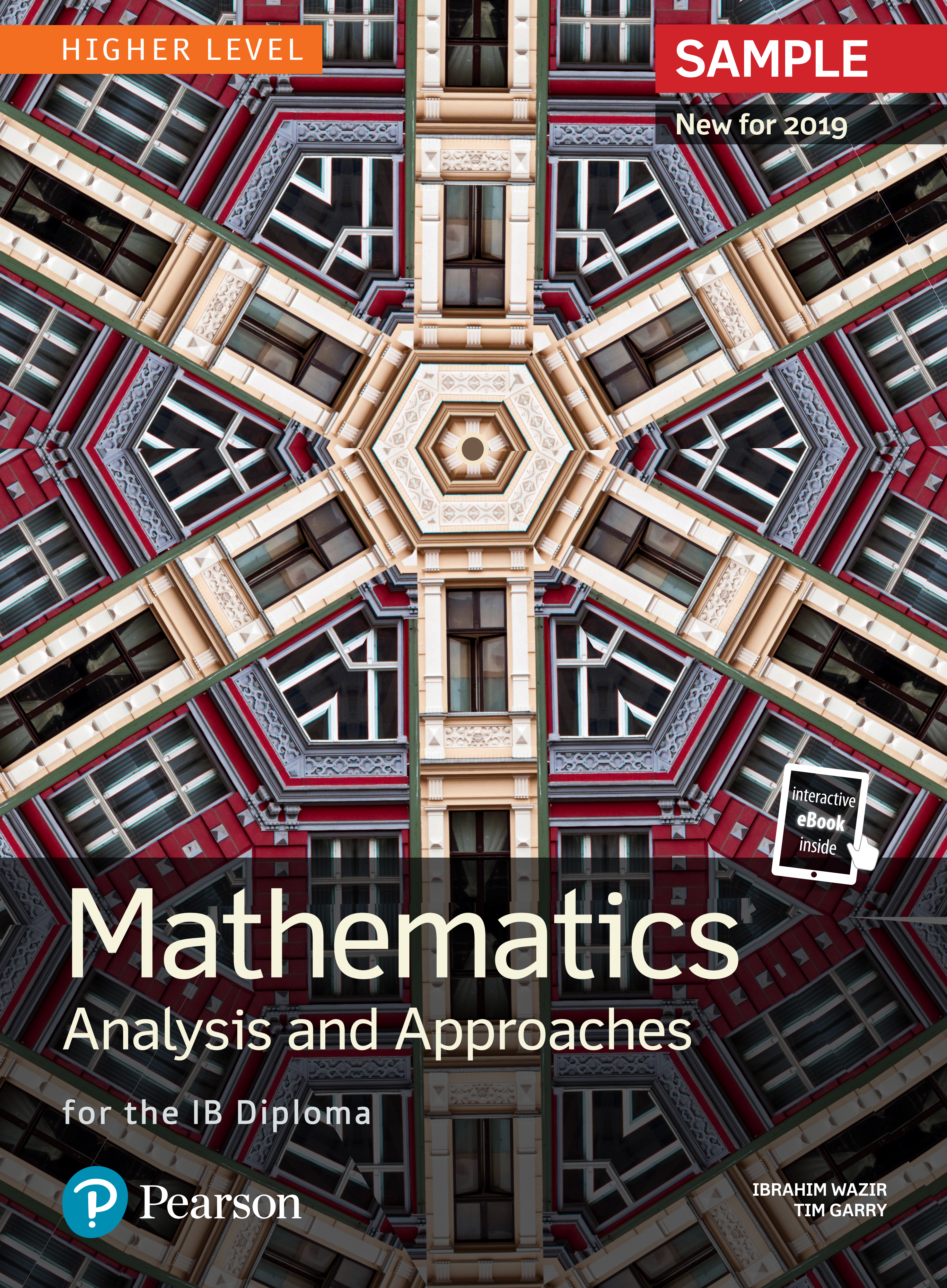


HIGHER LEVEL

SAMPLE

New for 2019

interactive eBook inside

Mathematics

Analysis and Approaches

for the IB Diploma

Pearson

IBRAHIM WAZIR
TIM GARRY

HIGHER LEVEL

Mathematics

Analysis and Approaches

for the IB Diploma

IBRAHIM WAZIR
TIM GARRY

SAMPLE

Contents

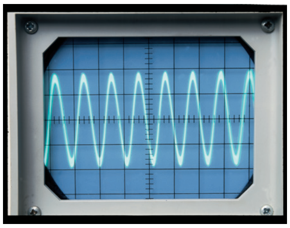
6

Trigonometric functions and equations

Learning objectives

By the end of this chapter you should be familiar with...

- angles measured in radians
- computing the length of an arc and the area of a sector
- the unit circle and the definitions for $\sin\theta$, $\cos\theta$ and $\tan\theta$
- knowing exact values of $\sin\theta$, $\cos\theta$ and $\tan\theta$ for $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$ and their multiples
- the Pythagorean identities and double angle identities for sine and cosine
- the relationships between $\sin\theta$, $\cos\theta$ and $\tan\theta$
- the graphs of $\sin\theta$, $\cos\theta$ and $\tan\theta$, and their amplitude and period transformations of graphs in the form $a\sin(b(x+c))+d$ and $a\cos(b(x+c))+d$
- applying trigonometry to real-life problems
- solving trigonometric equations in a finite interval
- the reciprocal trigonometric ratios $\sec\theta$, $\csc\theta$ and $\cot\theta$
- the Pythagorean identities involving $\tan\theta$, $\sec\theta$, $\csc\theta$ and $\cot\theta$
- the inverse functions $\arcsin x$, $\arccos x$, $\arctan x$; and their domains, ranges and graphs
- the compound angle identities for $\sin\theta$ and $\cos\theta$
- double angle identity for $\tan\theta$
- relationships between trigonometric functions and the symmetry of their graphs.



