SOLVING POLYNOMIAL EQUATIONS BY *EXCEL*

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(*Abstract.*) Mathematicians have been solving equations for thousands of years now. We have been successful in getting closed-form solutions for polynomials up to degree 4. Degree 5 equations require special functions other than radicals. Most numerical solutions have long execution times or require programming know-how for their application. We propose a method here of solving equations that is readily available and the results are obtained instantaneously, through the use of *Excel.*

Mathematicians have long been preoccupied with the search for solutions of polynomial equations. The degree of difficulty of a solution corresponds rather closely to the degree of the polynomial.

A polynomial equation is an expression of the form:

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 = 0,$$

where the a_i's are real constants and n is the degree of the polynomial. Linear equations ($n = 1$) are easily solved. By 2000 years ago, the Chinese already knew how to solve n equations in n unknowns by the method we call 'Gaussian elimination'. The quadratic equation ($n = 2$) is harder to solve because it requires the square root operation. But the solution was found by many cultures, as early as 2000 years ago by the Babylonians. The first really hard case is the cubic equation ($n = 3$), whose solution requires both square roots and cube roots. The solution was discovered by Italian mathematicians in the 1540s, together with the quartic equation ($n = 4$) in what are known as Cardan's method and Ferrari's method. This breakthrough renewed interest in polynomial equations.

For over 250 years, there was a lull in the progress of solving polynomial equations. The obstacle was the quintic equation ($n = 5$). By the 1820s, it became clear that it is not solvable in the same sense as before; it could not be solved by *radicals,* like the previous cases. It required a new and more abstract form of algebra. Very little progress was made in view of this. However, two very simple but important contributions were made by Descartes (1637). One was the use of exponents to represent the degree of a term. The other was his theorem that a polynomial $p(x)$ that is 0 when $x = a$ has a factor $(x - a)$. Dividing $p(x)$ of degree n by $(x - a)$ leaves a polynomial of degree n – 1. We will make use of this fact in our examples several times.

Solutions by *Excel* are very straightforward. It goes in two steps. First, we graph the polynomial function to find where the curve crosses the x-axis. The crossing points give us our initial guesses for the roots. Then, we call *Excel,* in particular, *Solver,* to zero in on that value of x where the polynomial goes to 0, or at least very close to 0. We do not need to use any formulas to find the roots. It is all done by *Solver,* which is very efficient that it yields the answers in no time at all.

The only drawback to this method is that sometimes complex roots are to be expected. In this case, we resort to the quadratic formula, which gives us a pair of conjugate roots. The real difficulty is when there are more than a pair of complex roots. Then we have no recourse but to use Cardan's formula or Ferrari's for the quartic. We illustrate this in one example (Ex. 5).

Ex. 1: The cubic $p(x) = x^3 + 2x^2 - x - 2.$

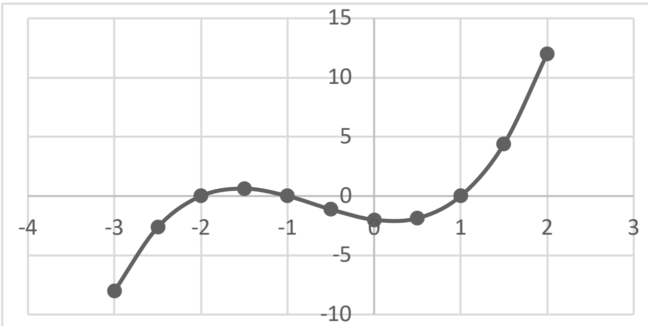

Figure 1. $p(x) = x^3 + 2x^2 - x - 2$

Calling *Solver* gives the solution $x = 1$. Dividing by $(x - 1)$ gives the quadratic: $x^2 + 3x + 2 = 0$, which yields the two other roots, -1 and -2.

