# THE VARIABLE ROTATION OF THE EARTH

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(*Abstract*.) The variability of the Earth's rotation is studied as to its causes and consequences. We show a simple derivation of the tidal force due to the Moon and also the Roche limit of a planet. We look at the Earth-Moon system and consider the effect of a lengthening of the day on the separation between the Earth and Moon. We consider the multiple moons hypothesis for the origin of the Moon. We quote some results from astronomical observations. [Paper presented at the *International Conference on Technology in Collegiate Mathematics,* held in San Diego, CA, on 03/07/25.]



In recent decades there have been a few developments in specialized instrumentation in atomic clocks, laser-ranging, and satellite gadgetry that led to an upsurge in publications in geophysics. Thus, the renewed interest in Earth science. In this communication, we wish to study the Earth's rotation. Its variability is due to several causes, foremost among them is that due to the tides. Tidal dissipation leads directly to a slowing down of the rotation, thus, to an increase in the length-of-day. By conservation of angular momentum this leads to a retreat of the Moon's orbit from the Earth. Tidal dissipation also may be applied in explaining why Mercury and Venus do not have moons. The same principle may be used in an alternative theory for the origin of the Moon – the multiple moons hypothesis.

We have evidence of the Earth's rotation by the changing view of the night sky every 24 hours (Figure 1). The constellations visible to the observer change with the hours.

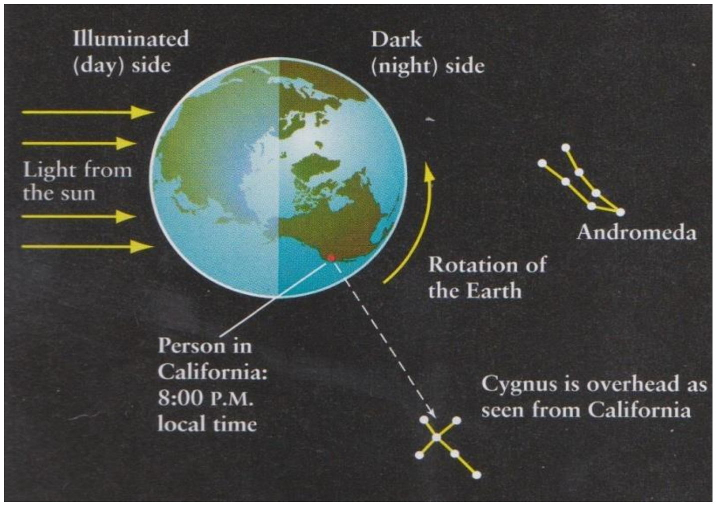


Figure 1. The view of the night sky changes due to rotation of the Earth.

The rotation of the Earth was first ably demonstrated by Leon Foucault in 1851 with his pendulum: a 62 lb bob, hanging in a 220-ft string at the Pantheon in Paris. At that latitude ($48.9^0$), the pendulum had a precession rate of $T = \frac{24}{\sin\phi} = 31.8\ hr,$ or equivalently, the pendulum appeared to swing its plane of oscillation by $11.3^0/hr$ (Figures 2 and 3).

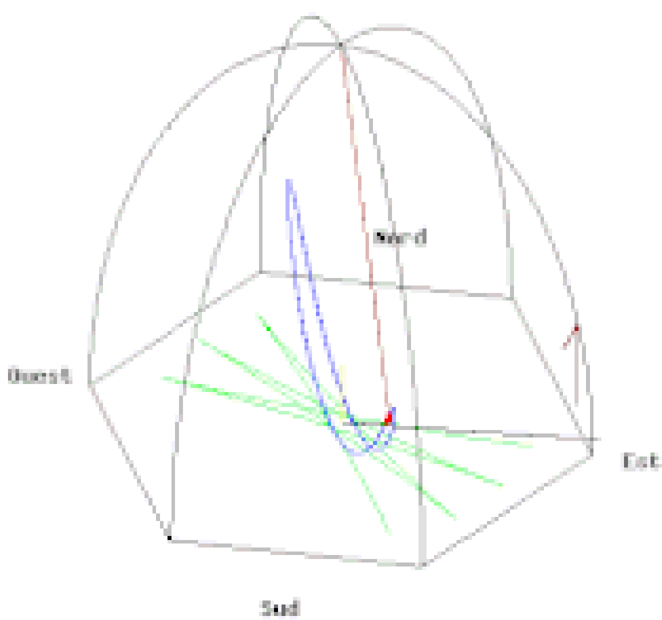


Figure 2. Le pendule Foucault

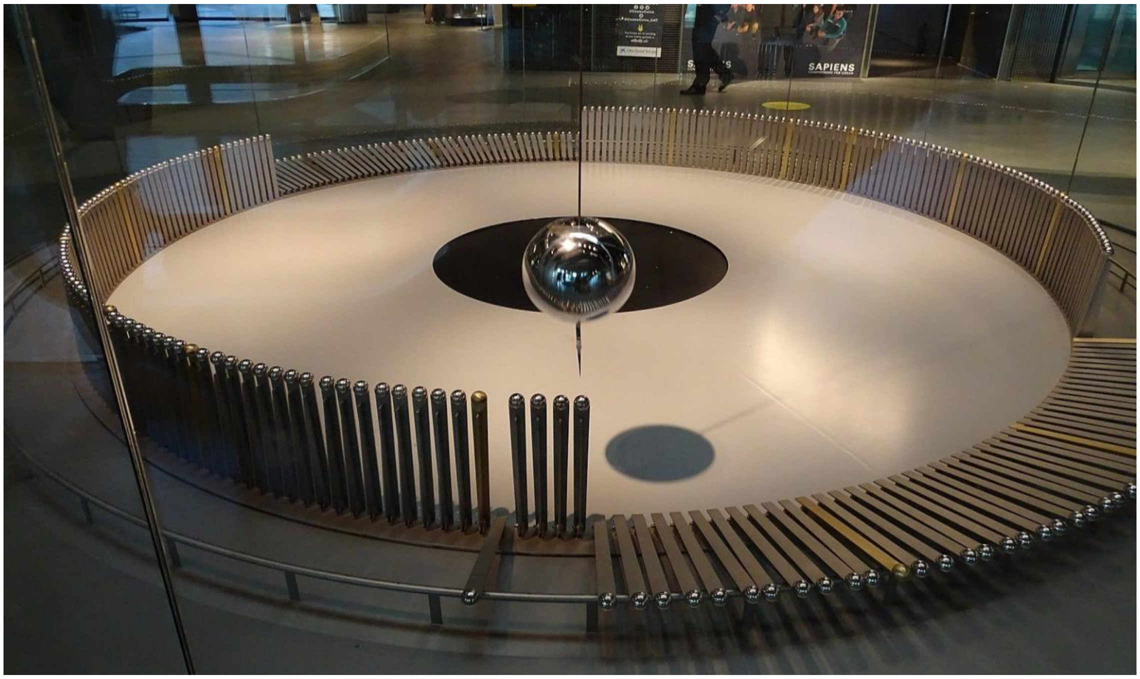


Figure 3. A modern-day Foucault pendulum.

The tilt of the Earth's axis, together with its revolution around the Sun, gives us the changing seasons (Figure 4).

