Residue Designs, LaTeXand GeoGebra
Computer Generated Conjecture

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What would it be like to have your students be problem posers, rather than just problem solvers? What if they could use advanced, free software to explore captivating mathematical patterns so that what they see allows them to generate the conjectures they will be proving? How motivating will it be when they are proving their own conjectures, rather than rehashing proofs from a textbook? One route for this kind of learning is to use Modular Residue Designs.

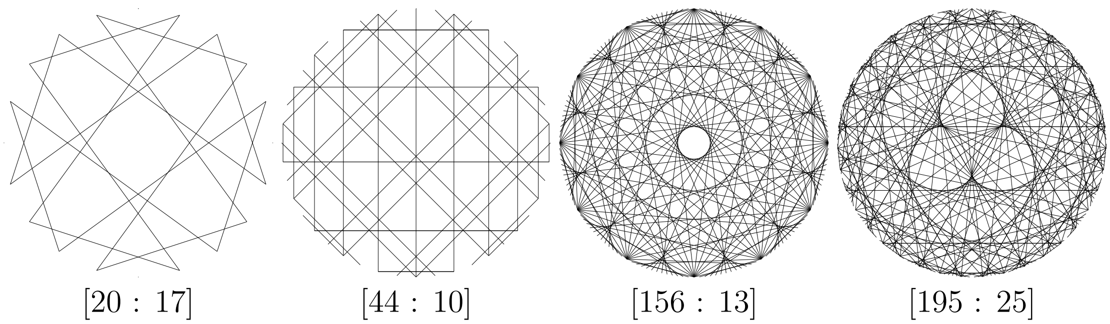


Figure 1: Modular Residue Designs: Levels of Complexity

1 History

Residue designs seem to have evolved from string art. They emerged in the middle of the last century, when they appeared as articles or chapters on recreational mathematics. As a beginning, for a modulus m, take a circle of radius 1 and place m equally spaced points around the circle, labeling them in order from 0 to $m-1$. In Figure 2 you can see the circle for $m = 5$. This will be the base from which residue designs will be drawn.

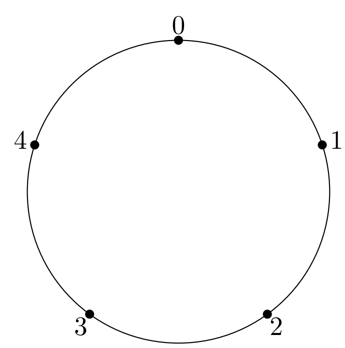


Figure 2: Circle with 5 points.

Thinking about $m = 5$, we can look at $k = 2$ and $a = 2$. Since $2 \cdot 2 = 4$ we have $2 \mapsto 4$. This is shown in the first diagram in Figure 3. In a similar what, when $a = 3$ we get $2 \cdot 3 = 6$, which reduces to 1 modulo 5, so $3 \mapsto 1$, as shown in the second diagram in 3. For $m = 5$ there are five different pertinent multiplications, giving five relations:

$$\begin{array}{ccc} 0 & \mapsto & 0 \\ 1 & \mapsto & 2 \\ 2 & \mapsto & 4 \\ 3 & \mapsto & 1 \\ 4 & \mapsto & 3 \end{array}$$

These relations give the lines shown in Figure 4. This will be denoted as the [5: 2] diagram. If you have a more general modulus m and you use a multiplier n to draw in the lines, you get the design $[m, n]$. Usually, however, we don't draw the arrowheads, the points on the circle, or even the circle itself. The full set of residue designs modulo 5 is shown in Figure 5.

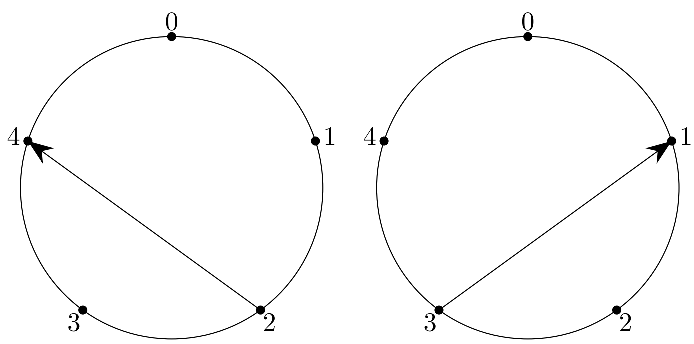


Figure 3: $2 \mapsto 4$ and $3 \mapsto 1$

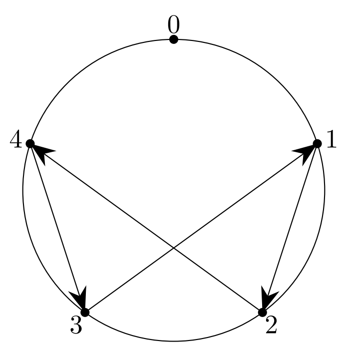


Figure 4: [5: 2]

With larger moduli, a variety of patterns present themselves. Because some of these patterns reflect algebraic properties such as commutativity, identities and inverses, articles about residue designs also showed up in mathematics education journals. The [5 : 1] design, for example, has no chords, as seen in Figure 5, because 1 is the multiplicative identity.

Figure 6 shows all of the modulo 10 designs. Within these designs, several different patterns appear which recur in many moduli. The [10 : 0] design has all chords going to 0. The [10 : 9] design has all horizontal chords. The [10 : 4] design has perpendicular chords. All the chords in the

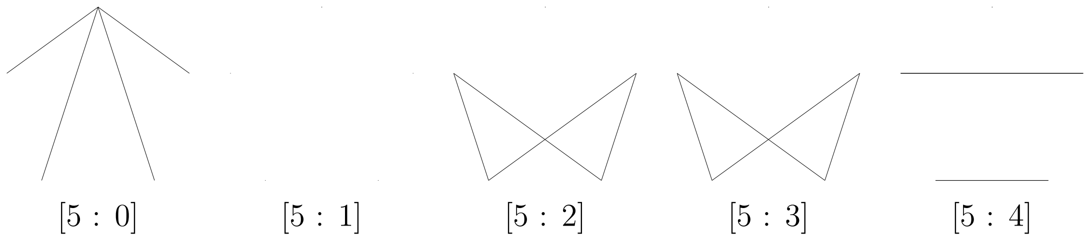


Figure 5: Modulo 5 designs

[10 : 6] design are diameters. The [10 : 3] and [10 : 7] designs look identical. All of the designs have bilateral symmetry, while only some of them have any sort of rotational symmetry.