Ex. 2: The cubic $p(x) = x^3 + 2.48x^2 - 4.3155x + 2.484406.$

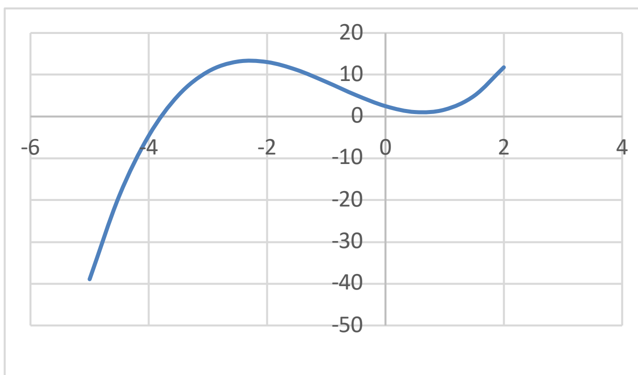

Figure 2. $p(x) = x^3 + 2.48x^2 - 4.3155x + 2.484406$

	x^3	x^2	x	Constant	Y
	1	2.48	-4.3155	2.484406	1.75E-06
x1	-3.79116				
	1	2.48	-4.3155	2.484406	

-3.79116		-3.79116	4.970815	-2.48440425		
	1	-1.31116	0.655315	1.74865E-06		
a	1		x2 = 0.655580+0.474900i			
b	-1.31116		x3 = 0.655580-0.474900i			
c	0.655315					

Solver gives the real root $x = -3.79116$. The associated quadratic yields the complex conjugates $0.655580 \pm i0.497490$.

Ex. 3: Quintic with 5 real roots $p(x) = x^5 + 3x^4 - 5x^3 - 15x^2 + 4x + 12$

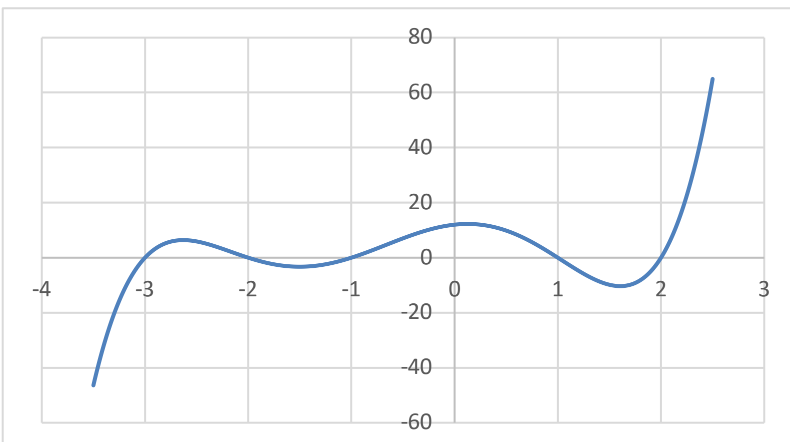

Figure 3. $p(x) = x^5 + 3x^4 - 5x^3 - 15x^2 + 4x + 12$

	x^5	x^4	x^3	x^2	x	Constant	y
Coef	1	3	-5	-15	4	12	0
1st Root	-3.00						
			Synthetic Division 1				
Coef	1	3	-5	-15	4	12	
-3.00		-3	0.00E+00	1.50E+01	0	-12	
	1	0.00	-5.00E+00	0.00E+00	4	0	

	x^4	x^3	x^2	x	Constant	y
Coef	1	0.00	-5.00E+00	0.00	4	0.00
2nd Root	-2					

Synthetic Division 2

Coef	1	0.00	-5.00E+00	0.0E+00	4
-2.00		-2.00	4.00E+00	2.00	-4
	1	-2.00	-1.00E+00	2.00	0

	x^3	x^2	x	Constant	y
Coef	1	-2.00	-1.00E+00	2.00	0.00
3rd Root	-1				

Synthetic Division 3

Coef	1	-2.00	-1.00E+00	2.00
-1.00		-1.00	3.00E+00	-2.00
	1	-3.00	2.00E+00	0.00

Quadratic Formula

a	b	c
1	-3.00	2.00

4th Root	2.00
5th Root	1.00

In summary, the roots are -3, -2, -1, 1, and 2.

