THE EARTH AS A SPINNING TOP

Jay Villanueva

Miami
Dade College

11011 SW 104 St.
Miami, FL 33176
jvillanu@mdc.edu

(*Abstract*.) The study of Earth's motions of rotation once a day, revolution once a year, and precession once in 26,000 years, is modeled by taking the Earth as an elastic symmetric sphere acted upon solely by gravitation. Using Lagrange's equations for the Euler angles of the Earth in a moving coordinate system, we find that the Earth also executes a nutation (a bobbing up and down of the symmetry axis) superimposed on the precession.

The major motions of the Earth are:

1. Rotation about its axis once a day
2. Revolution about the Sun once a yearn
3. Precession of its axis once in 26,000 year. n

We wish to study these motions from basic principles by considering the Earth as a symmetric ellipsoid and see if there are other motions we can derive. As a starting point, we would like to see by studying the motion of a symmetric top if we can apply the same analysis to the motions of the Earth.

We have evidence of the Earth's rotation by the changing view of the night sky every 24 hours (Figure 1). The constellations visible to the observer change with the hours.

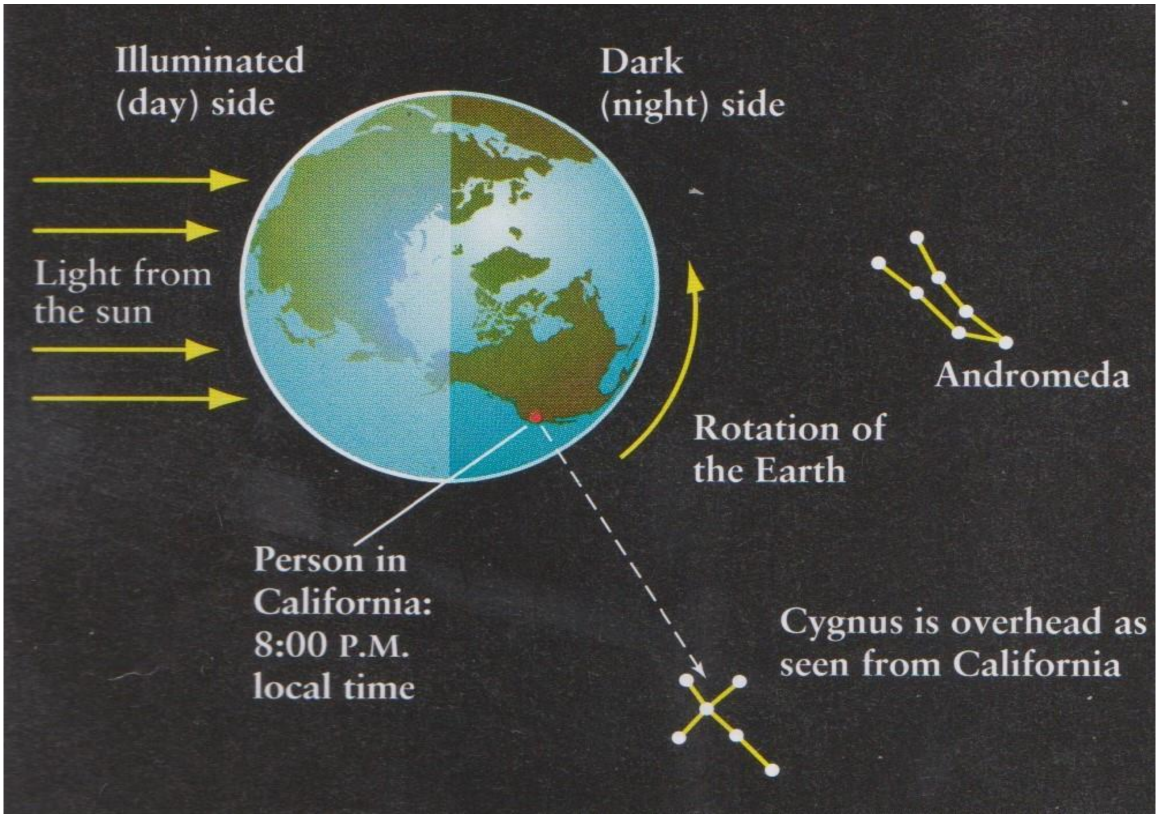

Figure 1. The view of the night sky changes due to rotation of the Earth.

The rotation of the Earth was first ably demonstrated by Leon Foucault in 1851 with his pendulum: a 62 lb bob, hanging in a 220-ft string at the Pantheon in Paris. At that latitude (48.9^0), the pendulum had a precession rate of $T = \frac{24}{\sin\phi} = 31.8^0$, or equivalently, the pendulum appeared to swing its plane of oscillation by $11.3^0/hr$ (Figures 2 and 3).