The oscilloscope shows the pressure of a sound wave versus time for a high-pitched sound. The graph is a repetitive pattern that can be expressed as the sum of different sine waves. A sine wave is any transformation of the graph of the trigonometric function $y = \sin x$ and takes the form $y = a\sin[b(x + c)] + d$.

Trigonometry developed from the use and study of triangles in surveying, navigation, architecture, and astronomy to find relationships between lengths of sides of triangles and measurement of angles. As a result, trigonometric functions were initially defined as functions of angles – that is, functions with angle measurements as their domains. With the development of calculus in the 17th century and the growth of knowledge in the sciences, the application of trigonometric functions grew to include a wide variety of periodic (repetitive) phenomena such as wave motion, vibrating strings, oscillating pendulums, alternating electrical current, and biological cycles. These applications of trigonometric functions require their domains to be sets of real numbers without reference to angles or triangles. Hence, trigonometry can be approached from two different perspectives – **functions of angles** or **functions of real numbers**. This chapter focuses on the latter – viewing trigonometric functions as defined in terms of a real number that is the **length of an arc** along the unit circle.

6.1 Angles, circles, arcs and sectors

An **angle** in a plane is made by rotating a ray about its endpoint, called the **vertex** of the angle. The starting position of the ray is called the **initial side** and the position of the ray after rotation is called the **terminal side** of the angle (Figure 6.1). An angle with its vertex at the origin and its initial side on the positive x-axis is in **standard position** (Figure 6.2a). A **positive angle** is produced when a ray is rotated in an anticlockwise direction, and a **negative angle** when rotated in a clockwise direction.

Two angles in standard position that have the same terminal sides regardless of the direction or number of rotations are called **coterminal angles**. Greek letters are often used to represent angles, and the direction of rotation is indicated by an arc with an arrow at its endpoint. The x and y axes divide the coordinate plane into four quadrants (numbered with Roman numerals). Figure 6.2b shows a positive angle α and a negative angle β that are coterminal in quadrant III.

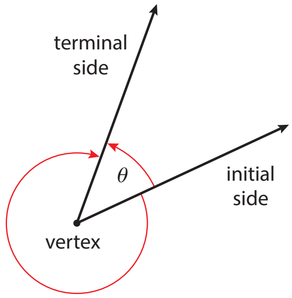


Figure 6.1 Components of an angle

$\beta = \alpha - 360$

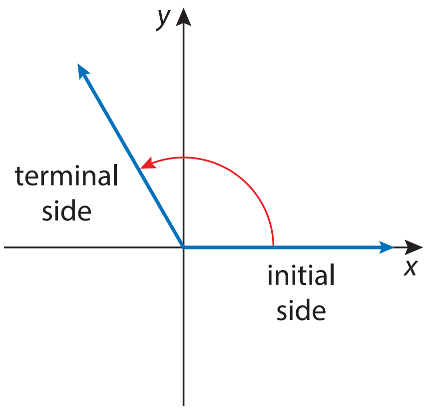


Figure 6.2a Standard position of an angle

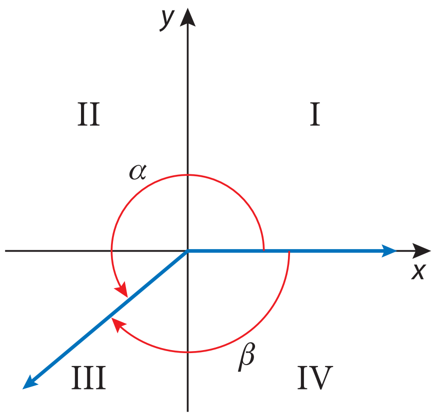


Figure 6.2b Coterminal angles

Measuring angles: degree measure and radian measure

A unit of one degree (1°) is defined to be $\frac{1}{360}$ of one anticlockwise revolution about the vertex. There is another method of measuring angles that is more natural. Instead of dividing a full revolution into an arbitrary number of equal divisions (e.g. 360), consider an angle that has its vertex at the centre of a circle (a **central angle**) and subtends (or intercepts) a part of the circle, called an **arc of the circle**. Figure 6.3 shows three circles with radii of different lengths ($r_1 < r_2 < r_3$) and the same central angle θ subtending (intercepting) the arc lengths s_1, s_2 and s_3. Regardless of the size of the circle (i.e. length of the radius), the ratio of arc length, s, to radius, r, for a given angle will be constant. For the angle θ in Figure 6.3, $\frac{s_1}{r_1} = \frac{s_2}{r_2} = \frac{s_3}{r_3}$. Because this ratio is an arc length divided by another length (radius), it is just an ordinary real number and has no units.