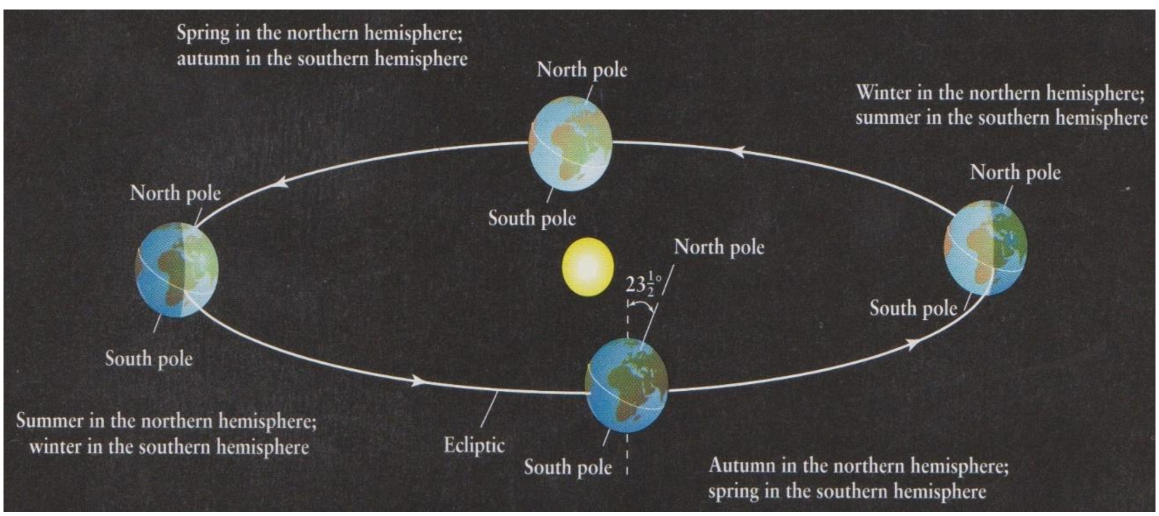


Figure 4. Revolution around the Sun

The North-South axis of the Earth is not fixed, rather, it precesses once in 25,800 years, like a wobbling top near the end of its spin (Figure 5).

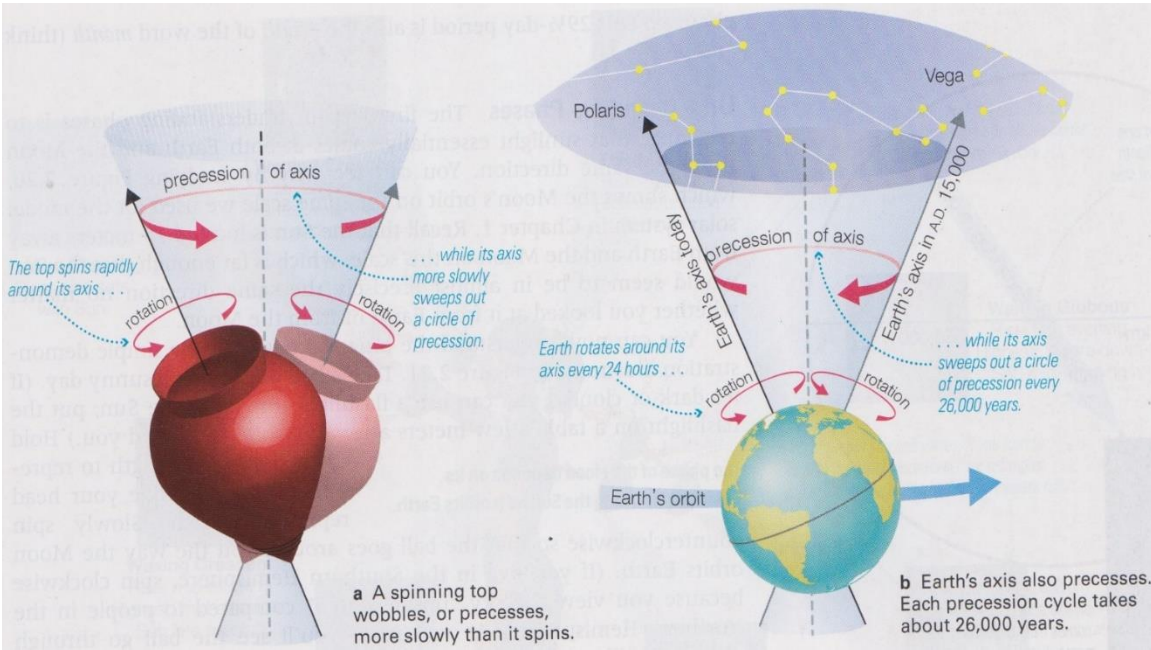


Figure 5. The Earth's axis precesses like a wobbling top at the end of its spin.

Because of this, the North star is not always Polaris. 4000 years ago, it was Thuban in Draco; in 15,000 years, it will be Vega in Lyra (Figure 6).

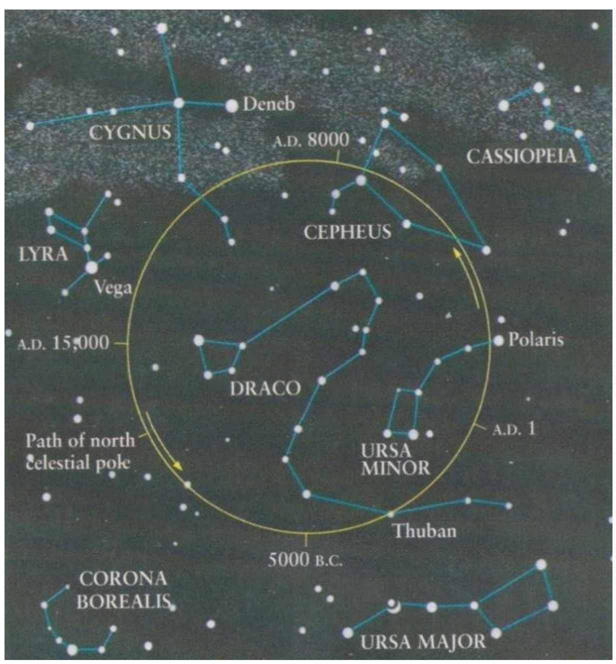


Figure 6. Precession of the axis causes the North star to change with the years.

We can understand the precession of a top's axis from basic physics. The torque acting on the top is (Figure 7):

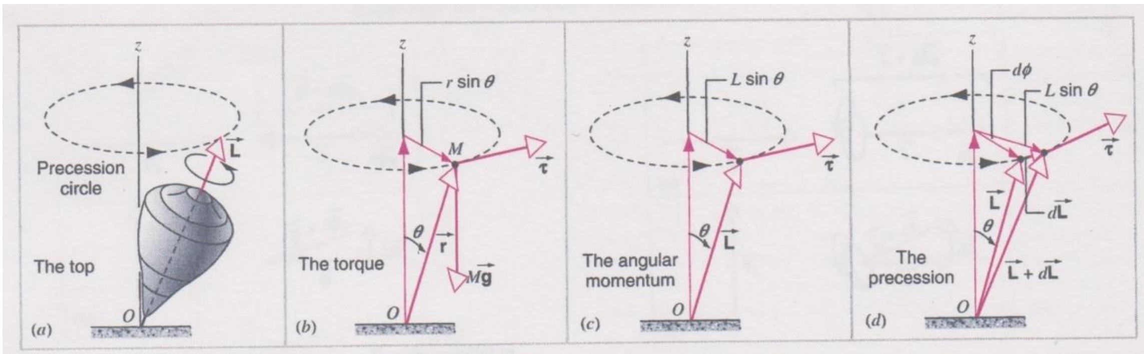


Figure 7. Torque on a spinning top.

$\vec{\tau} = \vec{r} \times \vec{F}$, perpendicular to both $\vec{r}$and$\vec{F}$. Its magnitude is $\tau = Mgr\sin\theta$. In time *dt,* the change in angular momentum is $dL = \tau dt = Mgrsin\theta\ dt$. The rate of precession $\omega_P$ is given by

$$\omega_P = \frac{d\phi}{dt} = \frac{dL/Lsin\theta}{dt} = \frac{\tau dt}{L\sin\theta \cdot dt} = \frac{Mgr\sin\theta}{L\sin\theta} = \frac{Mgr}{L}.$$

Astronomers tell us that for the spinning Earth, the gravitational pull of the Sun and Moon on the equatorial bulges causes the precession, with a half-angle of $23.5^0$ in 25,800 years. This is shown as follows.