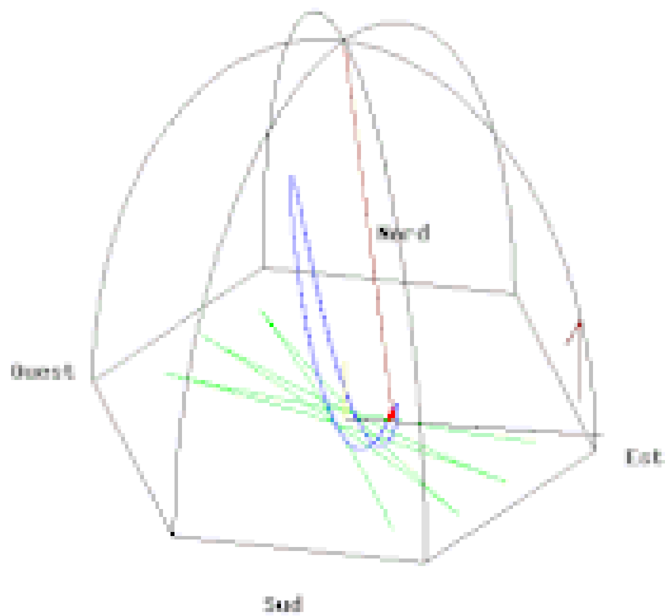

Figure 2. Le pendule Foucault

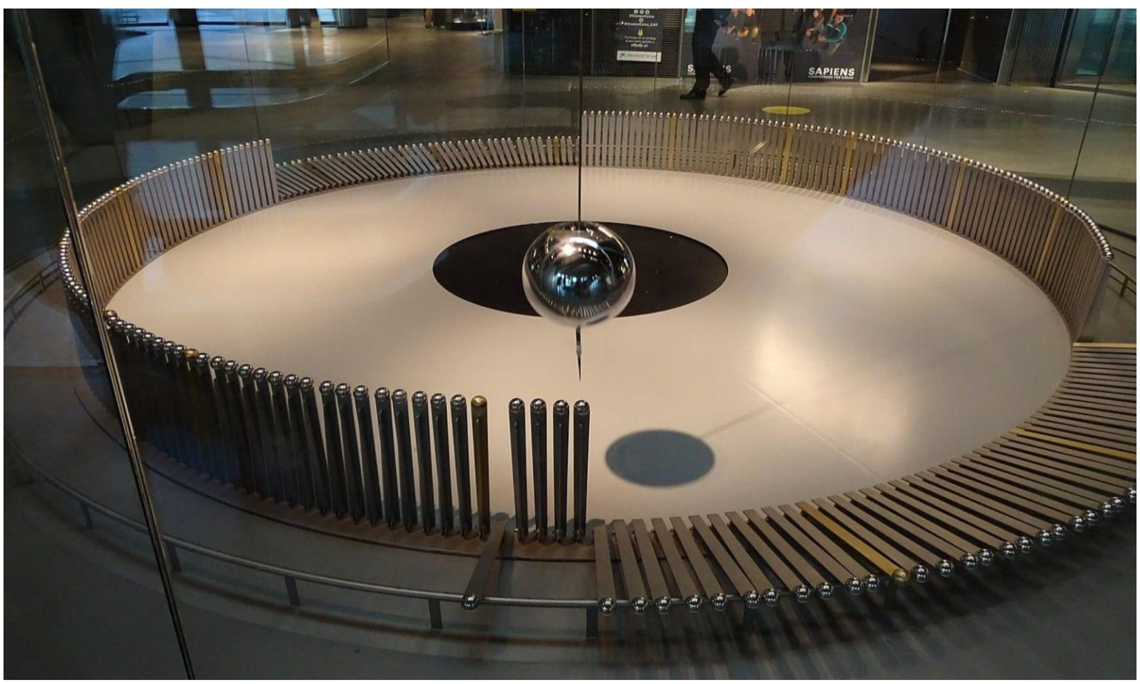

Figure 3. A modern-day Foucault pendulum.

The tilt of the Earth's axis, together with its revolution around the Sun, give us the changing seasons (Figure 4).