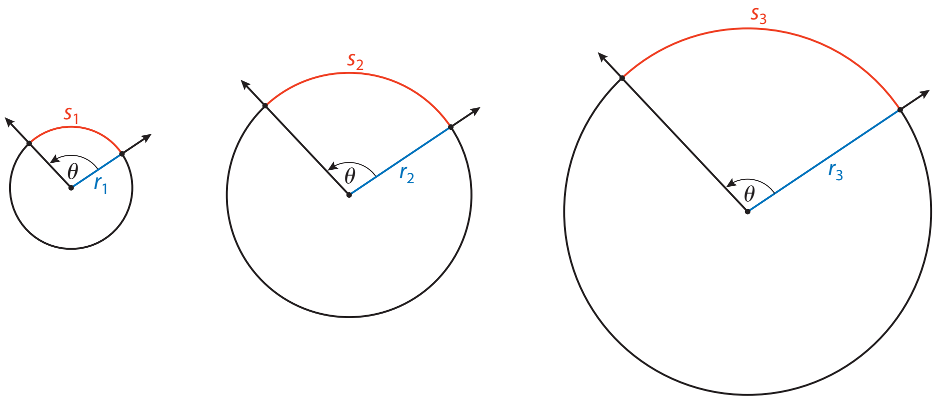
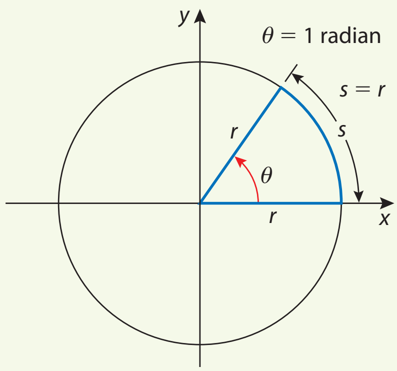


Figure 6.3 Different circles with the same central angle θ subtending different arcs, but the ratio of arc length to radius, $\frac{s}{r}$, remains constant.

Major and minor arcs
If a central angle is less than 180°, then the subtended arc is referred to as a minor arc. If a central angle is greater than 180°, then the subtended arc is referred to as a major arc.

When the measure of an angle is, for example, 5 radians, the word 'radians' does not indicate units, but indicates the method of angle measurement. If the measure of an angle is in units of degrees, we must indicate this by word or symbol. However, when radian measure is used it is customary to write no units or symbol eg. $\theta = 5$.

The ratio $\frac{s}{r}$ indicates how many radius lengths, r, fit into the length of arc s. For example, if $\frac{s}{r} = 2$, then the length of s is equal to two radius lengths. This accounts for the name **radian**.



One radian is the measure of a central angle θ of a circle that subtends an arc s of the circle that is exactly the same length as the radius r of the circle. That is, when $\theta = 1$ radian, arc length = radius.

The unit circle

When an angle is measured in radians it makes sense to draw it in standard position. It follows that the angle will be a central angle of a circle whose centre is at the origin. As Figure 6.3 illustrated, it makes no difference what size circle is used. The most practical circle to use is the circle with a radius of one unit, because the radian measure of an angle will simply be equal to the length of the subtended arc.

$$\text{Radian measure: } \theta = \frac{s}{r} \qquad \text{If } r = 1, \text{ then } \theta = \frac{s}{1} = s.$$

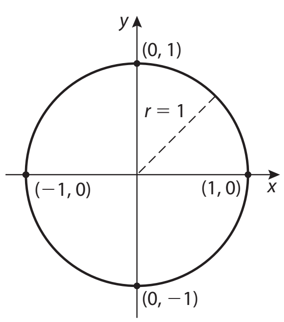


Figure 6.4 The unit circle

The circle with a radius of one unit and centre at the origin (0, 0) is called the unit circle (Figure 6.4). The equation for the unit circle is $x^2 + y^2 = 1$. Because the circumference of a circle with radius r is $2\pi r$, a central angle of one full anticlockwise revolution (360°) subtends an arc on the unit circle equal to 2π units. Hence, if an angle has a degree measure of 360°, its radian measure is exactly 2π. It follows that an angle of 180° has a radian measure of exactly π. This fact can be used to convert between degree measure and radian measure.

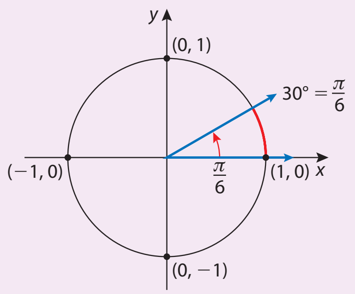
Example 6.1

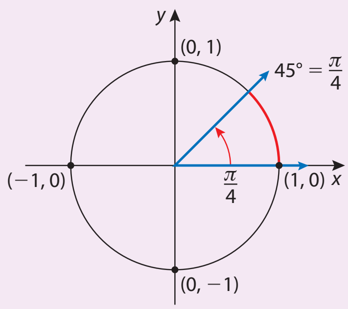
Convert 30° and 45° to radian measure and sketch the corresponding arc on the unit circle. Use these results to convert 60° and 90° to radian measure.

Solution:

Note that the 'degree' units cancel.

$30° = 30°\left(\frac{\pi}{180°}\right) = \frac{30°}{180°}\pi = \frac{\pi}{6}$ $45° = 45°\left(\frac{\pi}{180°}\right) = \frac{45°}{180°}\pi = \frac{\pi}{4}$





Since 60° = 2(30°) and $30° = \frac{\pi}{6}$, then $60° = 2\left(\frac{\pi}{4}\right) = \frac{\pi}{3}$.