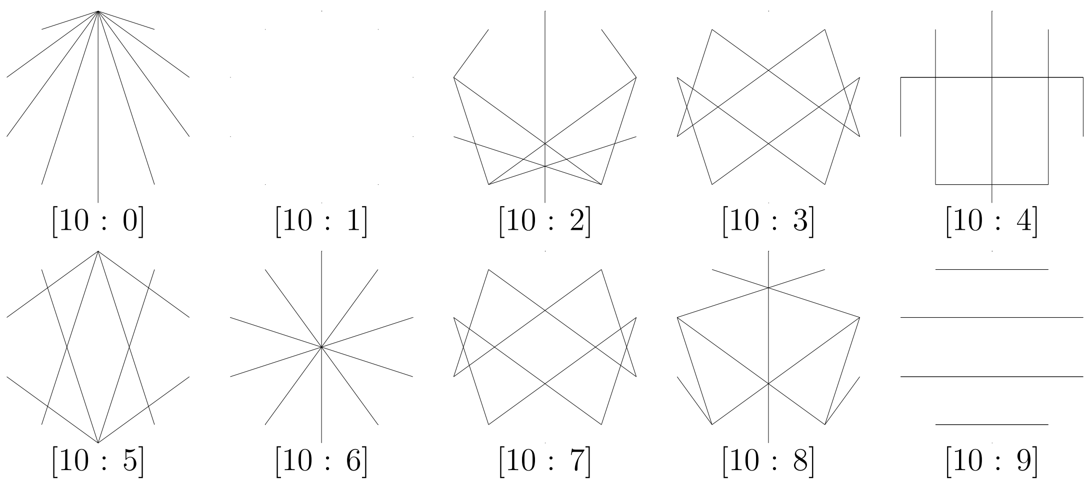


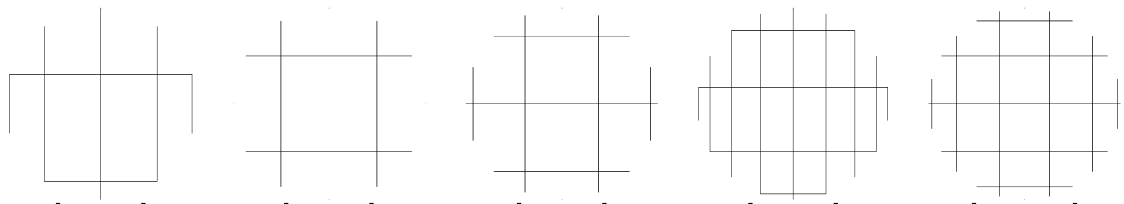
Figure 6: Modulo 10 designs

2 Explorations

For a long time, people could only generate residue designs by hand. This limited the ways they could be used for explorations. Now, however, there are free computer applications like LaTeX and GeoGebra which can quickly generate large numbers of designs. With a large number of designs to look at, for example, students might find several designs with perpendicular chords, as shown in Figure 7. Looking at these designs, they can recognize a pattern and make a conjecture that, for a residue design $[m : n]$, the chords will be perpendicular when $m = 2n + 2$. They can check other values of n and see that [8 : 3], [14 ; 6] and [20 ; 9] also match the pattern.

From this point, depending on the level of the students, they can explore why the designs with $m = 2n+2$ have perpendicular chords and even prove that the pattern will continue. They can also explore the patterns of how many horizontal and vertical lines will be in the $[m : n]$ design and see these numbers as functions of n.

As noted above, the mod 10 designs present other patterns for students to explore: why will all of the chords in the design $[m : n]$ be horizontal when $m = n + 1$?; when will the chords in a designs be only diameters and how many will show up in each design; when will the design have rotational symmetry; etc.

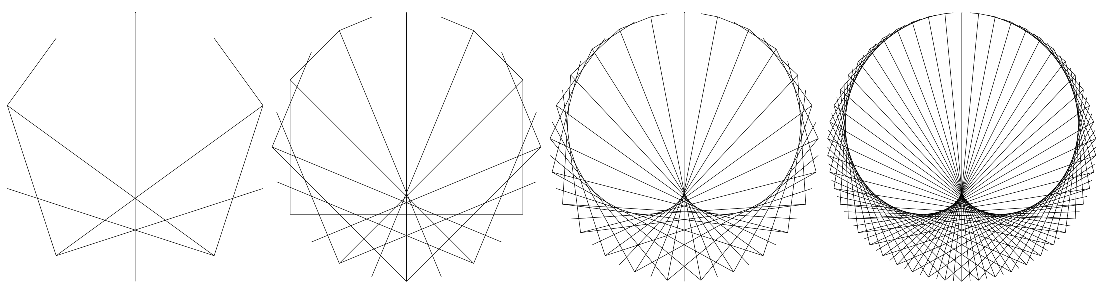


[10 : 4] [12 : 5] [16 : 7] [18 : 8] [24 : 11]

Figure 7: Perpendicular Chords

3 Cycloids

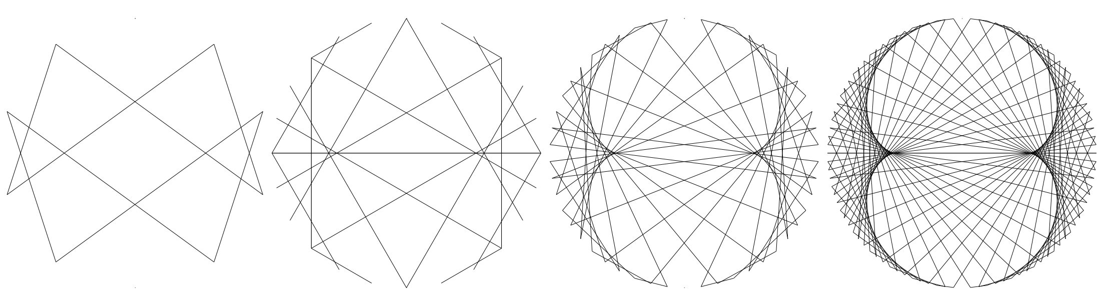
In the earliest writing about residue designs, one of the main topics was what happens when you pick a multiplier and then look at larger and larger moduli. Figure 8 shows four designs of increasing modulus where the multiplier is 2. The shape framed by the chords is a cardioid. It shows up in polar coordinates as the graph of $r = 1 + \sin(\theta)$, but it can also be generated by rolling one circle around another, fixed circle and tracing the path of a point on the rolling circle. In this way, it resembles designs drawn using the Spirograph toy.



