OBTAINING THE MAXIMUM DISTANCE OF A PROJECTILE IN AN ATMOSPHERE

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The problem that we shall examine in this paper is that of finding the launch angle to obtain the maximum distance of a projectile in an atmosphere with wind. We shall show the optimal launch angle depends on factors such as the characteristics of the projectile, the launch velocity, and the magnitude of the wind. We will approximate the flight trajectory and distance traveled under various launch angles and different wind conditions, where the wind shall be implemented as a vector field in three-dimensional space.

Using the mass and cross-sectional areas of the projectiles, the trajectories of the objects under the effect of wind are a set of differential equations whose solution will be approximated by numerical methods. These solutions can then be rendered in a 3D graphics package allowing students to study the flight paths of objects from any position and orientation in three-dimensional space.

This paper is geared towards people teaching differential equations, numerical analysis, and mathematical modeling.

Creating a Model of Flight

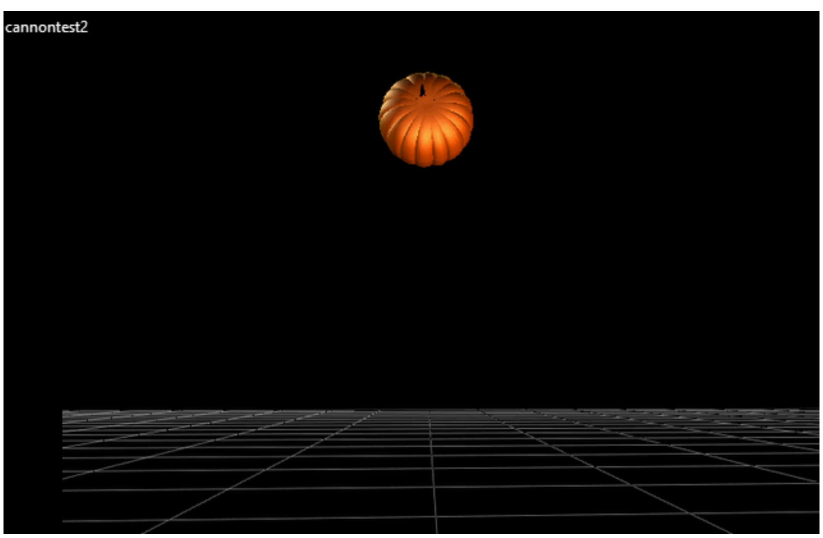

Figure 1-An Easy to See Projectile.

The projectile we will use in this paper will be that of a (simulated) pumpkin with the following parameters:

Cross-section=.0324294 meters
Coefficient of drag=.3
Mass=4 kg

Such an object was chosen for two reasons: 1) Pumpkins show up much better in 3D flight animations than just about any other object and 2) For years there was "Pumpkin Chunkin" competitions about how to launch a pumpkin the maximum distance.

To Create a Mathematical Model of Flight we need to consider the following parameters:

- Gravity
- Mass
- Cross-Sectional Area
- Shape of Surface
- Initial Velocity Vector
- Coefficient of Drag
- Rotation of Object
- Vector Field of Wind
- Rotation of Earth

Using these parameters, we shall give equations which define the equations for the trajectory of a projected object [2][3]. Since the derivation of these equations was discussed in detail in [2], we shall just present the equations that we need.

Notation for our Model of the Flight of a Projectile

v is the velocity vector
a is the acceleration vector
F_x, F_y and F_z are the forces in the x, y, and z-directions respectively (where x is distance, y is height, and z is the distance in the horizontal direction)
g is the force of gravity
m is the mass of the projectile in kg.
v_x, v_y, and v_z are the velocities in the x, y, and z directions
a_x, a_y, and a_z are the accelerations in the x, y, and z directions
Velocity and acceleration is measured in meters/sec.