Similarly, 90° = 2(45°) and $45° = \frac{\pi}{4}$, so $90° = 2\left(\frac{\pi}{4}\right) = \frac{\pi}{2}$.

Conversion between degrees and radians

Because 180° = π radians, then $1° = \frac{\pi}{180°}$ radians, and 1 radian $= \frac{\pi}{180°}$.

An angle with a radian measure of 1 has a degree measure of approximately 57.3° (accurate to 3 s.f.).

Knowing the four facts given in Example 6.1 can help you to quickly convert mentally between degrees and radians for many common angles.

Example 6.2

(a) Convert the following radian measures to degrees.
Express them exactly, if possible. Otherwise, express them accurate to three significant figures.

(i) $\frac{4\pi}{3}$ (ii) $-\frac{3\pi}{2}$ (iii) 5 (iv) 1.38

(b) Convert the following degree measures to radians. Express exactly.

(i) 135° (ii) −150° (iii) 175° (iv) 10°

Be sure to set your GDC to degree mode or radian mode, as appropriate. As you progress further in mathematics (especially calculus) radian measure is far more useful.

Solution:

(a) (i) $\frac{4\pi}{3} = 4\left(\frac{\pi}{3}\right) = 4(60°) = 240°$

(ii) $-\frac{3\pi}{2} = -\frac{3}{2}(\pi) = -\frac{3}{2}(180°) = -270°$

(iii) $5\left(\frac{180°}{\pi}\right) \approx 286.479° \approx 286°$ (3 s.f.)

(iv) $1.38\left(\frac{180°}{\pi}\right) \approx 79.068° \approx 79.1°$ (3 s.f.)

(b) (i) $135° = 3(45°) = 3\left(\frac{\pi}{4}\right) = \frac{3\pi}{4}$

(ii) $-150° = -5(30°) = -5\left(\frac{\pi}{6}\right) = -\frac{5\pi}{6}$

(iii) $175°\left(\frac{\pi}{180°}\right) \approx 3.0543 \approx 3.05$ (3 s.f.)

(iv) $10°\left(\frac{\pi}{180°}\right) \approx 0.174\,53 \approx 0.175$ (3 s.f.)

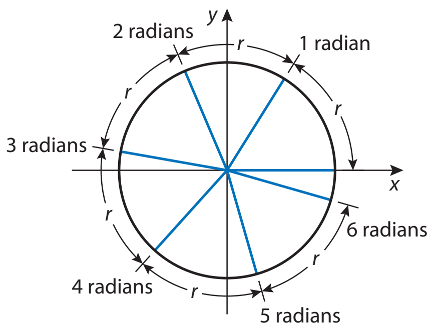


Figure 6.5 Arcs with lengths equal to the radius placed along the circumference of a circle.

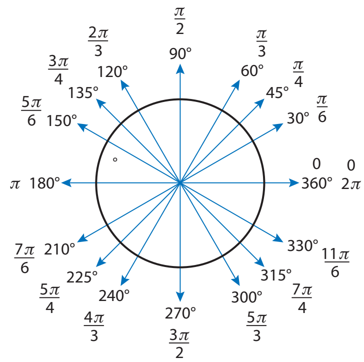


Figure 6.6 Degree and radian measure for common angles.

Because 2π is approximately 6.28 (3 s.f.), there are a little more than six radius lengths in one revolution, as shown in Figure 6.5.

Figure 6.6 shows all of the angles between 0° and 360° inclusive that are multiples of 30° or 45°, and their equivalent radian measure. You will benefit by being able to convert quickly between degree measure and radian measure for these common angles.

Arc length

For any angle θ, its radian measure is given by $\theta = \frac{s}{r}$. Simple rearrangement of this formula leads to another formula for computing arc length.

For a circle of radius r, a central angle θ subtends an arc of the circle of length s given by $s = r\theta$ where θ is in radian measure.

The units of the product $r\theta$ are equal to the units of r because in radian measure θ has no units.

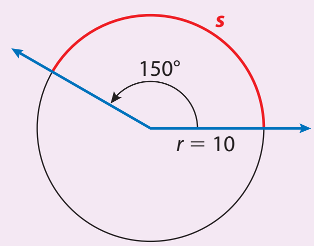
Example 6.3

A circle has a radius of 10 cm. Find the length of the arc of the circle subtended by a central angle of 150°.

Solution:

To use the formula $s = r\theta$, we must first convert 150° to radian measure.

$$150° = 150°\left(\frac{\pi}{180°}\right) = \frac{150\pi}{180} = \frac{5\pi}{6}$$



Substituting $r = 10$ cm into $s = r\theta$ gives:

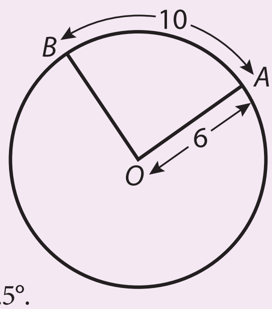
$$s = 10\left(\frac{5\pi}{6}\right) = \frac{25\pi}{3} \approx 26.179\,94\text{cm}$$

The length of the arc is 26.2 cm (3 s.f.).

Example 6.4

The diagram shows a circle of centre O with radius $r = 6$cm. Angle AOB subtends the minor arc AB such that the length of the arc is 10cm. Find the measure of angle AOB in degrees, accurate to 3 significant figures.