[10 : 2] [24 : 2] [50 : 2] [100 : 2]

Figure 8: Multiplier of 2

Figure 9 shows several designs where the multiplier is 3. In these designs, the chords frame a shape called a nephroid, which can be generated by rolling a circle around a fixed circle where the fixed circle has double the radius of the rolling circle. A point on the rolling circle will trace out the nephroid.



[10 : 3] [24 : 3] [50 : 3] [100 : 3]

Figure 9: Multiplier of 3

The cardioid and the nephroid are two examples of epicycloids, which are shapes that trace a point on a circle which rolls around the outside of a fixed circle. The exact shape of a given epicycloid is determined by the ratio of the radii of the two circles. As noted, a cardioid occurs when the ratio is 1, while a nephroid is traced when the ratio of the fixed radius to the rolling radius is 2 to 1. The cardioid is sometimes called an epicycloid of one cusp while the nephroid is also known as the epicycloid of two cusps.

When the multiplier in the residue design is increased further, still more epicycloids are generated. Figure 10 shows designs for multipliers of 4, 5, 6 and 7. These generate, respectively, epicycloids of 3, 4, 5 and 6 cusps, which are also shown in Figure 10.

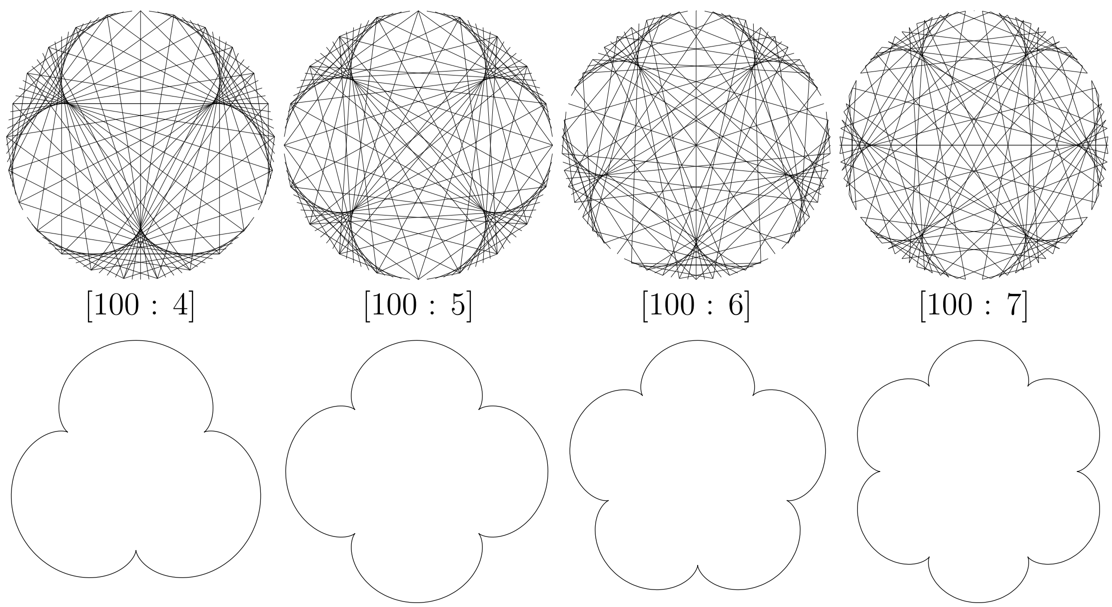


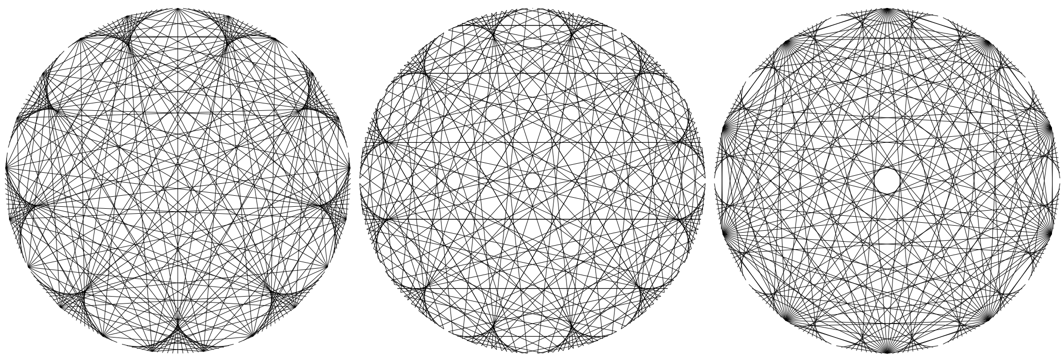
Figure 10: Designs with Increasing Multipliers and their Cycloids

Figure 11 shows designs with still larger multipliers and, in each case, the number of lobes and cusps in the epicycloid is one less than the multiplier n. As I pointed out above, the earliest articles I've seen on residue designs describe this result. Even though none of the writings I've seen gives a formal proof, it is provable.

What I have not seen published yet is a deeper connection between residue designs and epicycloids. These are difficult, I suppose, to see until you look at a large number of residue designs, which in turn isn't easy or likely until you have a fast method for generating residue designs. I found the pattern using GeoGebra. I chose a single multiplier but used a "slider" in GeoGebra for the modulus. I then automated the slider, which made a slide show of many different designs. Some of them stood out, as shown in Figure 12.