The force on an object due to the drag of the atmosphere is given as follows:

$$F_D=\frac{1}{2}\rho v^2 A c_D$$

where:

ρ= the density of the fluid
v=the velocity of the object
A=the cross-sectional area
C_D=the coefficient of drag

$c_D = c_D(R_e)$ where R_e is the Reynold's number which is a function of: the fluid's density, the velocity of the object, the body length parallel to the direction of fluid flow, and the viscosity (thickness of the fluid) (Note: c_D is not constant)

$$R_e = \frac{\rho v L}{\mu}$$

At sea-level the air density is ρ=1.225 kg/m^3.

Total velocity is defined by:

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

The wind affects the amount of drag on the object [3].

Define the velocity of the wind as follows:

$$v_w = \begin{bmatrix} v_{wx} \\ v_{wy} \\ v_{wz} \end{bmatrix}(\mathrm{t}) \tag{1}$$

This gives us the equations for the velocity and acceleration of the object under the effect of wind [3]:

$$v_{ax} = v_x + v_{wx} \tag{2}$$

$$v_{ay} = v_y + v_{wy} \tag{3}$$

$$v_{az} = v_z + v_{wz} \tag{4}$$

The Equations for Our Model of Flight of a Projectile [2][3]

$$a_x = \frac{-F_D v_{ax}}{m v_a} = \frac{dv_x}{dt} \tag{5}$$

$$a_y = \frac{-F_D v_{ay}}{m v_a} = \frac{dv_y}{dt} \tag{6}$$

$$a_z = -g - \frac{-F_D v_{az}}{m v_a} = \frac{dv_z}{dt} \quad (7)$$

$$F_D = \frac{1}{2}\rho v_a^2 A c_D \quad (8)$$

Where v_a is defined as follows:

$$v_a = \sqrt{v_{xa}^2 + v_{ya}^2 + v_{za}^2} \quad (9)$$

Approximating the solution of the above set of differential equations (1)-(9) one may approximate the position vector (x, y, z) as a function of time. Standard techniques are Euler's method and the Runge-Kutta methods.

Side Note: One can also consider the Magnus force of a spinning object where lift is in the direction orthogonal to direction of flight and spin axis.

A Python Program Approximating the Solution the Equations of (1)-(9)

The following program uses a 1st order Euler approximation of the above system of differential equations.

Highlighted are the launch angle, velocity, and the velocity of the wind.

****

```
from math import *

p=1.225
cs=.0324294 #units in square meters
cd=.3
c=.5*p*cs*cd
v0=300 #in meters per second
theta=33 #launch angle in degrees
thetarad=theta*pi/float(180.0)
vx0=v0*cos(thetarad)
vy0=v0*sin(thetarad)
vz0=0
mass=4 #in kg
lxwind=0 #in meters per second
lywind=0
lzwind=0
```

```
xn=0
yn=0
zn=0
h=.01
n=2500
vx=[]
vy=[]
vz=[]
x=[]
y=[]
z=[]
vx.append(vx0)
vy.append(vy0)
vz.append(vz0)
x.append(0)
y.append(0)
z.append(0)


for i in range(n):

    v=sqrt(pow(vx[i],2.0)+pow(vy[i],2.0)+pow(vz[i],2.0))
    vw=sqrt(pow(vx[i]+lxwind,2.0)+pow(vy[i]+lywind,2.0)+pow(vz[i]+lzwind,2.0))
    dx=-c*pow(vw,2.0)*(vx[i]+lxwind)/(vw*float(mass))
    vxn=vx[i]+dx*h
    vx.append(vxn)
    dy=-c*pow(vw,2.0)*(vy[i]+lywind)/(vw*float(mass))-9.81
    vyn=vy[i]+dy*h
    vy.append(vyn)
    dz=-c*pow(vw,2.0)*(vz[i]+lzwind)/(vw*float(mass))
    vzn=vz[i]+dz*h
    vz.append(vzn)

for i in range(n):

    xn+=vx[i]*h
    x.append(xn)
    yn+=vy[i]*h
    y.append(yn)
    zn+=vz[i]*h
    z.append(zn)

for i in range(n):

    print("at time %.2f x is %.2f y is %.2f and z is %.2f" %(i*h,x[i],y[i],z[i]))
```