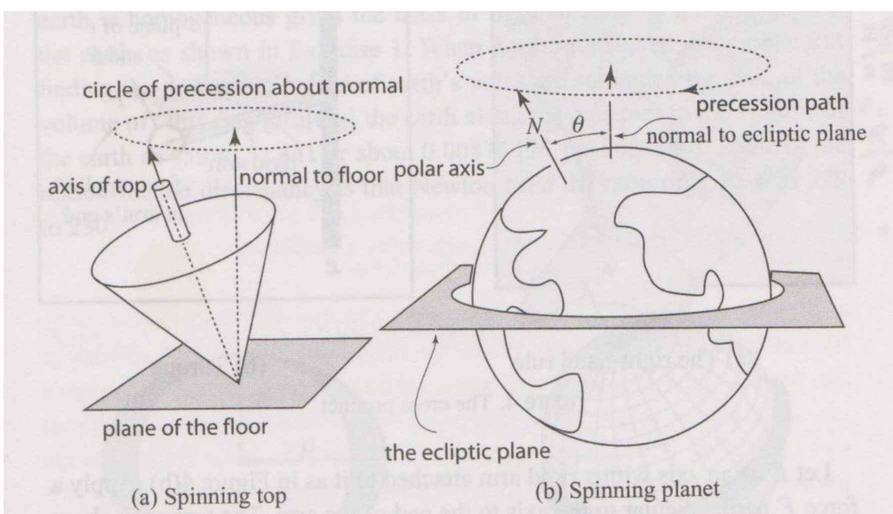


Figure 8. The Earth likened to a spinning top

Like a spinning top, the Earth's NS-axis precesses in a circle parallel to the ecliptic plane, as shown in Figure 8. For our first model (Simoson, 2010), we consider a uniform belt of mass around the equator. The polar radius of the Earth is $R = 6356.8\ km$; we represent the differential radius of $\Delta R = \rho - R = 21.4\ km$ as a homogeneous belt of mass around the equator, Figure 9. Applying the cross product, $\vec{r} =$ the radius vector from $O$ to the point mass $P$ against $\vec{F} =$ the gravitational force due to the Sun $S$, we find a torque tending to rotate the ecliptic plane towards the equatorial plane, as shown in Figure 10.

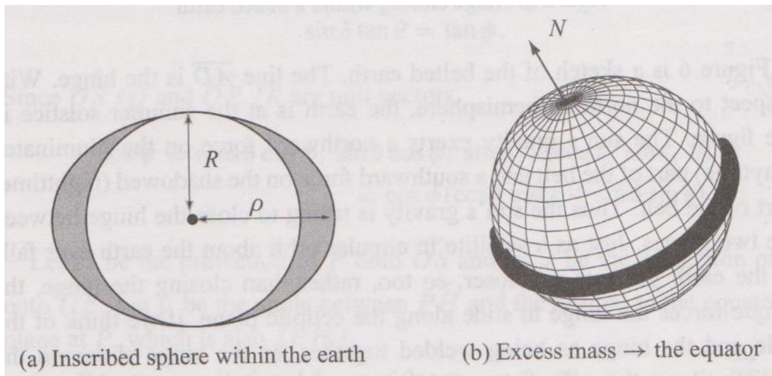


Figure 9. The equatorial bulge shown as a homogeneous belt around a spherical Earth

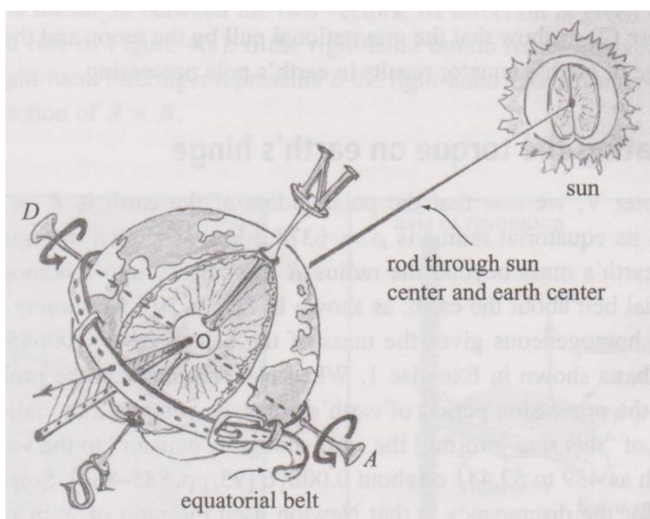


Figure 10. Gravitational torque on the equatorial belt of mass

Take the equatorial plane as the $xy$-plane, with origin $O$ at the Earth. The belt of mass is a circle with center $O$ passing through $A(R, 0, 0)$ and $B(0, R, 0)$. Let $\phi =$ the angle between

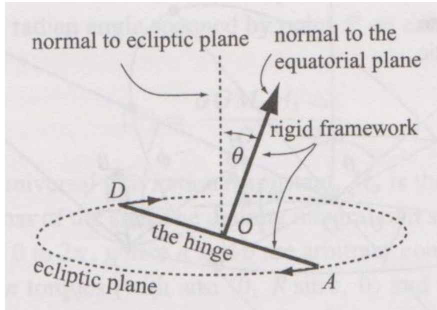


Figure 11. Motion between the ecliptic and equatorial planes due to the torque

the Sun $S$ and its projection $\overrightarrow{OE}$ onto the equatorial plane. Let $\delta =$ the angle between the $+x$ and $\overrightarrow{OE}$, $P = R(cos\ \epsilon, sin\ \epsilon, 0)$, $0 \le \epsilon \le 2\pi$, on the Earth's belt. Let $\psi = \angle SOP$, and $\Omega = \angle PSO$. Thus,

$$S = Q(\cos\delta\cos\phi, \sin\delta\ cos\ \phi, \sin\phi),\ 0 \le \delta \le 2\pi,$$

with $Q$ = the distance between the Earth and the Sun. We can show that

$$sin\ \delta\tan\theta = \tan\phi. \tag{1}$$

Since $\overrightarrow{OS}/Q$ and $\overrightarrow{OP}/R$ are unit vectors,

$$cos\ \psi = (cos\ \delta\ cos\ \phi, \sin\delta\ cos\ \phi, sin\ \phi) \cdot (cos\ \epsilon, sin\ \epsilon, 0)$$

$$= cos\ \phi(cos\ \delta\ cos\ \epsilon + sin\ \delta\ sin\ \epsilon). \tag{2}$$

Let $H$ = the projection of $P$ onto $\overrightarrow{OS}$, $J$ the projection of $H$ onto $\overrightarrow{OE}$. Let $\mu = \angle PHJ$, the angle between $\overrightarrow{PH}$ and the normal $\hat{n}$ to the equatorial plane at $P$. The gravitational force $F$ of the Sun, along $\overrightarrow{PS}$, may be decomposed into two normal components: one along $\overrightarrow{OS}$ and one along $\overrightarrow{PH}$. The latter component is that part of $F$ pulling the equatorial plane into the ecliptic plane. We want that part of this force normal to the equatorial plane, $\hat{n}$. Since

$\left|\overrightarrow{PH}\right| = Rsin\psi, \left|\overrightarrow{OH}\right| = Rcos\psi$, and $\left|\overrightarrow{HJ}\right| = |OH|sin\phi$, then

$$cos\ \mu = \left|\overrightarrow{HJ}\right|/\left|\overrightarrow{PH}\right| = cos\ \psi\ sin\ \phi. \tag{3}$$