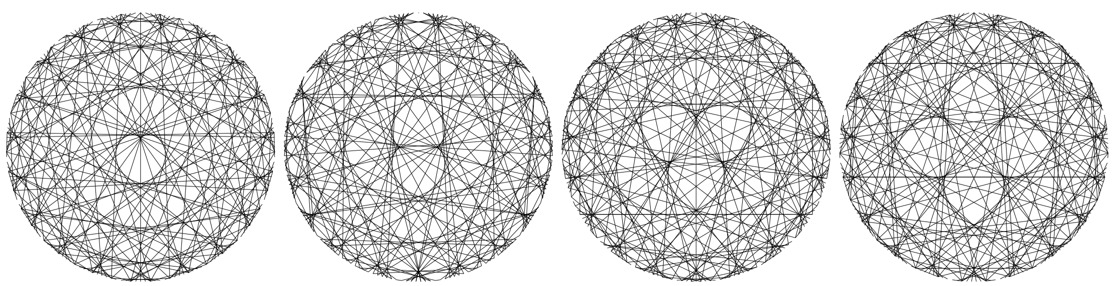
You may notice two epicycloids in each of these designs; you might say that one of them is on the outside border of the design while the other is in the center. The bordering epicycloid, in each case, has nineteen cusps. This is because, simply, they each have 20 for their multiplier. The central epicycloids, however, are more complex than the ones we have looked at so far. They tend to have more "depth" to them, especially since for many of them, the lines of the epicycloids intersect themselves.

These epicycloids are drawn in Figure 13. For the most part, they come from circles where the ratio of the radii is not just $a : 1$. Instead, they can be $a : b$ where the radius of the fixed circle is not always a whole number multiple of rolling radius. In fact, sometimes the radius of the rolling circle is larger.

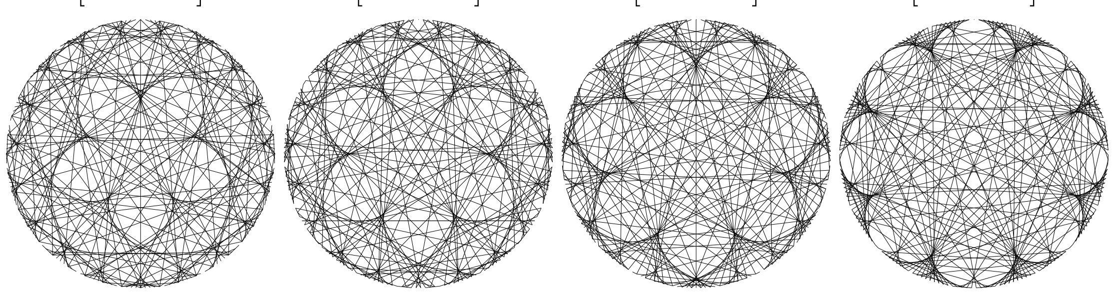


[210 : 10] [210 : 13] [210 : 21]

Figure 11: Still Larger Multipliers



[172 : 20] [173 : 20] [174 : 20] [175 : 20]



[176 : 20] [177 : 20] [178 : 20] [179 : 20]

Figure 12: More Complex Patterns

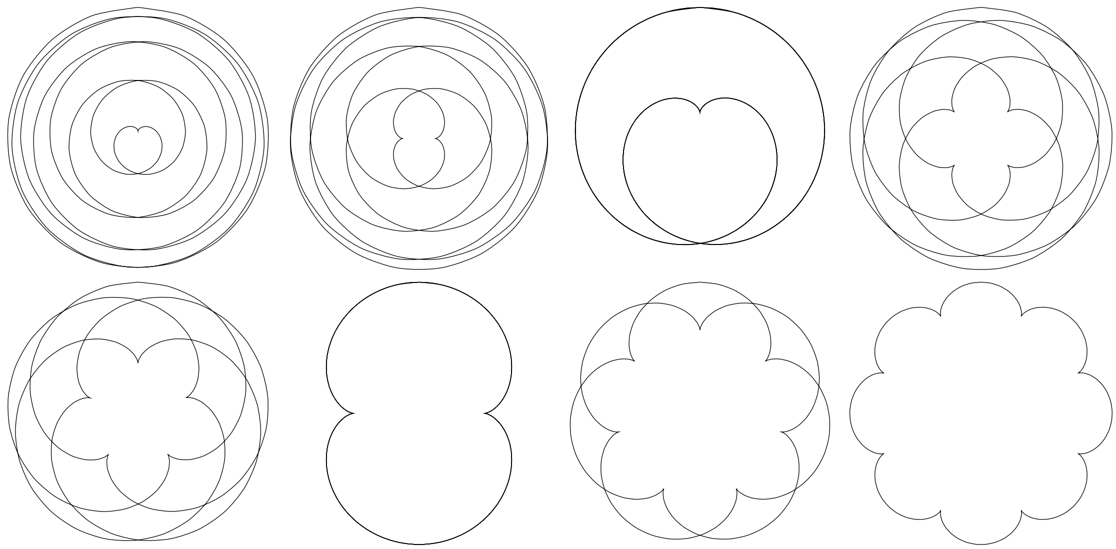


Figure 13: More Complex Epicycloids

What is the pattern in m and n that generates these epicycloids. In each design here,

$$m \quad = \quad 9 \times 20 - r$$

where r is a between number between 0 and 9. If, for comparison, you let $n = 25$ and run through the same values of k, you get the designs in Figure 14. They have exactly the same central epicycloid patterns as those found in Figure 12.

As a principle, when there are numbers k and r, with $0 < r < k$ so that $m = k \cdot n - r$, then the chords of the residue design $[m : m]$ are all tangent to an epicycloid whose fixed to rolling ratio is $k - r : r$. Or, to be more careful, they will when k and r are relatively prime.

You might have noticed, when looking at the epicycloids in Figure 13, that the third and sixth epicycloids rather stand out as being less complex than the others and rather lacking in rotational symmetry, especially compared to the corresponding residue designs. This is because 3 and 6 are not relatively prime to $k = 9$. In these cases, you need not just the epicycloid, but also two rotated copies of it to match the pattern in the corresponding residue design. Figure 15 shows this. Here the two extra copies of the epicycloid are drawn in red and green.

More generally, when k and r are not relatively prime, let $g = \gcd(k : r)$. The ratio $k - r : r$ can be simplified and the actual epicycloid will be correspondingly simpler. In the residue design there will then be g copies of the epicycloid rotated to be spaced evenly about the circle. The proof of this result moves the discussion to a generalization of residue designs, what Joseph Madachy called Lost Chord Designs.

4 Lost Chords

Lost chord designs showed up in Joseph Madachy's *Madachy's Mathematical Recreations* in 1979. He learned them from W. H. Cozens, but they may go back to curve stitching designs introduced by Mary Everest Boole. Here is the general idea.