Solution:



Rearrange the arc length formula, $s = r\theta$, giving $\theta = \frac{s}{r}$.

Remember that the result for θ will be in radians. Therefore, angle $AOB = \frac{10}{6} = \frac{5}{3}$ or $1.\dot{6}$ radians.

Now, we convert to degrees: $\frac{5}{3}\left(\frac{180°}{\pi}\right) \approx 95.49297°$.

The degree measure of angle AOB is approximately 95.5°.

Geometry of a circle

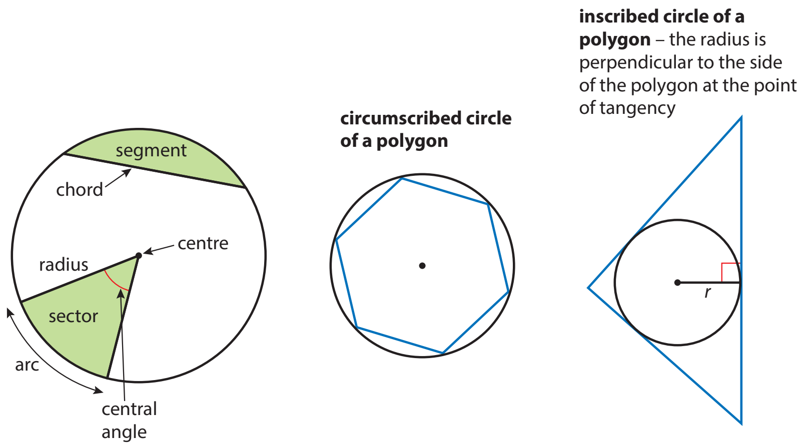


Figure 6.7 Circle terminology

Sector of a circle

A **sector of a circle** is the region bounded by an arc of the circle and the two sides of a central angle (Figure 6.7). The ratio of the area of a sector to the area of the circle (πr^2) is equal to the ratio of the length of the subtended arc to the circumference of the circle ($2\pi r$). If s is the arc length and A is the area of the sector, we can write the following proportion:

$$\frac{A}{\pi r^2} = \frac{s}{2\pi r}$$

Solving for A gives:

$$A = \frac{\pi r^2 s}{2\pi r} = \frac{1}{2}rs.$$

From the formula for arc length we have $s = r\theta$, with θ the radian measure of the central angle. Substituting $r\theta$ for s gives the area of a sector to be

$$A = \frac{1}{2}rs = \frac{1}{2}r(r\theta) = \frac{1}{2}r^2\theta.$$

Area of a sector
In a circle of radius r, the area of a sector with a central angle θ measured in radians is $A = \frac{1}{2}r^2\theta$.

The formula for arc length, $s = r\theta$, and the formula for area of a sector, $A = \frac{1}{2}r^2\theta$, are true only when θ is in radians.

This result makes sense because, if the sector is the entire circle, $\theta = 2\pi$ and area $A = \frac{1}{2}r^2\theta = \frac{1}{2}r^2(2\pi) = \pi r^2$, which is the formula for the area of a circle.

Example 6.5

A circle of radius 9cm has a sector whose central angle measures $\frac{2\pi}{3}$.
Find the exact values of:

(a) the length of the arc subtended by the central angle

(b) the area of the sector.

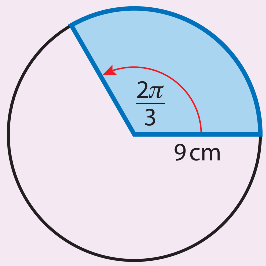
Solution:

(a) $s = r\theta \Rightarrow s = 9\left(\frac{2\pi}{3}\right) = 6\pi$

The length of the arc is exactly 6π cm.

(b) $A = \frac{1}{2}r^2\theta \Rightarrow A = \frac{1}{2}(9)^2\left(\frac{2\pi}{3}\right) = 27\pi$

The area of the sector is exactly 27π cm^2.



Exercise 6.1

1. Convert each angle into radians.

(a) 60° (b) 150° (c) −270° (d) 36°

(e) 135° (f) 50° (g) −45° (h) 400°

2. Convert each angle into degrees. If possible, express exactly, otherwise express accurate to three significant figures.

(a) $\frac{3\pi}{4}$ (b) $-\frac{7\pi}{2}$ (c) 2 (d) $\frac{7\pi}{6}$

(e) −2.5 (f) $\frac{5\pi}{3}$ (g) $\frac{\pi}{12}$ (h) 1.57

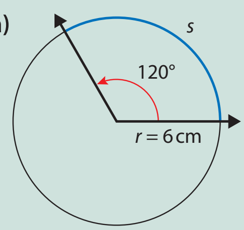
3. The measure of an angle in standard position is given. Find two angles (one positive and one negative) that are coterminal with the given angle.
If no units are given, assume the angle is in radian measure.