Since $Q \gg R$, we assume that $\left|\overrightarrow{PS}\right| = \left|\overrightarrow{OS}\right| = Q$ for all $P$ on the Earth's belt. By the law of sines in $\Delta SOP$, $sin\ \Omega/R = sin\ \psi/Q \rightarrow sin\ \Omega = Rsin\ \psi/Q$. Hence, the magnitude of the component of $F$ normal to $\overrightarrow{OS}$ is $|F|sin\ \Omega = R|F|sin\ \psi/Q$. Thus,

$$\hat{n} = (R\lceil F\rceil/Q)sin\psi\cos\mu\hat{k}, \text{ where } \hat{k} = (0, 0, 1). \tag{4}$$

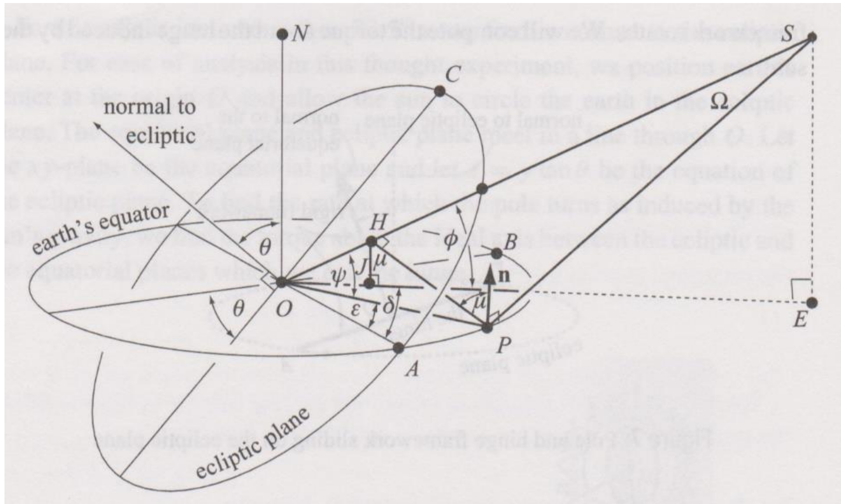


Figure 12. Relations between the ecliptic and equatorial planes

The torque about *OA* obtained by applying $\hat{n}$ *to P* is $R(0, \sin\epsilon, 0) \times \hat{n}$. Since the vectors are ⊥, the total torque of the Sun pulling on *P*, rotating the equatorial plane onto the ecliptic, is

$$\tau = R\, sin\, \epsilon\, |\hat{n}|\hat{\imath} = \frac{R^2 sin\, \epsilon\, sin\, \psi\, cos\, \mu}{Q} |F|\hat{\imath}, \hat{\imath} = (1, 0, 0).$$

$$= \frac{R^2}{Q}\, sin\, \phi\, \cos\phi\, (sin\, \epsilon\, cos\, \epsilon\, cos\, \delta + sin^2\, \epsilon\, sin\, \delta)\, |F|\hat{\imath}\, , \qquad (5)$$

using (3).

Since $Q \gg R$, we assume that $|F| \approx$ constant for any *P* along the belt. Let the point-mass at *P* be $\alpha M_e \Delta\epsilon / 2\pi$, where $\Delta\epsilon =$ the radian angle spanned by point *P* on Earth's equatorial belt. Thus,

$$|F| = \frac{\alpha G M_e M_S}{Q^2} \frac{\Delta\epsilon}{2\pi}, \qquad (6)$$

where $G =$ the universal gravitation constant, $M_e =$ the mass of the Earth, and $M_S =$ the mass of the Sun. The definite integral of $a\, sin\, \epsilon \cos\, \epsilon + b sin^2 \epsilon$, over $2\pi$, is $\pi b$, for arbitrary *a* & *b.* Thus, over $2\pi$, all the torques (7) & (8) sum up to

$$\tau = \frac{\alpha G M_e M_S R^2}{2Q^3} sin\, \phi\, cos\, \phi\, sin\, \delta\, \hat{\imath}$$

$$= \frac{\alpha G M_e M_S R^2}{2Q^3} sin^2 \phi\, cot\, \theta\, \hat{\imath}, \qquad (7)$$

using (1). Again, by (1), the average value of $sin^2 \phi$, as $\delta$ varies from 0 to $2\pi$, is $1 - cos\,\theta$.

Thus, averaged over the year,

$$\tau = \frac{\alpha G M_e M_S R^2}{2Q^3} cot\,\theta\,(1 - cos\,\theta)\,\hat{\imath}. \tag{8}$$

Now, to find the rate of precession of the Earth. Since the mass of the Earth's belt is negligible compared to the planet's mass, the rotational inertia may be taken about *O* as

$$I = \frac{2}{5} M_e R^2. \tag{9}$$

Let $\omega_S = 2\pi$ radians/day, the rotation rate of the Earth about its axis wrt the Sun. Let $\vec{\omega}(t) =$ the angular velocity of the Earth about its pole at time *t*. Since the pole is inclined by angle $\theta$ away from the +*z*-axis,

$$\vec{\omega}(t) = \omega_S(\sin(\beta_S t)\sin\theta, \cos(\beta_S t)\sin\theta, \cos\theta).$$

$$\dot{\vec{\omega}}(t) = \beta_S \omega_S \sin\theta\,(\cos(\beta_S t),\ -\sin(\beta_S t),\ 0).$$

Then, $$\dot{\vec{\omega}}(0) = \beta_S \omega_S \sin\theta\,\hat{\imath}\,.$$

With (9), the torque about *O* is

$$\tau = I\dot{\vec{\omega}}(0) = \frac{2\beta_S \omega_S M_e R^2}{5}\hat{\imath}\,. \tag{10}$$

From (8) and (10):

$$\frac{\alpha G M_e M_S R^2}{2Q^3} cot\,\theta\,(1 - cos\,\theta) = \frac{2\beta_S \omega_S M_e R^2}{5}.$$

And we find $\beta_S$:

$$\beta_S = \frac{5\alpha G M_S}{4\omega_S {Q_S}^3 sin\theta} cot\,\theta\,(1 - cos\,\theta) = 2.24 \times 10^{-12}\ radians/sec,$$

where $G = 6.67 \times 10^{-11} N\frac{m^2}{kg^2},\ M_S = 1.99 \times 10^{30}\ kg, Q_S = 1.496 \times 10^{11} m.$

Similarly, we may calculate the effect due to the Moon:

$$\beta_m = \frac{5\alpha G M_m}{4\omega_m {Q_m}^3 sin\theta} cot\,\theta\,(1 - cos\,\theta) = 5.07 \times 10^{-12}\ radians/sec,$$

where $\omega_m = 2\pi\frac{radians}{1.04} days, M_m = 7.35 \times 10^{22}\ kg, Q_m = 3.84 \times 10^{8} m.$

The combined influence of the Sun and moon on the pole precession is the sum:

$$\beta = \beta_S + \beta_m = 7.35 \times 10^{-12}\, radians/sec, \tag{11}$$

or about one rotation in 27,300 years, a relative error of 5.8% from the actual value of 25,800 years.

## 2.1 Tidal forces

Because the gravitational field of a spherical object is not uniform, but varies as $1/r^2$, the force exerted on an extended body varies across the body. For example, the force exerted by the Moon on the Earth is stronger on the parts of the Earth nearer the Moon than on the parts farther away (Figure 13).