Equally space points around the circle, as we have been doing with residue designs. Draw a chord of the circle running between the two points, a beginning point and an ending point. The next step is to draw a new chord by shifting the beginning point by one space to get the beginning point of the second chord and shifting the ending point by two spaces to get the ending point of

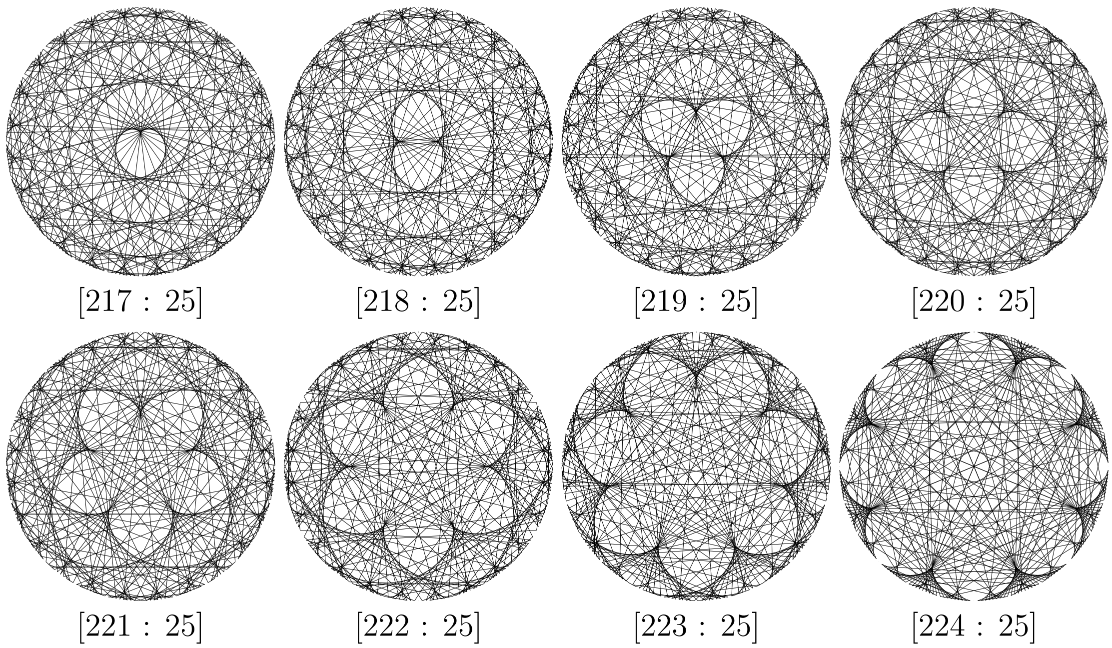


Figure 14: More Complex Patterns

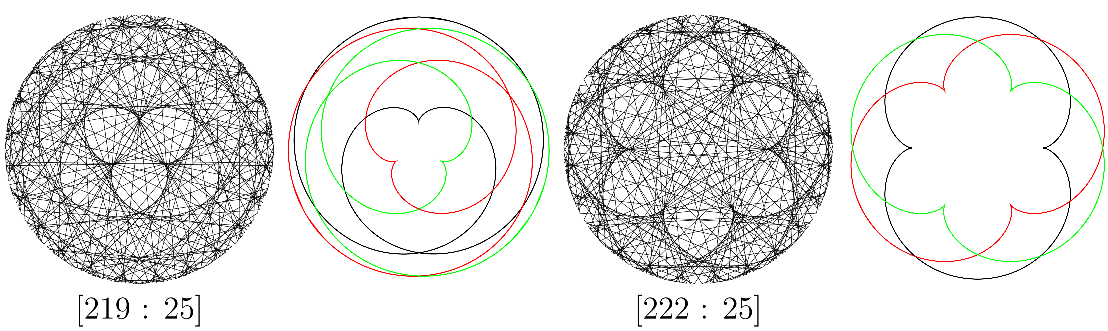


Figure 15: When k and r are not Relatively Prime

the second chord. A third chord is found by again shifting the beginning point by one space and the ending point by two spaces, and so on. This iterative process is continued to give a sequence of chords. Figure 16 shows the result of five such steps on a circle divided by 36 points. The process is

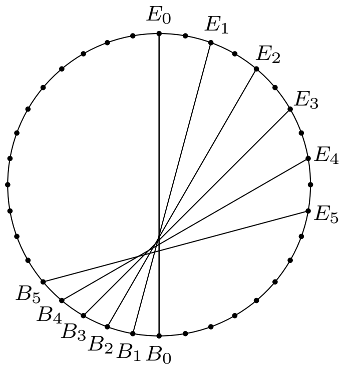


Figure 16: Plotting Lost Chords

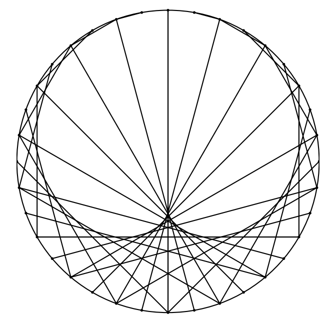


Figure 17: The (1 : 2 : 36) Design

continued recursively until the design returns to the original chord. After that, it will repeat without drawing any new chords. The design in Figure 17 shows the result when the points are 10° apart and the starting chord is the vertical diameter. Not coincidentally, this is also the [36 : 2] residue design.

The process can be generalized, first by changing the way that the beginning and ending points move in each iteration and, second, by changing the starting chord. A rotation of the starting chord by any central angle will cause a rotation of the design, while changing the length of the initial chord can actually change the design. As the figure above shows, the design you get by moving chords can actually match a residue design, but this won't necessarily happen. Theoretically, you can pick any chord and use any sequence of initial and ending chords to generate a design. We will, however, restrict our exploration to arithmetic sequences where the initial points are shifted by adding a each time and the ending points by adding b each time. It would seem, for example, that starting with a null chord with beginning and ending points both 0 and letting $a = 1$ and $b = n$ should give the same design as $[m : n]$.

A lost chord design is determined by the combination of m, the number of points on the circle, a and b, and the choice of the beginning chord. We will denote by $(m : a : b)$ the lost chord design where the the circle is subdivided by m points and the initial and terminal ends of the chords are moved, respectively, by a and b points each time. Additionally, the initial chord will be a null chord where both endpoints are at 0. An important question, then, is when will the design $(m : a : b)$ is the same as $[m : n]$ for some n.