Ex. 4: Quintic with 3 real roots $p(x) = 2x^5 - x^4 - 2x + 1$

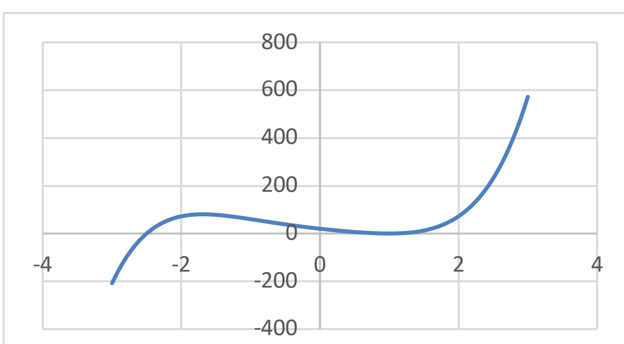

Figure 4. $p(x) = 2x^5 + x^4 + 9x^2 - 32x + 20$

	x^5	x^4	x^3	x^2	x	Constant	y
Coef	2	1	0	9	-32	20	0
1st Root	-2.5						
			Synthetic Division 1				
Coef	2	1	0	9	-32	20	
-2.500		-5.00	1.00E+01	-2.50E+01	40	-20	
	2	-4.00	1.00E+01	-1.600E+01	8	0	
	x^4	x^3	x^2	x	Constant		y
Coef	2	-4.0	1.00E+01	-16	8		0.00E+00
2nd Root	1						
			Synthetic Division 2				
Coef	2	-4.0	1.00E+01	-1.600E+01	8		
1.00		2.00	-2.00	8.00E+00	-8		
	2	-2.0	8.00E+00	-8.00E+00	0		
	x^3	x^2	x	Constant			y
Coef	2	-2.0	8.00E+00	-8.00E+00			0.00E+00
3rd Root	1						
			Synthetic Division 3				
Coef	2	-2.0	8.00E+00	-8.00E+00			
1.00		2.00	0.00E+00	8.00E+00			
	2	0.00	8.00E+00	0.00E+00			
			Quadratic Formula				
		a	**b**	**c**			
		2	0.00	8.00E+00			
4th Root	2i						
5th Root	-2i						

In summary, the roots are -2.5, 1, 1, and $\pm 2i$.

Ex. 5: Quintic with 1 real root $p(x) = 2x^5 + x^4 - 5x^3 - 5x^2 + 8x + 4$

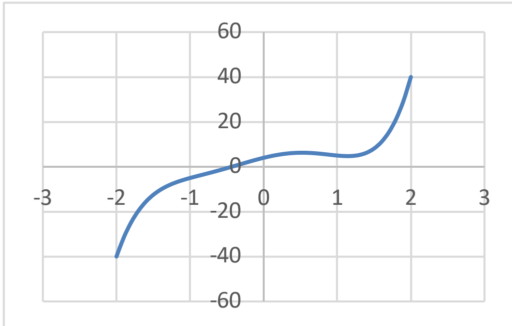

Figure 5. $p(x) = 2x^5 + x^4 - 5x^3 - 5x^2 + 8x + 4$

	x^5	x^4	x^3	x^2	x	Con s	y
Coef	2	1	-5	-5		4	- 1.65E -07
1st Root	-0.4340						
Coef	2	1	-5	-5		4	
- 0.4339 6		-0.87	- 0.057 3	2.19E+00		-4	
	2	0.13	-5.06	-2.81E+00		0	
	x^4	x^3	x^2	x	Cons		
Coef	2	0.13	-5.06	-2.81E+00			
	p	-2.53					
	q	-1.32					
	r	4.63					
		z^3	z^2	z	Cons		
		1	2.53	-18.5			
	Root	4.33					
	z - p	6.86					
	q	1.32					
	1/4z^2-r	0.0634					

	a	1.00
	b	-2.62
	c	1.91
2nd Root	1.31+0.89i	
3rd Root	1.31-0.89i	
	a	1
	b	2.62
	c	2.42
4th Root	-1.31+1.68i	
5th Root	-1.31-1.68i	

In summary, the roots are -0.4340, $1.31 \pm 0.89i$ and $-1.31 \pm 1.68i$.

Although we have written the roots up to two decimal places, of course, *Excel* stores the answers in many more decimal places.

Conclusions:

- For thousands of years the preoccupation of mathematics has been the solution of polynomial equations. Polynomials furnished us with many examples for they are among the simplest functions to work with.
- There are formulas for solving polynomial equations up to degree 4. We have the very useful quadratic formula. There are exact formulas for solving the cubic and quartic polynomial equations; they are just less familiar.
- There are no solutions to the quintic equation by radicals. We will need special elliptic modular functions to solve the quintic in closed form.
- The method we present here of solving polynomial equations is very straightforward and easy to apply. It is accomplished in two steps, both within *Excel.* First, we graph the function; we can use the plotting routine in *Excel.* Next, we call *Solver,* a built-in function in *Excel.* The real advantage of this method is the ready availability of the utility *Excel* in any personal computer.
- It will become evident to anyone using this method that it could be applied to other well-defined equations, not just polynomials. This then extends the domain of applicability of the method to a large degree.

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