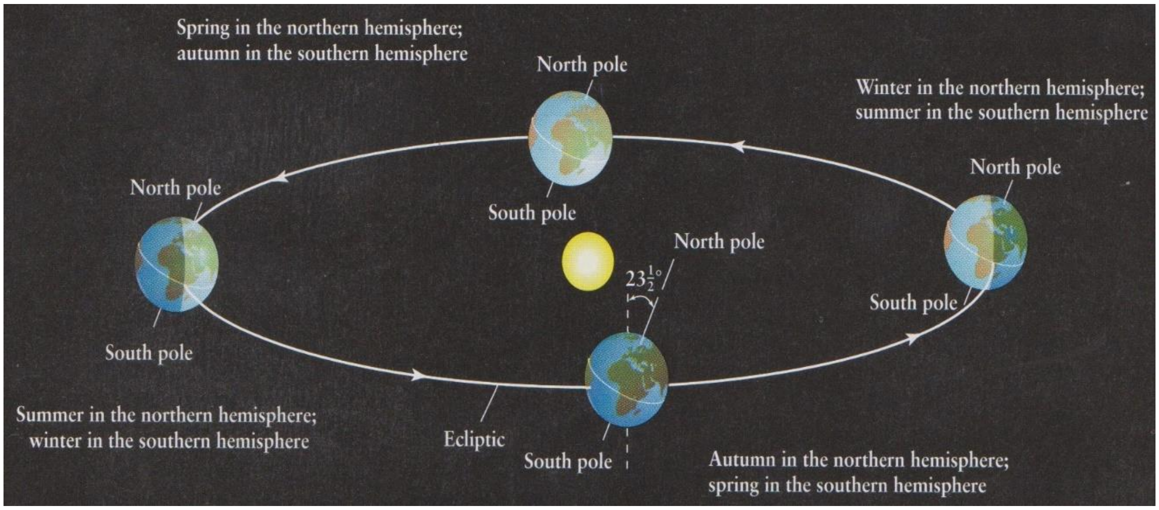

Figure 4. Revolution around the Sun

The North-South axis of the Earth is not fixed, rather, it precesses once in 26,000 years, like a wobbling top near the end of its spin (Figure 5).

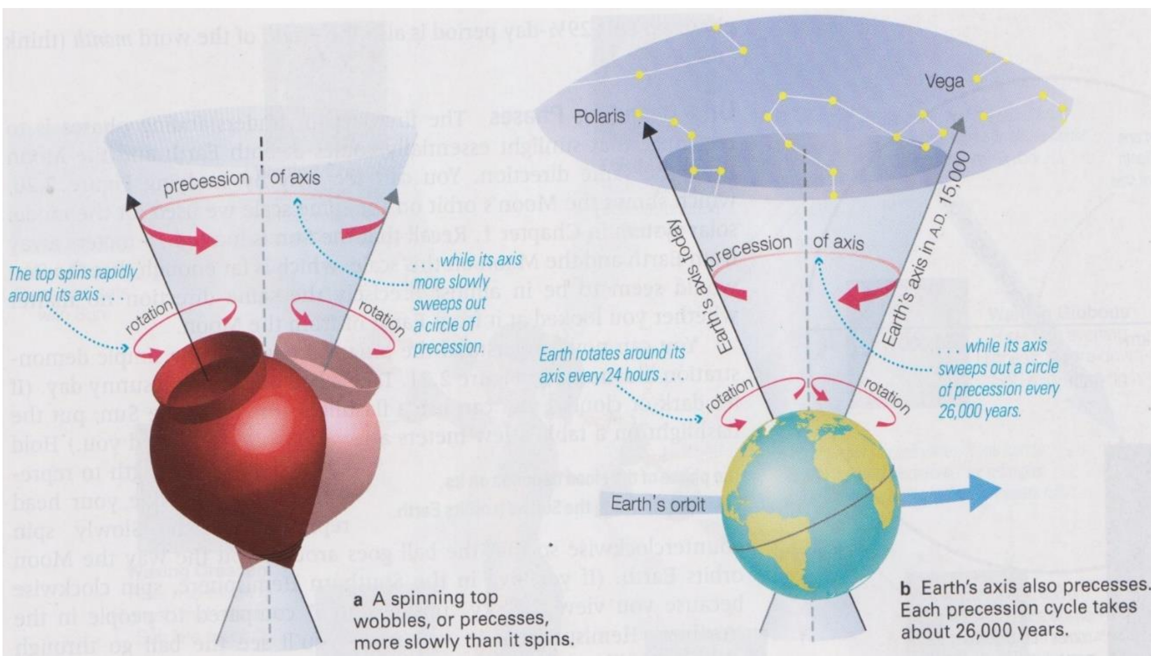

Figure 5. The Earth's axis precesses like a wobbling top at the end of its spin.

Because of this, the North star is not always Polaris. 4000 years ago, it was Thuban in Draco, in 15,000 years, it will be Vega in Lyra (Figure 6).

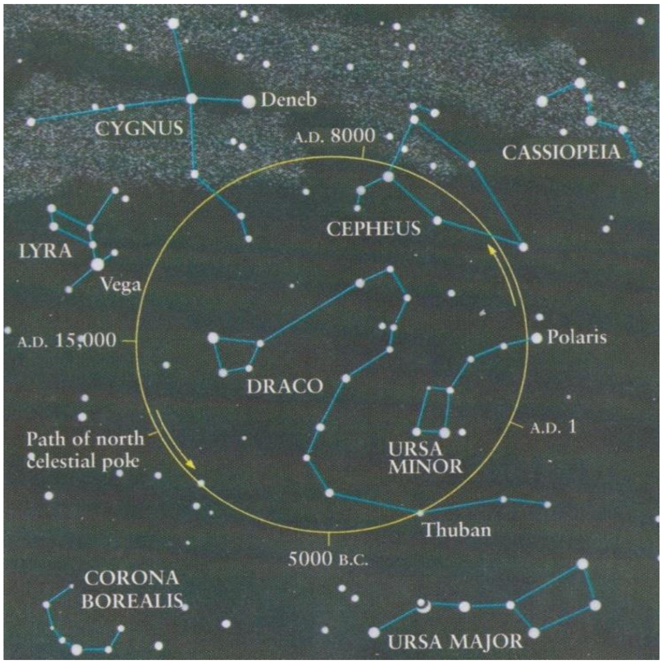

Figure 6. Precession of the axis causes the North star to change with the years.