Suppose that a and m are relatively prime. Then there are integers x and y, with $x > 0$ so that $ax + my = 1$. This means that after x iterations of moving the chord, the initial point will be at 1. The ending point will be at xb. If xb reduces to n modulo m then the chord from 1 to n will be in the design. By extension, the chord from i to ni will also be in the design, so $[m : n]$ is a subset of $(m : a : b)$.

On the other hand, we know that the chords of $(m : a : b)$ are determined by the relation $ai \mapsto bi$ for each i. In $[m : n]$,

$$\begin{aligned} (ai)n &= (an)i \\ &\equiv (axb)i \\ &\equiv (1b)i \\ &= bi. \end{aligned}$$

The chords in $(m : a : b)$ are, therefore, in $[m : n]$.

Now we can get back to our epicycloid designs. Suppose that k and r are relatively prime with $0 < r < k$. From this, we can write $1 = kx + ry$. Also, for some n, let $m = kn - r$. Then

$$\begin{aligned} 1 &= kx + ry \\ &= kx + (kn - m)y \\ &= kx + kny - my \\ &= k(x + ny) + m(-y) \end{aligned}$$

This means that k and m are relatively prime and, if we can find the value j that $r(x + ny)$ reduces to modulo m, it follows that $(m : k : r)$ is the design $[m : j]$. Notice that

$$\begin{aligned} r(x + ny) &= rx + rny \\ &= (kn - m)x + rny \\ &= knx - mx + rny \\ &= (kx + ry)n - mx \\ &= n - mx. \end{aligned}$$

This means that $j = n$ and $(m : k : r)$ is $[m : n]$.

From here, our next step is to connect the lost chord designs with the epicycloids.

5 Lost Chords and Epicycloids

The Lost Chord Designs always present at least one epicycloid. More specifically, when $a > b$, the chords in the design $(m : a : b)$ will always be tangent to an epicycloid whose ratio of fixed radius to rotating radius is $a - b : b$. Figure 18 shows examples where $a : b$ is $3 : 1$, $7 : 2$, and $4 : 3$. respectively.

To show this, we will start with a parametrization of the epicycloid.

$$\begin{aligned} x(t) &= k\sin(rt) + r\sin(kt) \\ y(t) &= k\cos(rt) + r\cos(kt) \end{aligned}$$

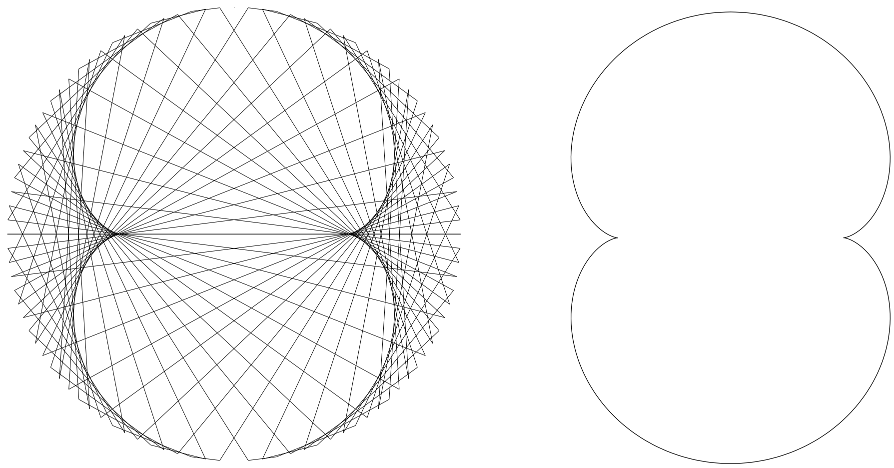
We can differentiate with respect to the parameter t to get

$$\begin{aligned} \frac{dy}{dt} &= -rk\sin(rt) - rk\sin(kt) \\ \frac{dx}{dt} &= rk\cos(rt) + rk\cos(kt) \\ \frac{dy}{dx} &= \frac{-rk\sin(rt) - rk\sin(kt)}{rk\cos(rt) + rk\cos(kt)} \\ &= -\frac{\sin(rt) + \sin(kt)}{\cos(rt) + \cos(kt)} \end{aligned}$$

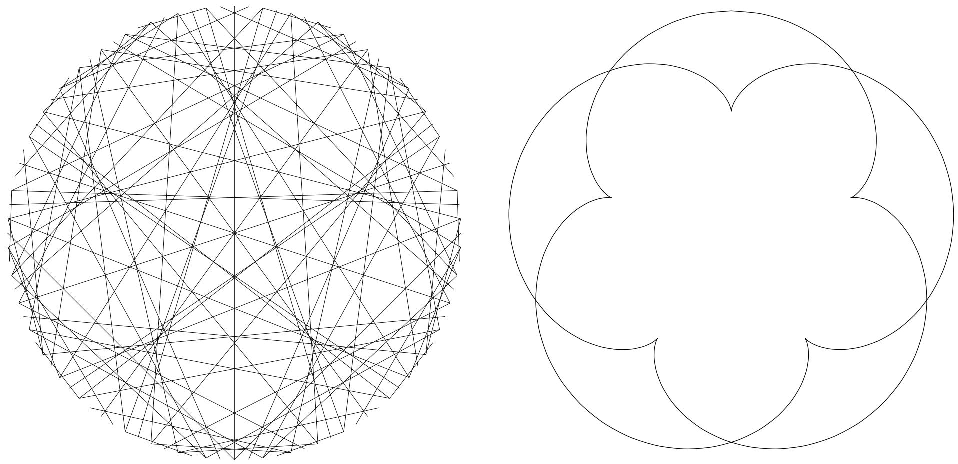
This gives the slope of the tangent line to the epicycloid at the point $(x(t), y(t))$.