****

Obtaining a Launch Angle to Obtain a Maximum Distance with No Wind

Using the parameters from our pumpkin:

Cross-section=.0324294 meters
Coefficient of drag=.3
Mass=4 kg

with the above parameters, our Python program, with launch velocities v0= 100 m/s, 300 m/s, and 500 m/s for angles from 10 to 60 degrees we get the results in Figures 2, 3, and 4:

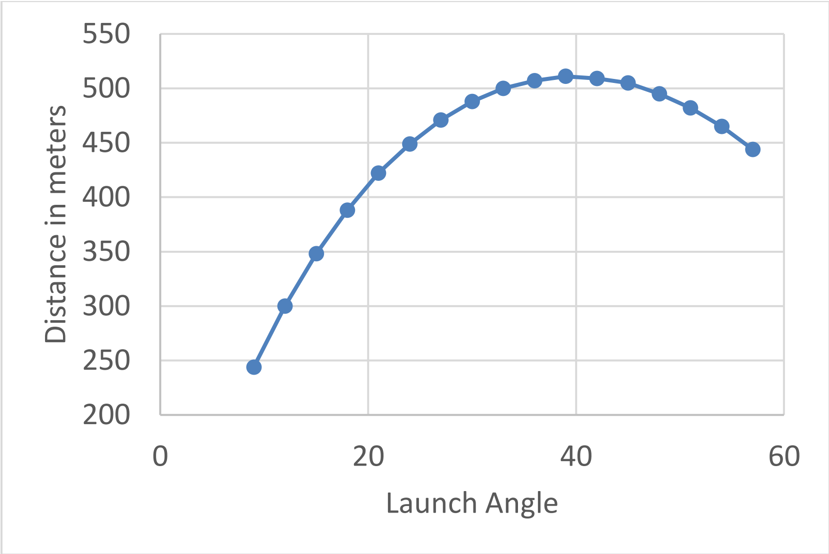

Figure 2-Distance of Projectile for Launch Velocity v0=100 m/s

We see the launch angle of 38° yields the maximum distance.

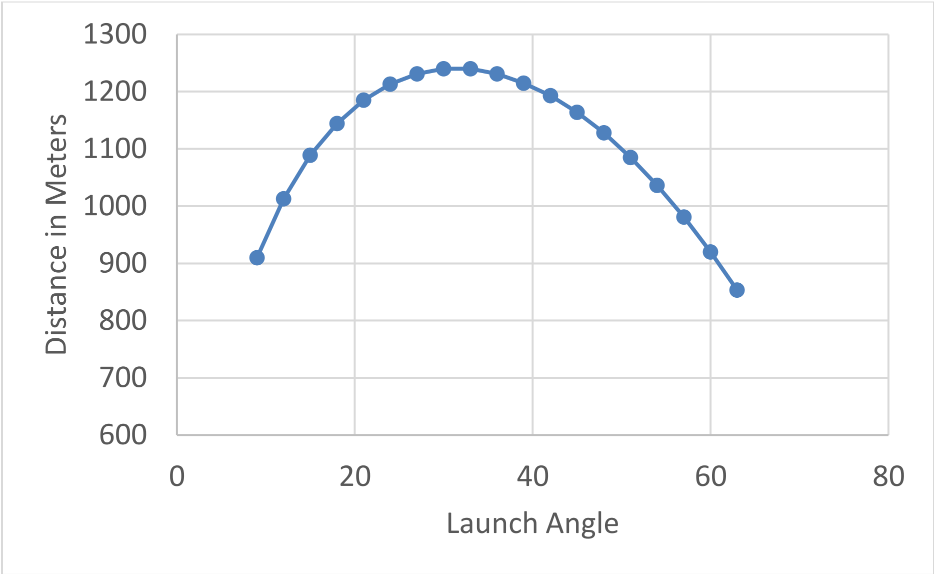

Figure 3-Distance of Projectile for Launch Velocity v0=300 m/s

We see the launch angle of 33° yields the maximum distance.