We can understand the precession of a top's axis from basic physics. The torque acting on the top is (Figure 6):

$$\vec{\tau} = \vec{r} \times \vec{F}, \quad \text{perpendicular to both } \vec{r} \text{ and } \vec{F}.$$

Its magnitude is $\tau = Mgr \sin\theta$. In time *dt,* the change in angular momentum is

$$dL = \tau dt.$$

The rate of precession ω_P is given by

$$\omega_P = \frac{d\phi}{dt} = \frac{dL}{L\sin\theta} = \frac{\tau dt}{L\sin\theta} = \frac{Mgr\sin\theta}{L\sin\theta} = \frac{Mgr}{L} = \frac{Mgr}{I\omega}.$$

Astronomers tell us that for the spinning Earth, the gravitational pull of the Sun and Moon on the equatorial bulges causes the precession, with a half-angle of 23.5^0 in 26,000 years.

The equations of motion of a rigid body in space are given by:

$$\frac{d\vec{P}}{dt} = \vec{F}, \quad \frac{d\vec{L}}{dt} = \vec{N}, \quad \vec{P} = M\vec{V}, \quad \vec{L} = \tilde{I} \cdot \vec{\omega}$$

$\vec{P}, \vec{L} =$ the linear & angular momenta; $\vec{F}, \vec{N} =$ the total force & the total torque about O; $\vec{V}$ = velocity of the c.m.; $\tilde{I}, \vec{\omega} =$ the inertia tensor & the angular velocity about O.

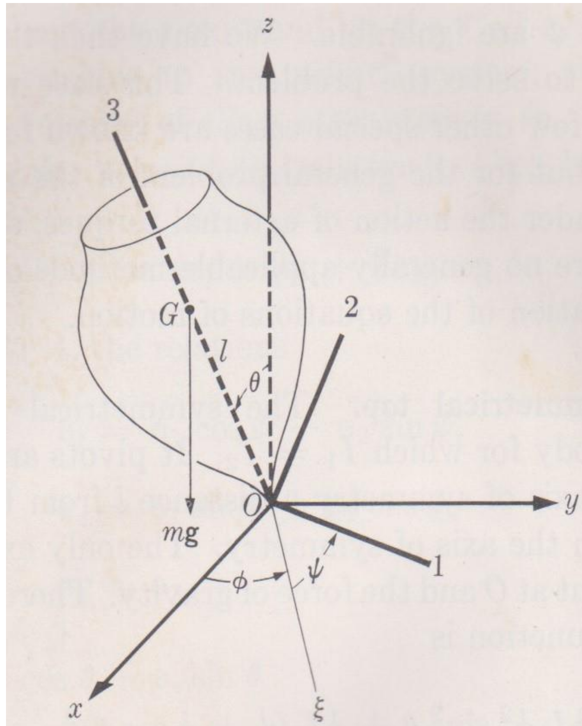

Figure 7. The body axes relative to the space axes.

$\tilde{I}$ changes as the body rotates; this may be avoided by referring the motion to axes fixed in the body (Figure 7):

$$\frac{d'\vec{L}}{dt} = \vec{\omega} \times \vec{L} = \vec{N} = \tilde{I} \cdot \frac{d\vec{\omega}}{dt} + \vec{\omega} \times \left(\tilde{I} \cdot \vec{\omega}\right).$$

Choose as body axes the principal axes of the body: $\hat{e}_1, \hat{e}_2, \hat{e}_3$ (Euler's equations of motion)

$$I_1\dot{\omega}_1 + (I_3 - I_2)\omega_3\omega_2 = N_1$$ (The fixed point O is taken as the origin

$$I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1 = N_2$$ for the body axes.)

$$I_3\dot{\omega}_3 + (I_2 - I_1)\omega_2\omega_1 = N_3.$$

Consider a freely rotating symmetrical body, with no applied. The last three equations become:

$$I_3\dot{\omega}_3 = 0, \qquad \dot{\omega}_1 + \beta\omega_3\omega_2 = 0, \qquad \dot{\omega}_2 - \beta\omega_3\omega_1 = 0, \qquad \beta = \frac{I_3 - I_1}{I_1}.$$

Eq. (i) → $\omega_3 = const.$; Eqs. (ii) & (iii) are a couple of 1st-order equations with solutions

$$\omega_1 = A_1 e^{pt}, \quad \omega_2 = A_2 e^{pt}, \quad \text{with} \quad p = \pm i\beta\omega_3, \qquad A_2 = \mp i A_1.$$

The complex conjugate pair of solutions may be superimposed to form real solutions

$$\omega_1 = A\cos(\beta\omega_3 t + \theta), \qquad \omega_2 = A\sin(\beta\omega_3 t + \theta).$$

The angular velocity $\vec{\omega}$ therefore precesses in a circle of radius A about $\hat{e}_3$, with angular velocity $\beta\omega_3$, in the same sense as ω_3 if $I_3 > I_1$, and opposite if $I_3 < I_1$.