Additionally, the endpoints for a chord in the $(m : k : r)$ lost chord design will have endpoints $(\sin(kt), \cos(kt))$ and $(\sin(rt), \cos(rt))$ where t is a whole number multiple of $\frac{360°}{m}$. It follows that the slope of the chord is

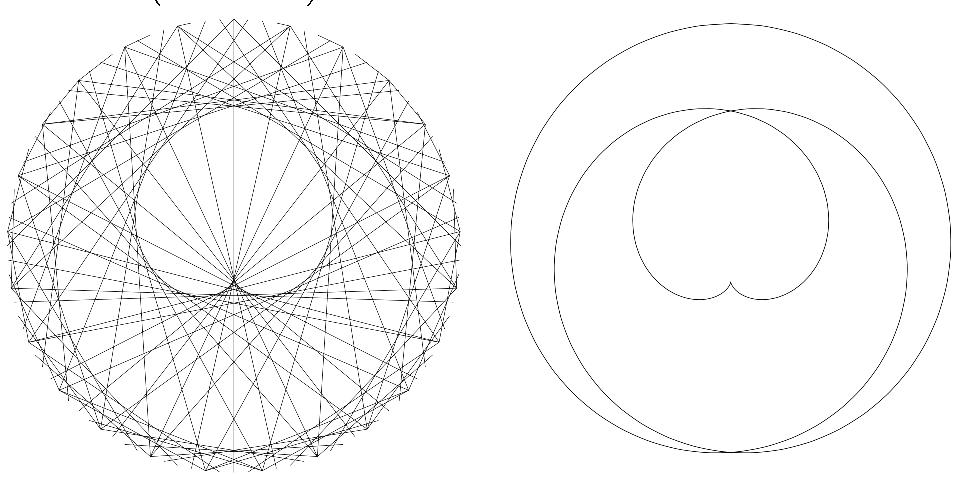
$$\frac{\Delta y}{\Delta x} = \frac{\cos(rt) - \cos(kt)}{\sin(rt) - \sin(kt)}$$



(100 : 3 : 1)



(100 : 7 : 2)



(100 : 4 : 3)

Figure 18: Lost Chord Designs and Epicycloids

Using trigonometric identities and algebraic manipulation, we have

$$
\begin{aligned}
\frac{\Delta y}{\Delta x} &= \frac{\cos(rt)-\cos(kt)}{\sin(rt)-\sin(kt)} \\
&= \frac{\cos(rt)-\cos(kt)}{\sin(rt)-\sin(kt)} \cdot \frac{(\sin(rt)+\sin(kt))(\cos(rt)+\cos(kt))}{(\sin(rt)+\sin(kt))(\cos(t)+\cos(kt))} \\
&= \frac{\left(\cos^2(rt)-\cos^2(kt)\right)(\sin(rt)+\sin(kt))}{(\sin^2(rt)-\sin^2(kt))(\cos(rt)+\cos(kt))} \\
&= \frac{((1-\sin^2(rt))-(1-\sin^2(kt)))(\sin(t)+\sin(kt))}{(\sin^2(rt)-\sin^2(kt))(\cos(rt)+\cos(kt))} \\
&= \frac{(\sin^2(kt)-\sin^2(rt))(\sin(rt)+\sin(kt))}{(\sin^2(rt)-\sin^2(kt))(\cos(rt)+\cos(kt))} \\
&= -\frac{\sin(rt)+\sin(kt)}{\cos(rt)+\cos(kt)} \\
&= \frac{dy}{dx}.
\end{aligned}
$$

This shows that the chord has the same slope as the tangent line. A little work shows that the chord will actually go through the point $(k\sin(rt)+r\sin(kt), k\cos(rt)+r\cos(kt))$ on the epicycloid, making the chord part of the tangent line. Additionally, the point on the epicycloid will break the chord into two segments whose ratio is $k:r$. Visually, this shows that the chords of the design will frame the epicycloid.

6 Greatest Common Divisors

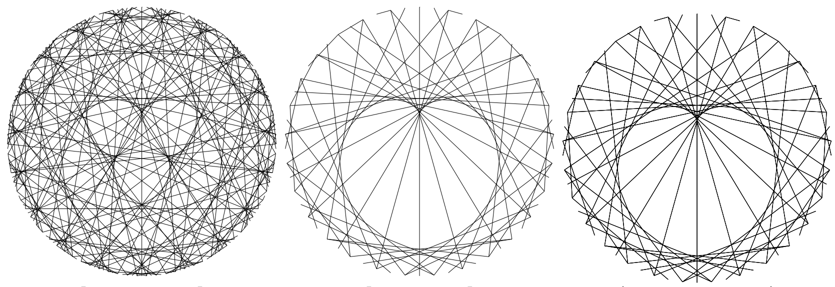
We saw, with [174 : 20] and [177 : 20] that when $m = kn - r$ where $\gcd(k, r) > 1$, the shape framed by the residue design is more complex than the epicycloid with fixed to rotating ratio of $k - r : r$, which simplifies. Correspondingly, the lost chord design $(m : k : r)$ matches the epicycloid, but doesn't give the full complexity of the residue design, since it only contains chords whose endpoints are multiples of $g = \gcd(k, r)$.

6.1 The Non-Relatively Prime Cases

Suppose that $\gcd(k, r) = g > 1$. Then, by the definition of m as $kn-r$. it follows that $\gcd(m, k) = r$. We can factor out g to give $m' = \frac{m}{g}$, $k' = \frac{k}{g}$ and $r' = \frac{r}{g}$. Then $m' = k'n - r'$. It follows that m' and $k\prime$ are relatively prime, as are k' and r'. Additionally, the design $(m : k : r$ is the same as $(m' : k' : r')$, which matches $[m' : n]$. We can see this in Figure 19 where $n = 20$, $k = 9$ and $r = 6$.

The chords the [58 : 20] design are exactly the chords in the (174 : 9 : 6) design, and they are the chords in [174 : 20] design where the initial edge is a multiple of $g = 3$. Figure 20 shows this set in red, as well as two other sets in purple and blue which are the chords from [174 : 20] where the initial points is congruent either to 1 or 2 modulo $g = 3$. These each give rotations of the epicycloid in [58 : 20] of either 120° or 240°. These three sets combined give the [174 : 20] and the three epicycloids combined give the pattern framed by the [174 : 20] design.

This separation illustrates the general setting when $g = \gcd(k, r) > 1$. The design $(m : k : r)$ is the same as the design $(m' : k' : r')$, which frames an epicycloid with ration $k' - r'$. The chords in $(m' : k' : r')$ are the chords in $[m : n]$ where the initial endpoint is a multiple of g. The other chords in $[m : n]$ can be separated into $g - 1$ subsets, where each of the subsets consists of chords where the intitial point is congruent to $a \pmod g$ for some $0 < a < g$. Each subset then frames a rotation of the epicycloid by a multiple of $\frac{360°}{g}$.