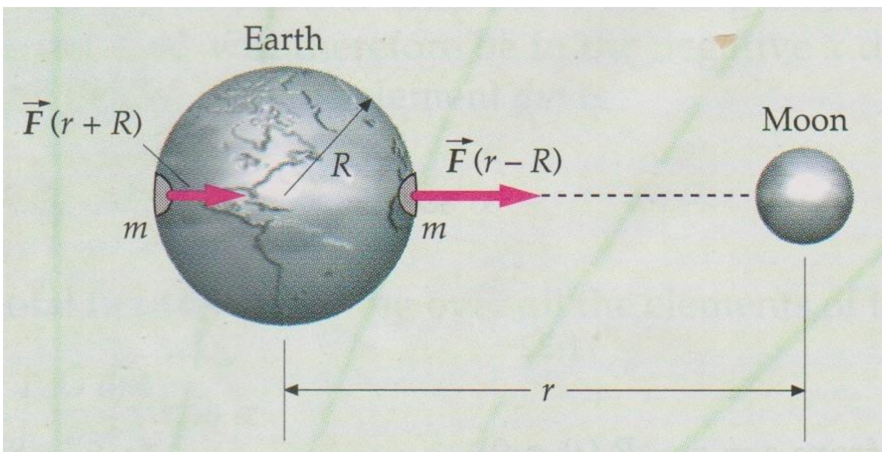


Figure 13. Finding the tidal force due to the Moon.

$$\Delta\vec{F} = \vec{F}(r-R) - \vec{F}(r+R)$$

$$= \frac{GMm}{(r-R)^2} - \frac{GMm}{(r+R)^2} = \frac{GMm[(r+R)^2 - (r-R)^2}{(r-R)^2(r+R)^2} = \frac{4GMmrR}{(r^2-R^2)^2}$$

$$\Delta\vec{F} \approx \frac{4GMmR}{r^3}, R \ll r. \tag{12}$$

Similarly, the tidal force due to the Sun is

$$\Delta\vec{F_S} \approx \frac{4GM_S mR}{{r_S}^3},$$

which is smaller than that due to the Moon by the factor

$$\frac{M_S}{{r_S}^3} : \frac{M}{r^3} = 0.458.$$

Although the Sun exerts a much greater force on the Earth than does the Moon, the differential force exerted by the Moon is much greater than that exerted by the Sun, because the distance to the Sun is much larger compared to the Moon. This tidal force is responsible for the observed tides.

## 2.2 The Roche limit

Most large astronomical objects are held together by gravity. If the tidal force (Eq. 12) on such an object is larger than the gravitational forces holding the body together, the object will fly apart.

Consider a planet of mass $M$. Because the tidal forces exerted by the planet vary as $M/r^3$, there is a minimum distance $r_m$ at which a satellite can exist. This minimum distance is called the *Roche limit*, after the French physicist Edouard Roche who studied it in 1848. We can estimate the Roche limit thus. Consider an object of mass $2m$ consisting of two uniform spheres each of mass $m$ and radius $a$ (Figure 14). These objects exert an attractive force $F$ that keeps them together:

$$F = \frac{Gm^2}{(2a)^2}.$$

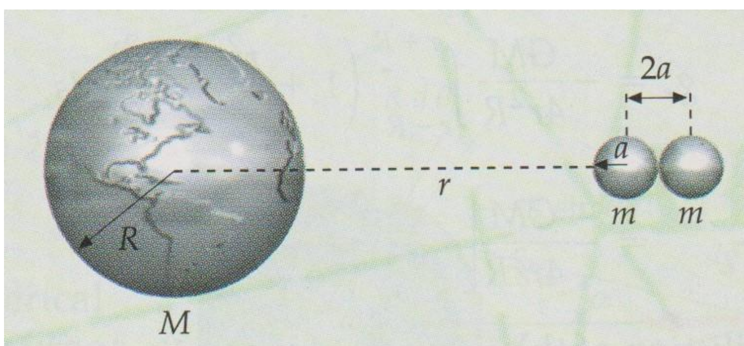


Figure 14. Finding the Roche limit of a planet.

When this object is at a distance $r$ from the planet of mass $M$, the tidal force is given by Eq. 12, with $R = a$. At the Roche limit, $r = r_m$, and the two forces are equal:

$$\frac{4GMmR}{r^3} = \frac{Gm^2}{(2a)^2}$$

or, $${r_m}^3 = \frac{16Ma^3}{m}.$$

Let $\rho_0$ = the density of the planet, with radius $R$. Then,

$$M = \rho_0 \frac{4}{3}\pi R^3, \quad m = \rho_s \frac{4}{3}\pi a^3$$

and,

$$r_m = \left(\frac{16\rho_0}{\rho_s}\right)^{1/3} R.$$

If the densities are equal, the Roche limit ≈ 2.5 × the radius of the planet. Natural satellites can exist only outside the Roche limit of a planet. Around Saturn, we find that inside the Roche limit are rings of small particles that cannot form a satellite held together by gravity. Artificial satellites can, of course, exist within the Roche limit of a planet because they are held together by nuts and bolts rather than by gravitational attraction.

3. The Earth-Moon system

To get a better idea how the gravitational force from the Moon varies over the Earth's surface, we get the potential at $P$ due to the gravity of the Moon plus the orbital motion of the Earth (Fig. 15, Stacey & Davis, 2008):

$$U = -\frac{Gm}{R'} = \frac{1}{2}\omega^2 r^2$$

$$= -\frac{Gm}{R}\left(1 + \frac{1}{2}\frac{m}{M+m}\right) - \frac{Gma^2}{R^3}\left(\frac{3}{2}cos^2\psi - \frac{1}{2}\right) - \frac{1}{2}\omega^2 a^2 sin^2\theta.$$

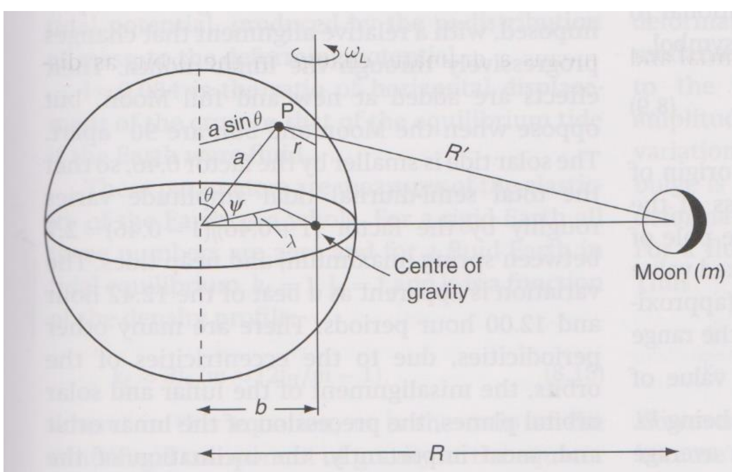


Figure 15. Geometry for the tidal potential of the Moon.

The first term is the gravitational potential at the center of the Earth due to the Moon, with a small correction, a constant independent of the location $P$ on the Earth, and so has no tidal effect. The third term is the rotational potential at $P$ due to the orbital rotation of the Earth about its center at speed $\omega_{orb} \gg \omega$. The second term is the tidal potential, a second-order zonal harmonic and represents the deformation of an equipotential surface to a prolate ellipsoid aligned with the Earth-Moon axis (Fig. 16).

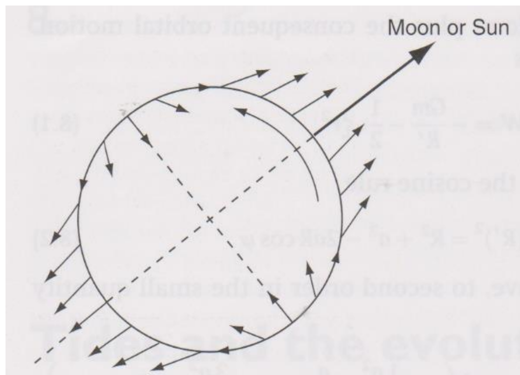


Figure 16. Tidal force at the surface of the Earth, $\vec{F} = -\nabla U$.