The magnitude of $\vec{\omega}$ is

$$\omega = [{\omega_3}^2 + A^2]^{1/2} = \; const.$$

The instantaneous axis of rotation, determined by $\vec{\omega}$, traces out a cone in the body (body cone) as it precesses around the axis of symmetry, with half-angle α_b (Figure 8)

$$\tan\alpha_b = \frac{A}{\omega_3}.$$

The angular momentum, $\vec{L} = const.$, since $\vec{N} = 0$. The angle α_s between $\vec{\omega}$ and $\vec{L}$ is given by

$$\cos\alpha_s = \frac{\vec{\omega}\cdot\vec{L}}{\omega L} = \frac{\vec{\omega}\cdot\tilde{I}\cdot\vec{\omega}}{\omega L} = \frac{2T}{\omega L} = const.$$

since T = *const.* The axis of rotation traces out a cone in space (the space cone), with half-angle α_s. Since

$$I = (\hat{e}_1\hat{e}_1 + \hat{e}_2\hat{e}_2)I_1 + \hat{e}_3\hat{e}_3 I_3 = I_1\tilde{1} + \hat{e}_3\hat{e}_3(I_3 - I_1).$$

$$2T = \omega^2 I_1[1 + \beta cos^2\alpha_b]$$

$$\vec{L} = \omega I_1[\hat{n} + \beta cos\alpha_b\hat{e}_3]\,,$$

$$\cos\alpha_s = \frac{1+\beta cos^2\alpha_b}{[1+(2\beta+\beta^2)cos^2\alpha_b]^{1/2}}\,.$$

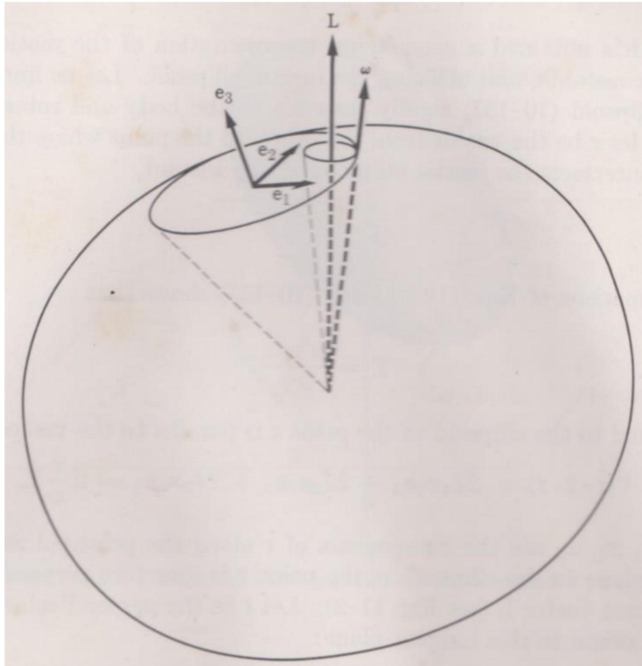

Figure 8. The body cone and space cone.

Now, we need to introduce a coordinate system carried by the body, where (x, y, z) are the space axes. Among the most useful are the Euler angles (θ, ϕ, ψ), akin but not identical to the pitch, roll, and yaw angles familiar to pilots of ships, planes, and transporters of objects with large vertical components (Figure 9).