[174 : 20] [58 : 20] (174 : 9 : 6)

Figure 19: Simplified Form

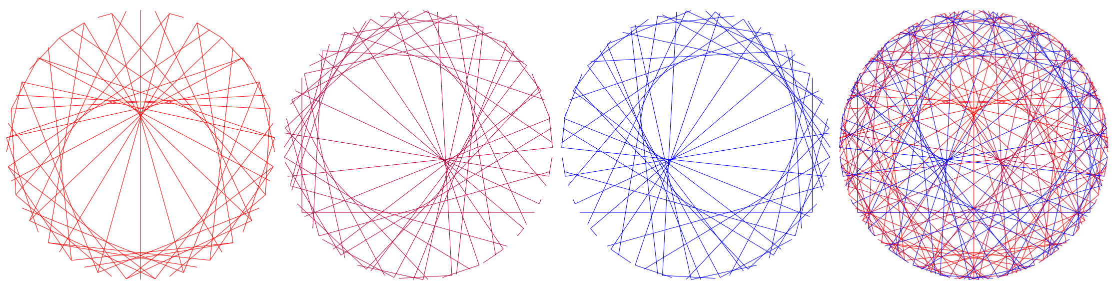


Figure 20: Three Separate Sets Combined

7 Using GeoGebra

GeoGebra is a powerful for exploring mathematics, including residue designs. The following links are for GeoGebra activities that allow you to explore residue designs, epicycloids and lost chord designs. You can use them as is or, if you know GeoGebra, modify them.

1. Residue Design: https://www.geogebra.org/classic/xknznmjk
2. Epicycloid: https://www.geogebra.org/classic/vntpxv5r
3. Lost Chord Deisgn: https://www.geogebra.org/classic/wsywpvpa
4. Flower Designs: https://www.geogebra.org/classic/nxdbuxmj

8 Using LaTeX

While LaTeX can be used to explore residue designs, it is very good for communicating them. The TikZ package provides tools for generating residue designs. The code that follows provides commands for generating residue designs and lost chord designs.

8.1 DrawMod

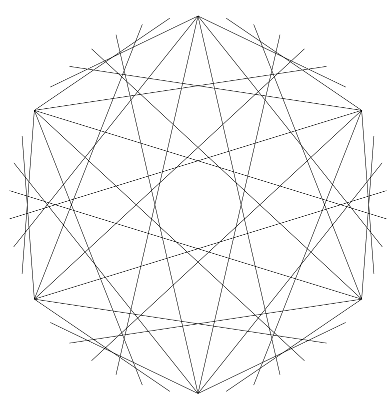
The DrawMod command will draw residue designs. The user has to include values for four parameters: the modulus, the multiplier, the color of the chords and the radius of the circle.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%% The \DrawMod command takes four arguments:
%%%  #1 is the modulus,
%%% #2 is the multiplier,
%%% #3 is the color of the chords,
%%% #4 is the radius of the design.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

\newcommand\DrawMod[4]{%

\foreach \x in {0, 1, ..., #1}{
\draw [color=#3, line width=0.1pt] ({#4*sin(360/#1*\x)},{#4*cos(360/#1*\x)}) --
({#4*sin(360/#1*\x*#2)},{#4*cos(360/#1*\x*#2)});
}

\large
\draw (0, -1.2*#4) node {[#1 : #2]};
}
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

The command needs to be used within a TikZ environment, such as

```
\begin{tikzpicture}
\DrawMod{42}{7}{black}{2.5cm}
\end{tikzpicture}
```

which produces the [42 : 7] design.



[42 : 7]

Figure 21: [42 : 7] Design

8.2 DrawChords

The DrawChords command produces a lost chord design which begins with a null chord at 0 as its initial chord. It takes five parameters: The modulus, the shift for the intial point, the shift for the terminal point, the color of the chords and the radius of the design.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%% The \DrawChords command takes four arguments:
%%%  #1 is the modulus,
%%% #2 shifts the initial point,
%%% #3 shifts the terminal point,
%%% #4 is the color of the chords,
%%% #5 is the radius of the design.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

\newcommand\DrawChords[5]{%

\foreach \x in {0, 1,  ..., #1}{
\draw [color=#4, line width=0.1pt] ({#5*sin(360/#1*\x*#2)},{#5*cos(360/#1*\x*#2)}) --
({#5*sin(360/#1*\x*#3)},{#5*cos(360/#1*\x*#3)});

\large
\draw (0, -1.15*#5) node {(#1 : #2 : #3)};

}}
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

Like the DrawMod command, the DrawChords command must be placed within a TikZ environment. As an example,

```
\begin{tikzpicture}
\DrawChords{50}{3}{2}{black}{2.5cm}
\end{tikzpicture}
```

will produce the (50 : 3 : 2) design.

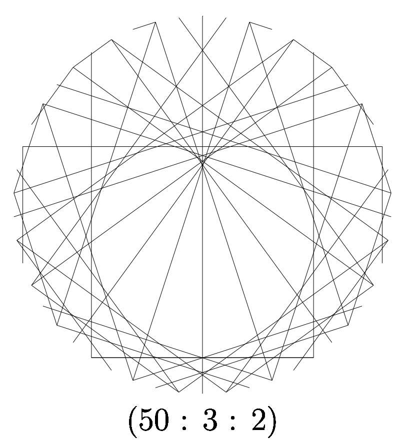


Figure 22: (50 : 3 : 2) Design

LaTeX is not as robust as GeoGebra in generating the designs. This makes sense, since the main goal of LaTeX is to to mathematical type setting. One important limitation is that it will run over run its memory allowance if the multiplier is too big. Also, as the modulus gets larger, more and more time is required to generate the designs. If there are dozens of designs with larger moduli, a document can take minutes to compile. GeoGebra, by contrast, can run through hundreds of designs very quickly. This means that GeoGebra is the better tool for exploring the designs.

9 Conclusion

Residue designs provide a great way to engage students in mathematical explorations. They have a low entry level and a very high ceiling when it comes to student engagement. The visual appeal can draw some students in, but there is also great depth and breadth to the mathematics that can be explored. GeoGebra and LaTeX provide valuable tools for generating the designs, which in turn increases their availability for exploration. These explorations can provide students with a personal motivation for proofs, since they will be proving conjectures that are their own.

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