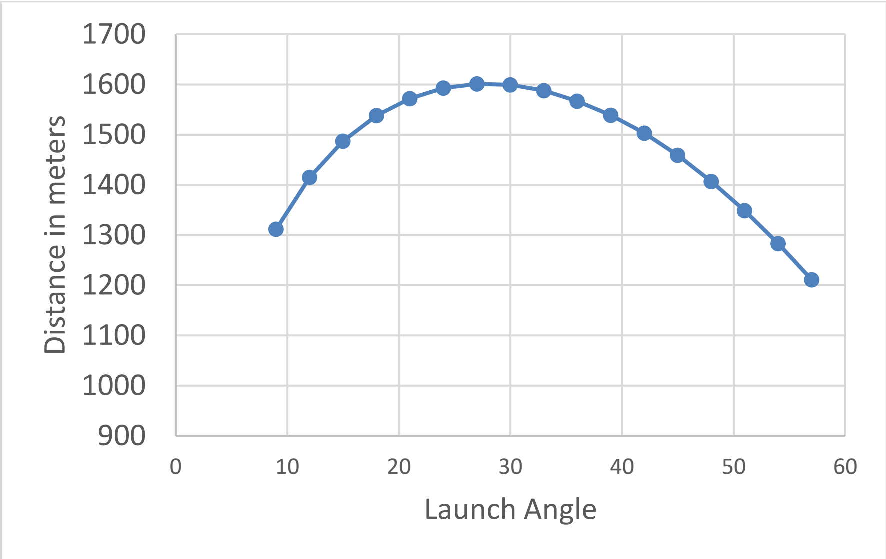

Figure 4-Distance of Projectile for Launch Velocity v0=500 m/s

We see the launch angle of 28° yields the maximum distance.

Summary: For our given pumpkin:

Launch Velocity	Launch Angle Yielding Maximum Distance
100 m/s	38°
300 m/s	33°
500 m/s	28°

Figure 5-Table Relating Launch Velocity and Launch Angle which Yields a Maximum Distance.

So, for our projectile, we see the launch angle needed to obtain maximum distance in our atmosphere is not 45° and depends on the initial velocity of the projectile.

Note: If we are dealing with a point mass, it can be shown a launch angle of 45° does indeed yield a maximum distance.

Showing That Wind Affects the Launch Angle Needed to Obtain a Maximum Distance

With a wind of 10 m/s headwind in the x direction and launch velocity of v0=300 m/s (using the same parameters as in the previous example) we get:

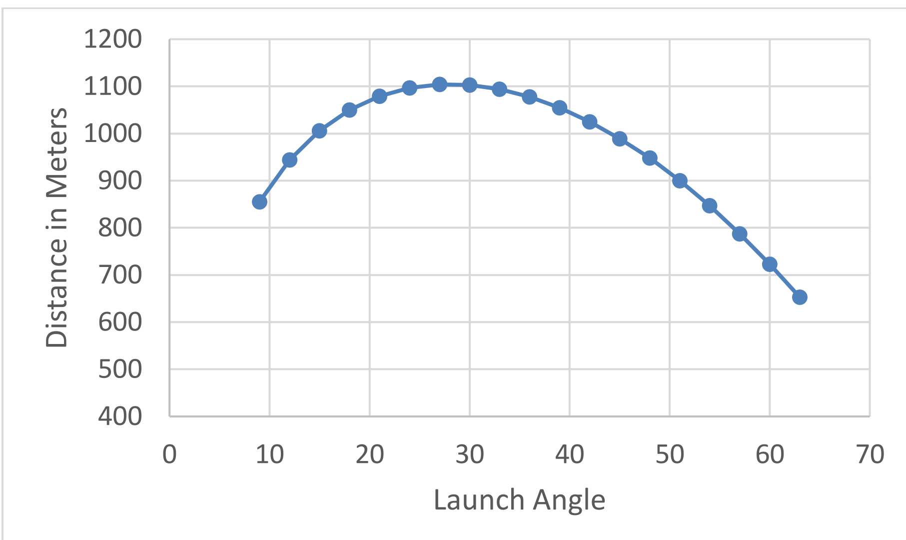

Figure 6-Distance of Projectile for Launch Speed v0=300 m/s and a headwind of 10 m/s.

Comparing Figures 3 and 6 we see:

HeadWind m/s	Launch Angle Yielding Maximum Distance
0 m/s	33°
10 m/s	30°

Figure 7-Table Relating Wind Velocity and Launch Angle which Yields a Maximum Distance.

Hence the velocity and vector of the wind does indeed affect the optimal launch angle.

Two factors that we have not considered here are:

- The spin of the projectile and
- The surface characteristics of the object.

Figure 8-The Dimples and Spin of a Golf Ball Effect its Distance

Not About Distance-But Another Application of our Equations and our Python Program: Terminal Velocity

If we were to shoot a skydiver out of a cannon (do not try this at home), where the skydiver is of average height (approximately 6') and average mass (approximately 80 kg), at 300 m/s straight up we want to find what their terminal velocity is. Using the above parameters in our Python program, we get the following data from our program: (again, y is the height above the ground in meters):

```
at time 18.36 x is 0.00 y is 86.55 and z is 0.00
at time 18.37 x is 0.00 y is 86.06 and z is 0.00
at time 18.38 x is 0.00 y is 85.56 and z is 0.00
at time 18.39 x is 0.00 y is 85.07 and z is 0.00
at time 18.40 x is 0.00 y is 84.57 and z is 0.00
at time 18.41 x is 0.00 y is 84.07 and z is 0.00
at time 18.42 x is 0.00 y is 83.58 and z is 0.00
at time 18.43 x is 0.00 y is 83.08 and z is 0.00
at time 18.44 x is 0.00 y is 82.59 and z is 0.00
at time 18.45 x is 0.00 y is 82.09 and z is 0.00
at time 18.46 x is 0.00 y is 81.60 and z is 0.00
at time 18.47 x is 0.00 y is 81.10 and z is 0.00
at time 18.48 x is 0.00 y is 80.60 and z is 0.00
at time 18.49 x is 0.00 y is 80.11 and z is 0.00
at time 18.50 x is 0.00 y is 79.61 and z is 0.00
at time 18.51 x is 0.00 y is 79.12 and z is 0.00
at time 18.52 x is 0.00 y is 78.62 and z is 0.00
at time 18.53 x is 0.00 y is 78.12 and z is 0.00
at time 18.54 x is 0.00 y is 77.63 and z is 0.00
at time 18.55 x is 0.00 y is 77.13 and z is 0.00
```

Figure 9-The Poor Skydiver Hits Terminal Velocity

Using the above data, we can see the skydiver's terminal velocity is about 110 miles/hour.

Given the parameters I used, the theory would say 112 miles/hour [3].

Summary

To attain the angle necessary to obtain a maximum distance in an atmosphere, we need to consider at a minimum:

- The atmosphere itself
- The mass, cross-sectional area, and density of the object
- The initial velocity
- The direction and velocity of the wind
- Alterations to the surface of the projectile

References:

[1] James Stewart, *Calculus* 7th Edition, Brooks-Cole, Belmont, CA, 2012.

[2] "The Effect of Wind on the Trajectory of Golf Balls" Proceedings of the 29th ICTCM https://www.pearson.com/content/dam/one-dot-com/one-dot-com/us/en/files/Paul-Bouthellier-ICTCM29.pdf

[3] Grant Palmer, *Physics for Game Programmers*, Apress, New York, NY 2005.