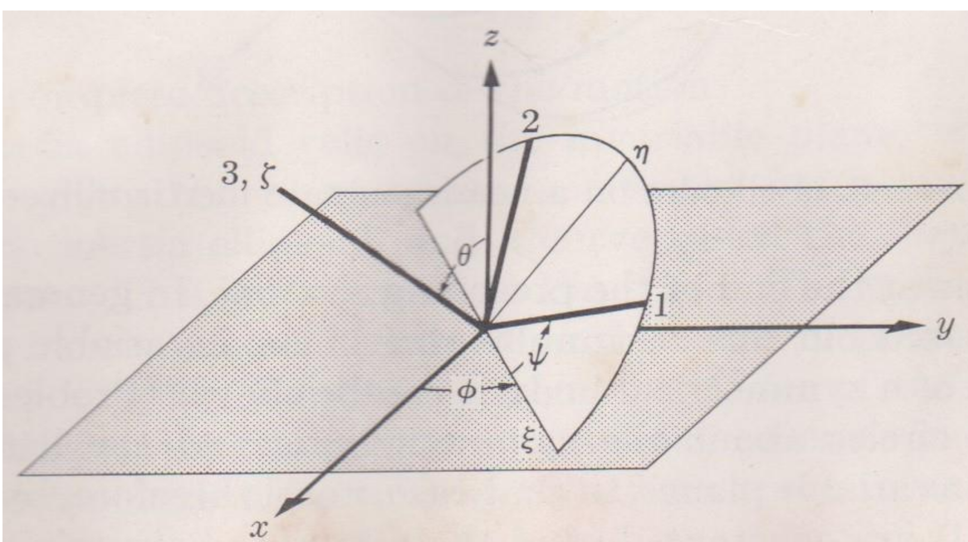

Figure 9. The Euler angles θ, ϕ, ψ.

$\theta =$ angle between the *z*- and the 3-axes;

the 1,2-plane intersects the *xy*-plane in the line of nodes, ξ

$\phi =$ measured in the *xy*-plane from the *x*-axis to ξ;

$\psi =$ measured in the 1,2-plane from ξ to the 1-axis.

As the body moves, θ, ϕ, ψ change with time. If θ alone changes, while ϕ, ψ are fixed, the body rotates around ξ with angular velocity $\dot{\theta}\hat{e}_\xi$. If ϕ alone changes, the body rotates around the z-axis with angular velocity $\dot{\phi}\hat{k}$. If ψ alone changes, the body rotates around the 3-axis with angular velocity $\dot{\psi}\hat{e}_3$. Thus,

$$\vec{\omega} = \dot{\theta}\hat{e}_\xi + \dot{\phi}\hat{k} + \dot{\psi}\hat{e}_3.$$

From the figure, we have the relations:

$$\begin{aligned}
\widehat{e_\xi} &= \hat{e}_1 cos\psi - \hat{e}_2 sin\psi, \\
\widehat{e_\eta} &= \hat{e}_1 sin\psi + \hat{e}_2 cos\psi, \\
\widehat{e_\zeta} &= \hat{e}_3, \\
\hat{k} &= \hat{e}_\zeta cos\theta + \hat{e}_\eta sin\theta \\
&= \hat{e}_1 sin\theta sin\psi + \hat{e}_2 sin\theta cos\psi + \hat{e}_3 cos\theta.
\end{aligned}$$

Therefore, $\vec{\omega}$ may be expressed along the principal axes:

$$\begin{aligned}
\omega_1 &= \dot{\theta} cos\psi + \dot{\phi} sin\theta sin\psi, \\
\omega_2 &= -\dot{\theta} sin\psi + \dot{\phi} sin\theta cos\psi \\
\omega_3 &= \dot{\psi} + \dot{\phi} cos\theta.
\end{aligned}$$

The kinetic energy:

$$T = \frac{1}{2} I_1 {\omega_1}^2 + \frac{1}{2} I_2 {\omega_2}^2 + \frac{1}{2} I_3 {\omega_3}^2$$

is a complicated expression in $\dot{\theta}, \dot{\phi}, \dot{\psi}, \theta, \psi$. The Euler angles are not orthogonal, cross terms appear in $\dot{\theta}\dot{\phi}$ and $\dot{\phi}\dot{\psi}$. For a symmetrical body ($I_1 = I_2$), T simplifies to:

$$T = \frac{1}{2} I_1 \dot{\theta}^2 + \frac{1}{2} I_2 \dot{\phi}^2 sin^2\theta + \frac{1}{2} I_3 \left(\dot{\psi} + \dot{\phi} cos\theta\right)^2.$$

The generalized forces Q_θ, Q_ϕ, Q_ψ are the 'torques' about the $\xi -, z -$,3 −a xes. Note that the variables ϕ and ψ are ignorable. We have three constants of the motion. The Lagrangian

$$L = \frac{1}{2} I_1 \dot{\theta}^2 + \frac{1}{2} I_2 \dot{\phi}^2 sin^2\theta + \frac{1}{2} I_3 \left(\dot{\psi} + \dot{\phi} cos\theta\right)^2 - mglcos\theta$$

Lagrange's equations give us three integrals of the motions:

$$\frac{dp_\psi}{dt} = \frac{\partial L}{\partial \psi} = 0, \quad \rightarrow \quad p_\psi = I_3(\dot{\psi} + \dot{\phi} cos\theta),$$

$$\frac{dp_\phi}{dt} = \frac{\partial L}{\partial \phi} = 0, \quad \rightarrow \quad p_\phi = I_1\dot{\phi} sin^2\theta + I_3 cos\theta(\dot{\psi} + \dot{\phi} cos\theta), \qquad (*)$$

$$\frac{dE}{dt} = -\frac{\partial L}{\partial t} = 0, \quad \rightarrow \quad E = \frac{1}{2}I_1\dot{\theta}^2 + \frac{1}{2}I_2\dot{\phi}^2 sin^2\theta + \frac{1}{2}I_3(\dot{\psi} + \dot{\phi} cos\theta)^2 + mglcos\theta$$

or,
$$E = \frac{1}{2}I_1\dot{\theta}^2 + \frac{(p_\phi - p_\psi cos\theta)^2}{2I_1 sin^2\theta} + \frac{p_\psi{}^2}{2I_3} + mglcos\theta.$$