Actually, the tidal bulge is slightly delayed by turbulent drag in the sea and by anelasticity of the solid part of the Earth, by a small angle $\delta \approx 2.9^0$, so that the global high tide occurs at points that were directly in line with the Moon $(2.9/360) \times 24\ hr = 12\ min$ ago (Fig. 17). It is conventional to give the tidal potential its own symbol

$$U_2 = -\frac{Gma^2}{R^3}\left(\frac{3}{2}cos^2\psi - \frac{1}{2}\right). \tag{13}$$

The deformation of both the solid Earth and the oceans modifies the tidal potential, described by dimensionless numbers *h and k,* introduced by A.E.H. Love, and *l,* by T. Shida, so that the tidal potential at the Earth's surface is now

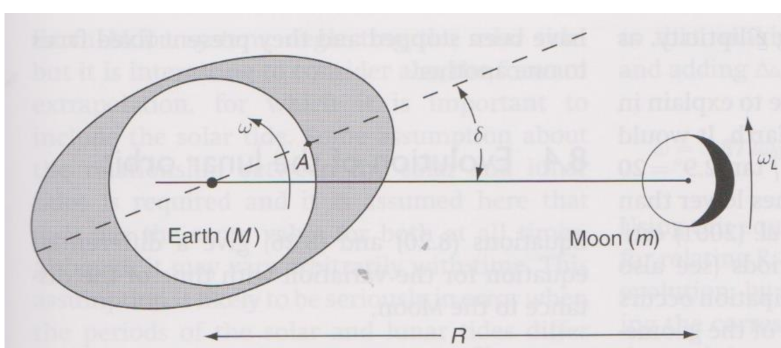


Figure 17. The tidal bulge on the Earth

$$U_2 = -\frac{k_2 G m a^2}{R^3}\left[\frac{3}{2}cos^2(\psi - \delta) - \frac{1}{2}\right]. \tag{14}$$

This diminishes with distance as $r^{-3}$. Thus, there is a tidal torque exerted on any mass $m^*$ at point *r:*

$$\begin{aligned}\tau &= -m^* \frac{\partial U}{\partial(\psi - \delta)} \\ &= -\frac{3k_2 G m m* a^5}{R^3 r^3}\cos(\psi - \delta)\sin(\psi - \delta) \\ &= -\frac{3k_2 G m^2 a^5}{R^6}\cos\delta\sin\delta \approx -\frac{3k_2 \delta G m^2 a^5}{R^6}.\end{aligned} \tag{15}$$

For the Moon, $m = m^*, R = r,$ and $\psi = 0.$ This torque acts in the direction that would reduce $\delta,$ i.e., it tries to make the Moon 'catch up' with the tidal bulge of the Earth. The bulge appears to the Moon to be ahead of its own orbital motion, but with respect to the Earth, which is rotating faster, the bulge is seen to be delayed. Hence, the effect of the lag of the bulge, caused by frictional losses of the tides of the Earth, is to apply an accelerating torque to the orbital motion of the Moon. The equal torque exerted by the Moon on the bulge tends to pull the bulge into line with the Moon and so acts as a brake on the Earth's rotation (Fig. 18).

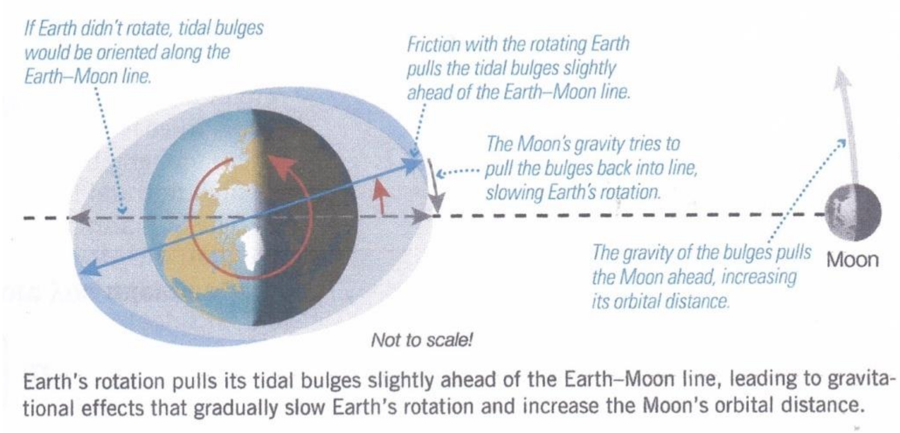


Figure 18. The tidal bulge in relation to the Earth and Moon.

## 3.1 Conservation of angular momentum

Angular momentum is conserved. The angular momentum gained by the Moon is the same as that lost by the Earth since the torques are equal. However, energy is lost from the motion. The Moon's orbit gains energy (kinetic plus potential), at a rate $\omega_{orb}\tau$. And the Earth's rotation loses energy at a greater rate, $\omega\tau$. The net rate of loss of energy is

$$(\omega - \omega_{orb})\tau = 3.06 \times 10^{12} W,$$

i.e., the torque times the angular speed of the tide relative to the Earth. Adding the energy dissipation by the solar tide, we have a total of $3.70\times 10^{12} W$.

Observed values of the tidal parameters in Eq. (15) are: $k_2 = 0.245, \delta = 2.89^0$, by satellite, giving a value of the lunar torque, $\tau = 4.40 \times 10^{16}\, kg\, m^2 s^{-2}$. This is equal to the rate of increase in orbital angular momentum of the Earth-Moon system,

$$L = \frac{mM}{m+M}\omega_{orb}R^2.$$

To identify separately the effects on $\omega_{orb}$ and $R$, we use Kepler's third law,

$$\omega_{orb}R^2 = G(M+m)$$

from which

$$L = \frac{G^{2/3}mM}{(m+M)^{1/3}}\omega_{orb}{}^{-1/3} = \frac{G^{1/2}mM}{(m+M)^{1/2}}R^{1/2},$$

so that

$$\tau = \frac{dL}{dt} = -\frac{1}{3}\frac{G^{\frac{2}{3}}mM}{(m+M)^{\frac{1}{3}}}\omega_{orb}{}^{-\frac{4}{3}}\frac{d\omega_{orb}}{dt} = \frac{1}{2}\frac{G^{\frac{1}{2}}mM}{(m+M)^{\frac{1}{2}}}R^{-\frac{1}{2}}\frac{dR}{dt}. \quad (16)$$

These give

$$\frac{d\omega_{orb}}{dt} = -1.2\times 10^{-23} rad\, s^{-2} (25\, arcsec/century^2) \quad (17)$$

$$\frac{dR}{dt} = 1.17\times 10^{-9} m\, s^{-1} (3.7\, cm/yr). \quad (18)$$

These results are derived from satellite observations and also from lunar-ranging measurements that give confirmation of Eq. (18). The decreasing angular speed of the Moon (Eq. (17) has been observed since the 1800s by conventional astronomy, but not with the precision that we now have with atomic clocks and satellite ranging.