(a) 30° (b) $\frac{3\pi}{2}$ (c) 175° (d) $-\frac{\pi}{6}$

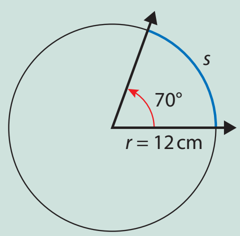
(e) $\frac{5\pi}{3}$ (f) 3.25

4. Find the length of the arc s in each diagram.

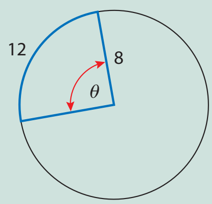
(a)



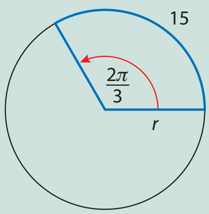
(b)



5. Find the angle θ in the diagram in both radians and degrees.

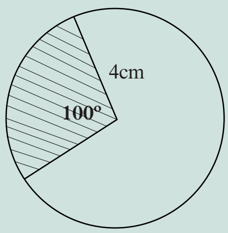


6. Find the radius r of the circle in the diagram.

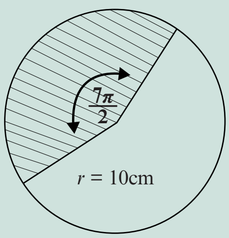


7. Find the area of the sector in each diagram.

(a)



(b)



8. An arc of length 60 cm subtends a central angle α in a circle of radius 20 cm. Find the measure of α in both degrees and radians, accurate to 3 s.f.

9. Find the length of an arc that subtends a central angle of 2 radians in a circle of radius 16 cm.

10. The area of a sector of a circle with a central angle of 60° is 24 cm^2. Find the radius of the circle.

11. A bicycle with tires 70 cm in diameter is travelling such that its tires complete one and a half revolutions every second. That is, the angular velocity of the tire is 1.5 revolutions per second.

(a) What is the angular velocity of a tire in radians per second?

(b) At what speed is the bicycle travelling along the ground? (This is the linear velocity of any point on the tire that touches the ground.)

12. A bicycle with tires 70 cm in diameter is travelling along a road at 25 $km\,h^{-1}$. What is the angular velocity of a tire of the bicycle in radians per second?

13. Given that ω is the angular velocity in radians per second of a point on a circle with radius r cm, express the linear velocity, v, in centimetres per second, of the point as a function in terms of ω and r.

14. A chord of length 26 cm is in a circle of radius 20 cm. Find the length of the arc that the chord subtends.

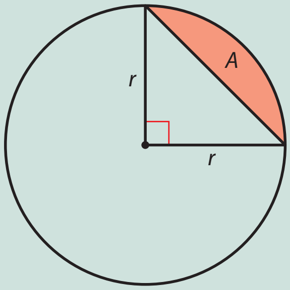
15. A circular irrigation system consists of a 400m pipe that is rotated around a central pivot point. If the irrigation pipe makes one full revolution around the pivot point each day, how much area, in square meters, does it irrigate each hour?

16. **(a)** Find the radius of a circle circumscribed about a regular polygon of 64 sides, if one side measures 3cm.

(b) Calculate the difference between the circumference of the circle and the perimeter of the polygon.

17. What is the area of an equilateral triangle that has an inscribed circle with an area of $50\pi \text{ cm}^2$, and a circumscribed circle with an area of $200\pi \text{ cm}^2$?

18. The sector of the circle here is subtended by two perpendicular radii. The area of the segment is A square units.
Find an expression for the area of the circle in terms of A.



6.2 The unit circle and trigonometric functions

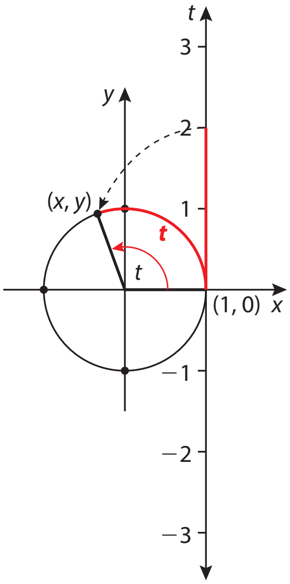


Figure 6.8 The wrapping function

Several important functions can be described by mapping the coordinates of points on the real number line onto the points of the unit circle.

Suppose that the real number line is tangent to the unit circle at the point (1, 0), and that zero on the number line matches with (1, 0) on the circle, as shown in Figure 6.8. Because of the properties of circles, the real number line in this position will be perpendicular to the x-axis. The scales on the number line, the x-axis and the y-axis need to be the same. Imagine that the real number line is flexible like a string and can wrap around the circle, with zero on the number line remaining fixed to the point (1, 0) on the unit circle. When the top portion of the string moves along the circle, the wrapping is anticlockwise ($t > 0$), and when the bottom portion of the string moves along the circle, the wrapping is clockwise ($t < 0$). As the string wraps around the unit circle, each real number t on the string is mapped onto a point (x, y) on the circle. Hence, the real number line from 0 to t makes an arc of length t starting on the circle at (1, 0) and ending at the point (x, y) on the circle. For example, since the circumference of the unit circle is 2π, the number $t = 2\pi$ will be wrapped anticlockwise around the circle to the point (1, 0). Similarly, the number $t = \pi$ will be wrapped anticlockwise halfway around the circle to the point $(-1, 0)$ on the circle. And the number $t = -\frac{\pi}{2}$ will be wrapped clockwise one-quarter of the way around the circle to the point $(0, -1)$ on the circle. Note that each number t on the real number line is mapped (corresponds) to *exactly one* point on the unit circle, thereby satisfying the definition of a function – consequently this mapping is called a **wrapping function**.