Set $E' = E - \frac{p_\psi{}^2}{2I_3}$, $\qquad V' = \frac{(p_\phi - p_\psi cos\theta)^2}{2I_1 sin^2\theta} + mglcos\theta.$

Then

$\dot{\theta} = \left\{\frac{2}{I_1}[E - V']\right\}^{1/2}$ and θ is given by:

$$\int_{\theta_0}^{\theta} \frac{d\theta}{[E - V']^{1/2}} = \left(\frac{I_1}{2}\right)^{1/2} t \quad \rightarrow \quad \theta(t) = \theta(\phi, \psi, t).$$

The effective potential energy, V', is plotted as a function of θ, for $\omega_3 \neq 0$ (Figure 10). The torque associated with V' is

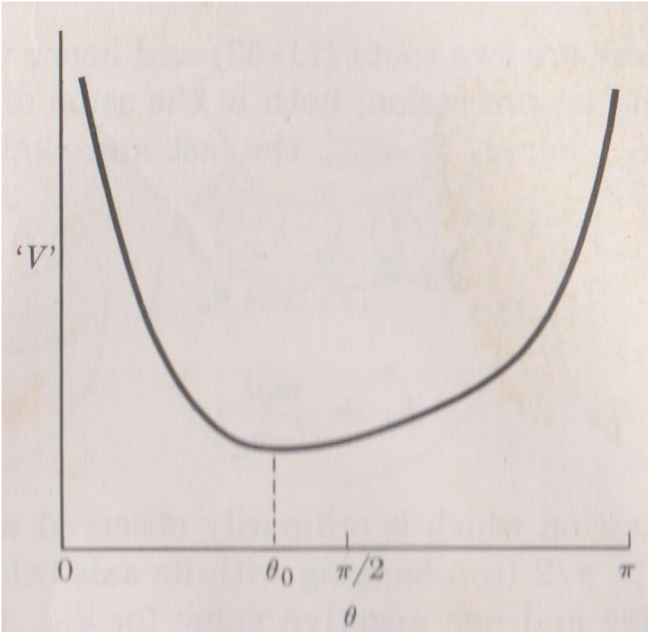

Figure 10. The effective potential energy, V'.

$$N' = -\frac{\partial V'}{\partial \theta} = mglsin\theta - \frac{(p_\phi - p_\psi cos\theta)(p_\psi - p_\phi cos\theta)}{I_1 sin^3\theta}\ .$$

We find that $N' > 0$ for $\theta \simeq 0$ and $N' < 0$ for $\theta \simeq \pi$. Hence, V' has a minimum at θ_0:

$$mglI_1 sin^4\theta_0 - \left(p_\phi - p_\psi cos\theta_0\right)\left(p_\psi - p_\phi cos\theta_0\right) = 0.$$

Solving for $\left(p_\phi - p_\psi cos\theta_0\right)$, with $p_\psi = I_3\omega_3$:

$$\left(p_\phi - p_\psi cos\theta_0\right) = \frac{1}{2} I_3\omega_3 \frac{sin^2\theta_0}{cos\theta_0}\left[1 \pm \left(1 - \frac{4mglI_1}{{I_3}^2{\omega_3}^2}\right)^{1/2}\right].$$

For $\theta_0 < \pi/2$, there is a minimum spin angular velocity below which the top cannot precess uniformly at an angle θ_0:

$$\omega_{min} = \left(\frac{4mglI_1}{{I_3}^2} cos\theta_0\right)^{1/2}.$$

For $\omega_3 > \omega_{min}$, there are two values of θ_0, both in the same direction as ω_3:

$$\dot{\phi}_0\ (fast) = \frac{I_3}{I_1}\frac{\omega_3}{cos\theta_0}\ , \qquad \dot{\phi}_0\ (slow) = \frac{mgl}{I_3\omega_3}\ .$$

It is the slow precession that is usually observed with a rapidly spinning top.

The more general motion ($p_\phi \neq p_\psi$) involves a nutation or oscillation of the 3-axis in the $\theta -$ direction as it precesses. The axis oscillates between angles $\theta_1$and θ_2 from:

$$E' = +\frac{\left(p_\phi - p_\psi cos\theta\right)^2}{2I_1 sin^2\theta} + \frac{{p_\psi}^2}{2I_3} + mglcos\theta,$$

where p_ϕ, p_ψ, E' are determined from the initial conditions. The cubic equation has two real roots; the third one is unphysical (cos$\theta > 1$). During nutation, the precession velocity varies according to Eq. (*):

$$\dot{\phi} = \frac{p_\phi - p_\psi cos\theta}{I_1 sin^2\theta}\ .$$

The motion is as shown in the figure (Figure 11), depending on the direction of the initial velocity imparted to the 3-axis. A third case arises when the top, spinning about its axis with velocity ω_3, is held with its axis initially at rest at an angle θ_1 and then released (shown as case *c*).