The corresponding slowing of the Earth's rotation is given by

$$I_z\frac{d\omega}{dt} = -\tau,$$

from which

$$\left(\frac{d\omega}{dt}\right)_{lunar\ tide} = -5.4\times 10^{-22}\, rad\, s^{-2}.$$

Adding the solar contribution of $(0.459)^2 = 0.21 = 21\%$, the total tidal braking of the Earth's rotation is

$$\left(\frac{d\omega}{dt}\right)_{Total\ tide} = -6.5 \times 10^{-22}\ rad\ s^{-2}. \quad (19)$$

This corresponds to a length-of-day increase of 2.4 *ms/century*. The tidal strain raised in one body by proximity to another is proportional to the ratio of the masses. Therefore, the tidal deformation of the Moon by the Earth is about $(80)^2$ times the tide in the Earth. And, tidal friction in the Moon, if it were rotating, would be $(80)^4$ times as strong. This is why the Moon's axial rotation coincides with its orbital period. Tidal friction has completely stopped its rotation relative to the Earth. The same is true for other close satellites in the Solar System. Io presents the same face to Jupiter. For Pluto and its large satellite, Charon, the relative rotations of both bodies have stopped and they present fixed faces to one another.

Tides and tidal dissipation are important to planetary systems. The inner planets and the Moon are solid bodies and their tides dissipate rotational energy. This has the effect of slowing and eventually stopping relative rotations. The large tide raised by the Earth on the Moon has completely stopped its rotation relative to the Earth. The tide raised by the Moon in the Earth is slowing down the Earth's rotation with a consequent transfer of angular momentum to the lunar orbit, causing the Moon's orbit to expand at a rate that is currently 3.8 *cm/yr*. Over geologic time this means a dramatic change in the orbit, as will be discussed in greater detail subsequently. There we will present the case for two moons surviving independently for the first 600 *million years* and being brought together by tidal friction 3.9 *bya*.

Venus and Mercury, being moonless, are a different situation but still explainable by tidal dissipation. Being much closer to the Sun they have strong solar tides, which have so slowed their rotations that they present almost constant faces to the Sun. Any satellites would have been orbiting faster than the planetary rotations, with tidal friction opposing their motions and causing them to spiral inwards. Orbital angular momentum transferred to planetary rotation, would be lost to the solar tidal friction. For these planets, any early satellites have merged with the parent planets. There is a very strong dependence of tidal friction on the separation, *r*. Tidal amplitude varies as $r^{-3}$, and dissipation as the square of amplitude, $r^{-6}$, or even more strongly if dissipation is non-linear. The motion of a satellite spiraling in towards its parent planet would accelerate rapidly if tidal friction is effective at all.

Tidal friction by the Moon on the Earth is slowing the Earth's rotation due to a consequent transfer of angular momentum to the lunar orbit, thus the Moon's orbit increases. The variation with time of the distance to the Moon is given by Eqs. (15) and (16),

$$R^{11/2}\frac{dR}{dt} = 6k_2\delta G^{\frac{1}{2}}a^5(m/M)(M+m)^{\frac{1}{2}}. \quad (20)$$

Making the simplest assumption that $(k_2\delta)$ is a constant, independent of the speed or amplitude of the tide, then the RHS of Eq. (19) is constant,

$$R^{11/2}\frac{dR}{dt} = {R_0}^{11/2}\frac{dR_0{}'}{dt}, \tag{21}$$

where $R_0$ and $R_0{}'$ are the present values of $R$ and $dR/dt$. Integrating from an initial distance,

$$\frac{2}{13}\left({R_0}^{13/2} - {R_i}^{13/2}\right) = {R_0}^{11/2}R_0'T\,. \tag{22}$$

The high power of $R$ ensures that, for $R_i \ll R_0,$ the early orbital evolution was very rapid and so the time scale hardly depends on the assumption made about $R_i$. The inferred total time for orbital evolution is

$$T \approx \frac{2}{13}R_0/R_0' = 1.6 \times 10^9\, yrs. \tag{23}$$

The geology of the Moon has been stable for much longer than this and is incompatible with a close approach to the Earth 1.6 *bya.* Also there is no evidence on the Earth for such a dramatic event. The assumption about $(k_2\delta)$ being constant must be reexamined. Tidal friction was much weaker in the past than linear extrapolation from present conditions suggests.

## 3.2 The multiple moons hypothesis

We now look at the multiple moons hypothesis where we consider three moons, formed in orbits with radii differing by factors of 1.6 in accordance with the Bode-Titius rule, assuming the same principle applies to satellites. That is, the satellites remain independent as long as their orbital radii exceeds 1.6, but if they come within that separation then gravity will cause them to collapse on the time scale of planetary accretion, $\approx 600{,}000{,}000\, yrs$. Moon samples brought back by the Apollo missions showed that the Moon suffered a major cataclysm between its formation and an intense bombardment within this time interval. The Bode rule is accurately obeyed by the major satellites of Jupiter.

Of the three initial moons the largest was probably innermost and subject to much stronger tidal friction than the others. Its orbit expanded, reducing its separation from the middle one, so that they quickly came together than the Bode's ratio, causing the smaller one to be subjected to orbit modification and eventual assimilation by the major one. That process was rapid enough to be observationally indistinguishable from the original planetary accretion. But the existence of the middle moon is necessary to explain the 600 million year delay before the major moon approached the outer one within the Bode ratio. Now to calculate the time required by tidal friction to increase its orbital radius by the factor 1.6.

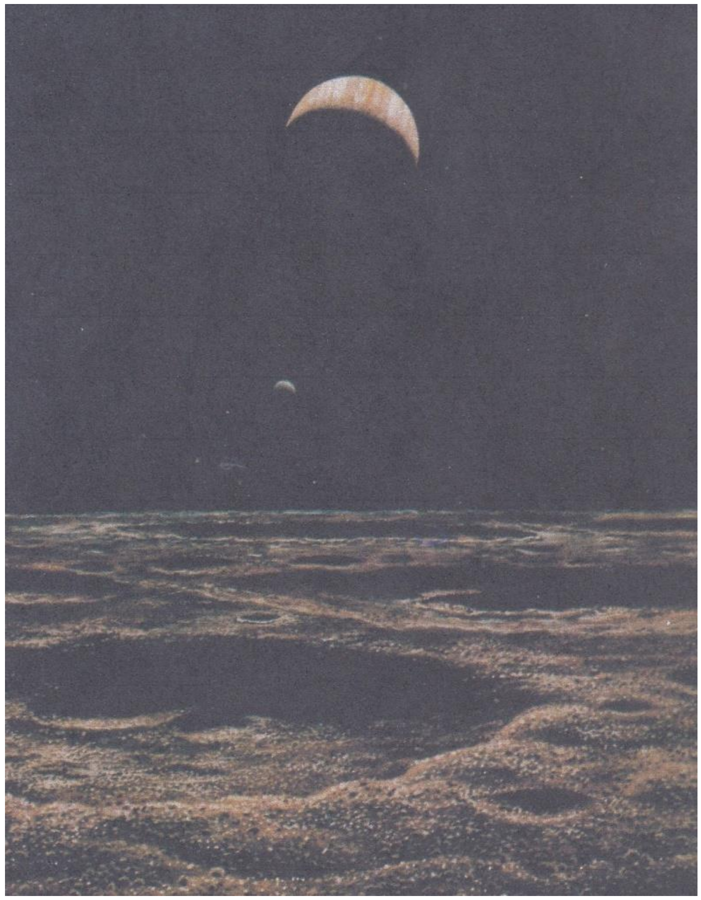

Figure 19. The multiple moons hypothesis

We rewrite Eq. (20) as a logarithmic derivative, with $m \ll M$:

$$\frac{d\,lnR}{dt} = \frac{6G^{1/2}a^5}{M^{1/2}} k_2 \delta m R^{-13/2}. \quad (24)$$

This applies to two moons $m_1, m_2$ at orbital radii $R_1, R_2 > R_1$, so that

$$\frac{d\ln(R_1/R_2)}{dt} = \frac{6G^{\frac{1}{2}}a^5}{M^{\frac{1}{2}}} k_2 \delta (m_1 {R_1}^{-13/2} - m_2 {R_2}^{-13/2}). \quad (25)$$

The ratio $R_1/R_2$ increases with time, i.e., the orbits become closer on a logarithmic scale, if

$$m_1/m_2 > (R_1/R_2)^{3/2}. \quad (26)$$

If the satellites are to be brought together, then Eq. (25) must be satisfied at the critical ratio $R_1/R_2 = 1/1.6$, so that a necessary condition for them to merge is $m_1/m_2 > 0.047$. The inner one must have more than 5% of the mass of the outer one.

