F_{2\cdot n-1}$. We note $(n-1)+n = 2\cdot n-1$. To cite an example from Table 2, let us consider the hypotenuse 89 corresponding to the Primitive Pythagorean Triple (39,80,89). Note that $F_{11} = 89$ and $F_{6-1}{}^2 + F_6{}^2 = F_5{}^2 + F_6{}^2 = 5^2 + 8^2 = 25 + 64 = 89 = F_{11} = F_{2\cdot 6-1} = F_{12-1}$. We observe $5+6=11$. The general proof of this identity can be found by googling Fibonacci Identities and is an exercise in the excellent and classic number theory text by David Burton referenced in the appended bibliography at the conclusion of the paper.

The second conjecture was furnished by one of our active participants at a workshop on problem solving and refers to the hypotenuses in Table 3 above. The conjecture generated is that all the hypotenuses formed are multiples of five. This is indeed a valid conjecture and is easily proven based on the period of the unit's digits in the Lucas sequence. If we refer to the Lucas sequence, we note that the set of unit's digits forms a period of length one dozen before repeating in the same pattern forever: namely $\{1,3,4,7,1,8,9,7,6,3,9,2,...\}$. One hence needs to consider one dozen scenarios considering only the unit's digits. We are taking the sums of the squares of the second and third terms and considering only the unit's digits in each case.

Case 1: $\{1,3,4,7\}: 3^2 + 4^2 = 9 + 16 = 25.$

Case 2: $\{3,4,7,1\}: 4^2 + 7^2 = 16 + 49 = 65.$

Case 3: $\{4,7,1,8\}: 7^2 + 1^2 = 49 + 1 = 50.$

Case 4: $\{7,1,8,9\}: 1^2 + 8^2 = 1 + 64 = 65.$

Case 5: $\{1,8,9,7\}: 8^2 + 9^2 = 64 + 81 = 145.$

Case 6: $\{8,9,7,6\}: 9^2 + 7^2 = 81 + 49 = 130.$

Case 7: $\{9,7,6,3\}: 7^2 + 6^2 = 49 + 36 = 85.$

Case 8: $\{7,6,3,9\}: 6^2 + 3^2 = 36 + 9 = 45.$

Case 9: $\{6,3,9,2\}: 3^2 + 9^2 = 9 + 81 = 90.$

Case 10: $\{3,9,2,1\}: 9^2 + 2^2 = 81 + 4 = 85.$

Case 11: $\{9,2,1,3\}: 2^2 + 1^2 = 4 + 1 = 5.$

Case 12: $\{2,1,3,4\}: 1^2 + 3^2 = 1 + 9 = 10.$

Since an integer is a multiple of five if and only if the units digit is either 0 or 5, our conjecture is proven.

5. References

1. Burton, David M., *Elementary Number Theory*, Allyn and Bacon, Boston, MA. (1976).

2. *Mathworld – A Wolfram Resource*, Wolfram Research, Champaign-Urbana, IL (2017).

3. *OEIS, The Online Encyclopedia of Integer Sequences*, Sequence A104449, The OEIS Foundation (2020).

4. Robbins, Neville, *Beginning Number Theory*, Wm. C. Brown Publishers, Dubuque, IA (1993).

5. Rosen, Kenneth H., *Elementary Number Theory and its applications,* Fifth Edition, Pearson Addison Wesley (2005).

6. Schiffman, Jay L., *Exploring The "Reverse" Lucas sequence 3, 1, 4, 5, 9, …*, ICTCM 2020 Conference Proceedings, P. 1-20, Pearson Education (2020).

7. *Voyage 200 handheld,* Texas Instruments, Dallas, TX. (2020).