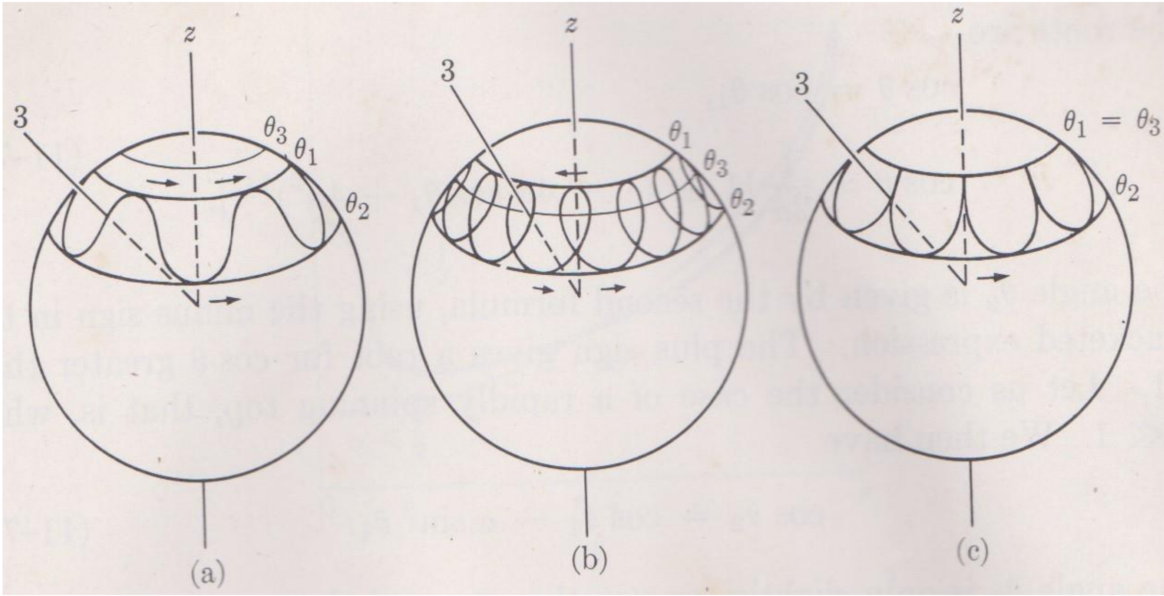

Figure 11. Nutations of the symmetry axis.

Conclusions:

- The major motions of the Earth are: rotation about its axis once a day, revolution about the Sun once a year, and precession of its axis about a celestial point o nce in 26,000 years;
- The Earth motions are often referred to those of a spinning top;
- Looking more closely at a symmetric spinning top, we find that the precessing motion falls out naturally. There is also nutation or a small bobbing up and down of the body's symmetry axis, superposed on the precession, of the order of 9" of arc with a period of 18.6 years;
- We found these results by solving Lagrange's equations for the Euler angles of a symmetric top, with a fixed point and acted only by gravity;
- We would like to model the Earth as a symmetrpolar radius ical spinning top. We anticipate that the analog of the top's weight could be the asphericity of the oblate spheroid, with equatorial radius 6.3784 $\times 10^8 cm$ and polar radius 2.15 $\times 10^6 cm$ shorter. Among other things, we wish to find out which of the three cases of nutation is actually being executed by the Earth;
- All of these motions can be developed physically from a playing top or gyroscope, as have been ably demonstrated on YouTube.

References

1. EU Condon & H Odishaw, 1967. *Handbook of Physics.* NY: McGraw-Hill Book Co.
2. R Hide & JO Dickey, 1991. “The variable rotation of the Earth,” in *Science* 253 (5020), p29-37.
3. DR Inglis, 1957. *Reviews of Modern Physics* p9-20.
4. K Lambeck, 1980. *The Earth’s Variable Rotation.* Cambridge University Press.
5. JB Marion & WF Hornyak, 1982. *Physics.* NY: Saunders College Publishing.
6. R Resnick, D Halliday, & KS Krane, 2002. *Physics.* NY: Wiley.
7. EJ Routh, 1955. *The Dynamics of Rigid Bodies.* NY: Dover.
8. FD Stacey, 1977. *Physics of the Earth.* NY: Wiley.
9. KR Symon, 1960. *Mechanics.* MA: Addison-Wesley, Inc.
10. PA Tipler, 1999. *Physics for Scientists & Engineers.* NY: Freeman & Co.