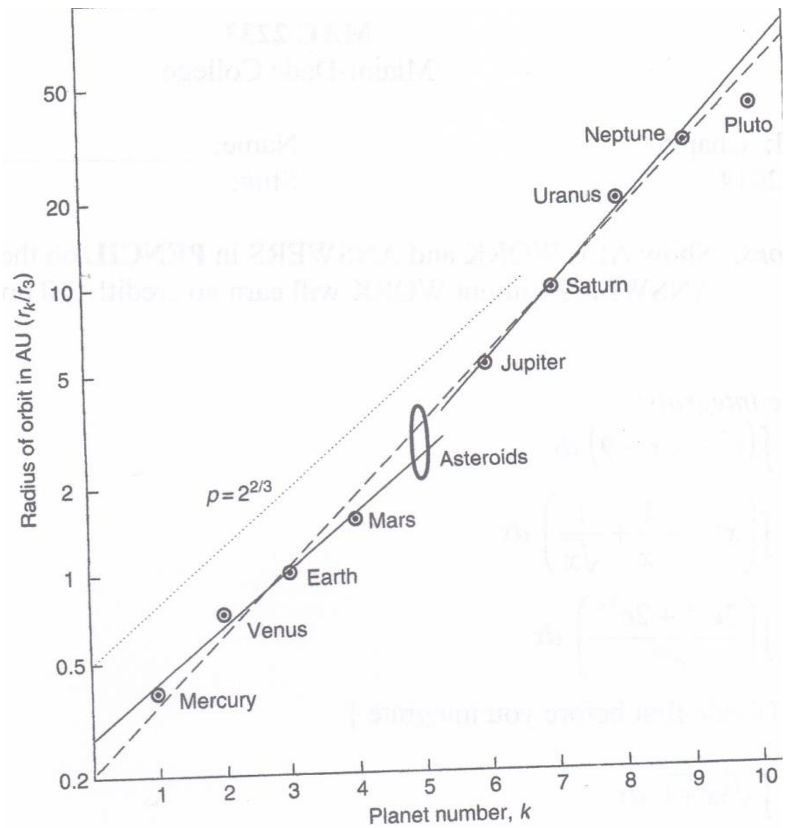


Figure 20. The Bode-Titius Rule

Integrating Eq. (20) from an initial radius $R_1$ to $R_1{}^*$ over $\Delta T = 6 \times 10^8\ yrs$, with $R_0{}'$ (= $3.7\ cm/yr$ at present) reduced by the factor (0.4/2.9) to account for the lower value of $\delta$ for the early Earth, we have

$$\left(\frac{R_1{}^*}{R_0}\right)^{13/2} - \left(\frac{R_1}{R_0}\right)^{\frac{13}{2}} = \frac{13}{2}\frac{0.4}{2.9}\frac{R_0{}'}{R_0}\Delta T = 0.05.$$

For any value of $R_1/R_0 < 0.4$, $R_1{}^*/R_0 = 0.63$ and $R_1/R_0 \approx 1.0$, placing the smaller moon almost in the present lunar orbit. When the impacts occurred, about 3.9 *bya,* the Moon was in an orbit of at least 38 Earth radii and possibly 45 (compared with the present 60.3 $R_0$).

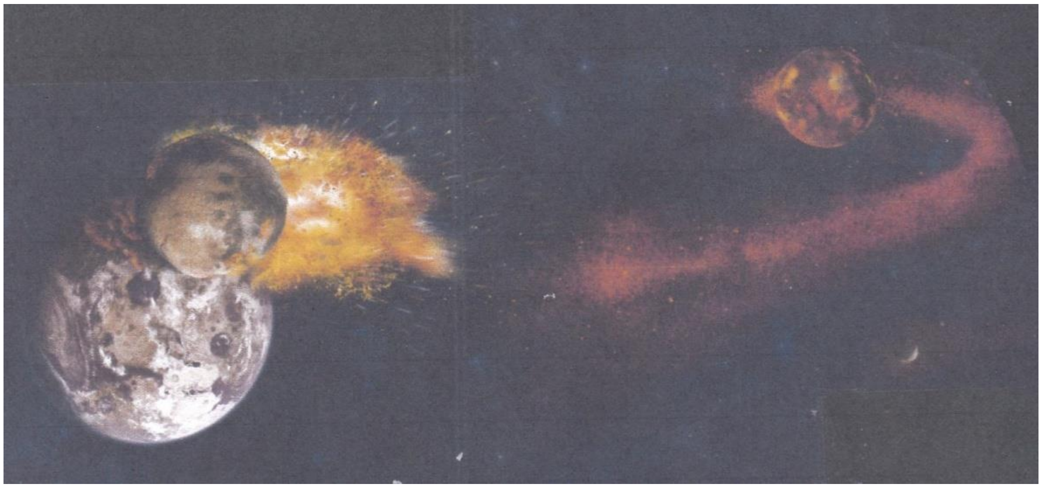

Figure 21. The fragments of the collision remained in orbit for subsequent collection.

The assumption that the orbital radii of the moons were initially separated by the Bode ratio 1.6 means that this is the factor by which the Moon's orbit evolved over the 600 million year period. The rate of orbital evolution is a very strong function of orbital radius, so that, if the phase lag $\delta$ is specified, the beginning and end radii are fixed. Assuming $\delta = 0.4^0$, the Moon started at $25R_E$ and reached $40R_E$ at the time of the cataclysm, 600 *my* later. If it had been much less than $25R_E$, then the initial radius would have increased very rapidly to a factor of 1.6 in much less time than 600 *my*. No plausible value of $\delta$ would have that process possible. This result is incompatible with the giant impact hypothesis for the origin of the Moon, according to which it would have formed inside about $4R_{E.}$ Our results show that a close approach of the Moon has not occurred at any time in Earth history.

The late intense bombardment of the Moon can be explained without requiring any bombardment of the Earth. The fragments that impacted the Moon were in Earth orbit, as was the body from which they were derived. The fragments remained in Earth orbit for 600 *my* of relative lunar quiescence before they were disturbed by the approaching Moon. The approach was slow enough to break the small moon into several pieces, which probably made further disrupting approaches before impacting. Since the colliding bodies were both in orbit, the speeds involved were more modest than that with an asteroid in a sharp elliptical orbit and would not have unduly affected the major Moon's orbit and would have spared the Earth from any bombardment.

Conclusions:

- There is a closest approach for an orbiting body to come near its parent body, called its Roche limit, due to the balance between gravitational forces and elastic forces within the body.

- The equatorial bulge of the Earth is delayed in its rotation by turbulent drag in the sea and the anelasticity of the solid Earth. And yet this tidal bulge is ahead of the Moon's orbital motion. This is shown by astronomical measurements, especially by satellite.
- This tidal friction in the Earth is what is mostly responsible for the increase in the length of day. The l.o.d. increases by 2.4 *ms/century.*
- Due to tidal friction the Moon is receding from the Earth by 3.8 *cm/yr.* This figure is directly confirmed by laser ranging.
- Tidal dissipation is important to planetary and satellite systems. In particular, it explains readily the phase lag of the tidal bulge of the Earth, the locked faces of some moons of planets, and the absence of moons for Mercury and Venus.
- The multiple moons hypothesis is based on well-known physical principles and gives a viable theory for the origin of the Moon, like the giant impact hypothesis.

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