A function f such that $f(x) = f(x + p)$ is a **periodic function**. If p is the least positive constant for which $f(x) = f(x + p)$ is true, then p is called the **period** of the function.

Before we leave our mental picture of the string (representing the real number line) wrapping around the unit circle, consider any pair of points on the string that are exactly 2π units from each other. Let these two points represent the real numbers t_1 and $t_1 + 2\pi$. Because the circumference of the unit circle is 2π, these two numbers will be mapped to the same point on the unit circle. Furthermore, consider the infinite number of points whose distance from t_1 is any integer multiple of 2π, i.e. $t_1 + k \cdot 2\pi$, $k \in \mathbb{Z}$, and again all of these numbers will be mapped to the same point on the unit circle. Consequently, the wrapping function is not a one-to-one function. Output for the function (points on the unit circle) is unchanged by the addition of any integer multiple of 2π to any input value (a real number). Functions that behave in such a repetitive (or cyclic) manner are called periodic.

We are surrounded by periodic functions, e.g. average daily temperature; sun rise and the day of the year; animal populations over many years; the height of tides and the position of the moon; and, an electrocardiogram.

Trigonometric functions

The x and y-coordinates of the points on the unit circle can be used to define six **trigonometric functions**: the **sine**, **cosine**, **tangent**, **cosecant**, **secant** and **cotangent** functions. These are often abbreviated as **sin**, **cos**, **tan**, **csc** (or **cosec**), **sec** and **cot** respectively.

Let t be any real number and (x, y) a point on the unit circle to which t is mapped. Then the function definitions are:

$\sin t = y$

$\cos t = x$

$\tan t = \frac{y}{x}, x \neq 0$

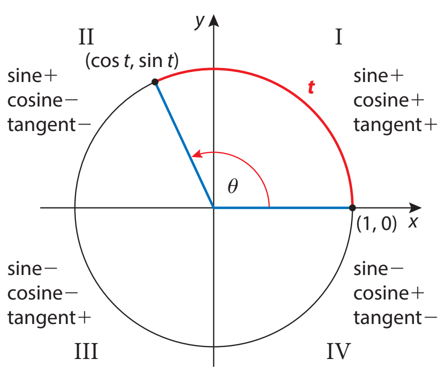
$\csc t = \frac{1}{y}, y \neq 0$

$\sec t = \frac{1}{x}, x \neq 0$

$\cot t = \frac{x}{y}, y \neq 0$

When the real number t is wrapped to a point (x, y) on the unit circle, the value of the y-coordinate is assigned to the sine function; the x-coordinate is assigned to the cosine function; and the ratio of the two coordinates $\frac{y}{x}$ is assigned to the tangent function. Sine, cosine and tangent are often referred to as the **basic trigonometric functions**.

Cosecant secant and cotangent are each a reciprocal of one of the basic trigonometric functions and are often referred to as the **reciprocal trigonometric functions**. All six are defined by means of the length of an arc on the unit circle as follows.



On the unit circle: $x = \cos t$, $y = \sin t$.

Figure 6.9 Signs of the trigonometric functions depend on the quadrant where the arc t terminates.

When trigonometric functions are defined as circular functions based on the unit circle, radian measure is used. The values for the domain of the sine and cosine functions are real numbers that are arc lengths on the unit circle. As we know the arc length on the unit circle subtends an angle in standard position whose radian measure is equivalent to the arc length (see Figure 6.9).

Evaluating the trigonometric functions for any value of t involves finding the coordinates of the point on the unit circle where the arc of length t will 'wrap to' (or terminate) starting at the point (1,0). It is useful to remember that an arc of length π is equal to one half of the circumference of the unit circle. All of the values for t in this example are positive, so the arc length will wrap along the unit circle in an anticlockwise direction.

Example 6.6

Evaluate the six trigonometric functions for each value of t.

(a) $t = 0$ (b) $t = \frac{\pi}{2}$ (c) $t = \pi$

(d) $t = \frac{3\pi}{2}$ (e) $t = 2\pi$

Solution:

(a) An arc of length $t = 0$ has no length so it 'terminates' at the point (1, 0). By definition:

$\sin 0 = y = 0$ $\csc 0 = \frac{1}{y} = \frac{1}{0}$ is undefined

$\cos 0 = x = 1$ $\sec 0 = \frac{1}{x} = \frac{1}{1} = 1$

$\tan 0 = \frac{y}{x} = \frac{0}{1} = 0$ $\cot 0 = \frac{x}{y} = \frac{1}{0}$ is undefined

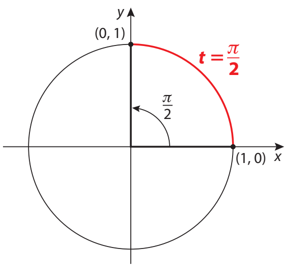


Figure 6.10 Arc length of $\frac{\pi}{2}$ or one-quarter of an anticlockwise revolution.

(b) An arc of length $t = \frac{\pi}{2}$ is equivalent to one-quarter of the circumference of the unit circle (Figure 6.10) so it terminates at the point (0, 1). By definition:

$\sin\frac{\pi}{2} = y = 1$ $\csc\frac{\pi}{2} = \frac{1}{y} = 1$

$\cos\frac{\pi}{2} = x = 0$ $\sec\frac{\pi}{2} = \frac{1}{x}$ is undefined

$\tan\frac{\pi}{2} = \frac{y}{x} = \frac{1}{0}$ is undefined $\cot\frac{\pi}{2} = \frac{x}{y} = 0$

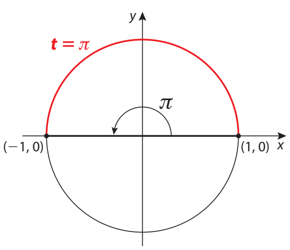


Figure 6.11 Arc length of π or one-half of an anticlockwise revolution.

(c) An arc of length $t = \pi$ is equivalent to one-half of the circumference of the unit circle (Figure 6.11) so it terminates at the point (21, 0). By definition:

$\sin\pi = y = 0$ $\csc\pi = \frac{1}{y}$ is undefined

$\cos\pi = x = -1$ $\sec\pi = \frac{1}{-1} = -1$

$\tan\pi = \frac{y}{x} = \frac{0}{-1} = 0$ $\cot\pi = \frac{x}{y}$ is undefined

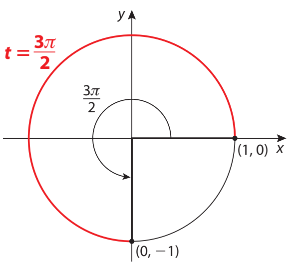


Figure 6.12 Arc length of $\frac{3\pi}{2}$ or three-quarters of an anticlockwise revolution.

(d) An arc of length $t = \frac{3\pi}{2}$ is equivalent to three-quarters of the circumference of the unit circle (Figure 6.12), so it terminates at the point (0, −1). By definition:

$\sin\frac{3\pi}{2} = y = -1$ $\csc\frac{3\pi}{2} = \frac{1}{y} = -1$

$\cos\frac{3\pi}{2} = x = 0$ $\sec\frac{3\pi}{2} = \frac{1}{x}$ is undefined

$\tan\frac{3\pi}{2} = \frac{y}{x} = \frac{-1}{0}$ is undefined $\cot\frac{3\pi}{2} = \frac{x}{y} = 0$

(e) An arc of length $t = 2\pi$ terminates at the same point as an arc of length $t = 0$ (Figure 6.13), so the values of the trigonometric functions are the same as found in part (a):

$\sin 0 = y = 0$ $\quad\quad$ $\csc 0 = \frac{1}{y}$ is undefined

$\cos 0 = x = 1$ $\quad\quad$ $\sec 0 = \frac{1}{x} = 1$

$\tan 0 = \frac{y}{x} = \frac{0}{1} = 0$ $\quad\quad$ $\cot 0 = \frac{x}{y}$ is undefined

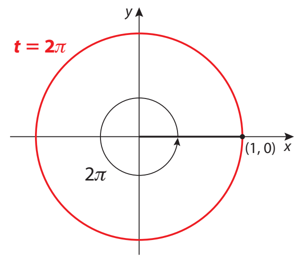


Figure 6.13 : Arc length of 2π or one full anticlockwise revolution.

From our previous discussion of periodic functions, we can conclude that all of the trigonometric functions are periodic. Given that the sine and cosine functions are generated directly from the wrapping function, the period of each of these functions is 2π. That is:

$$\sin t = \sin(t + k \cdot 2\pi),\ k \in \mathbb{Z} \text{ and } \cos t = \cos(t + k \cdot 2\pi),\ k \in \mathbb{Z}$$

Since the cosecant and secant functions are the respective reciprocals of sine and cosine, the period of cosecant and secant will also be 2π.

Initial evidence from Example 6.6 indicates that the period of the tangent function is π. That is,

$$\tan t = \tan(t + k \cdot \pi),\ k \in \mathbb{Z}$$

We will establish these results graphically in the next section. Also note that since the trigonometric functions are periodic (i.e. function values repeat) then they are not one-to-one functions. This is an important fact when establishing inverse trigonometric functions (see Section 6.6).

Domains of the six trigonometric functions

$f(t) = \sin t$ and $f(t) = \cos t$

domain: $\{t : t \in \mathbb{R}\}$

$f(t) = \tan t$ and $f(t) = \sec t$

domain:

$\{t : t \in \mathbb{R}, t \neq \frac{\pi}{2} + k\pi, k \in \mathbb{Z}\}$

$f(t) = \cot t$ and $f(t) = \csc t$

domain:

$\{t : t \in \mathbb{R}, t \neq k\pi, k \in \mathbb{Z}\}$

Evaluating trigonometric functions

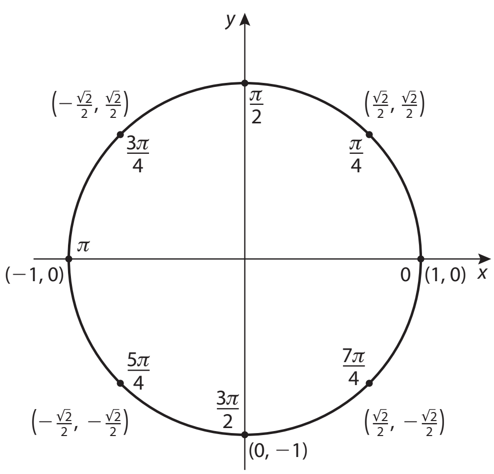


Figure 6.14 Arc lengths that are multiples of $\frac{\pi}{4}$ divide the unit circle into eight equally spaced points.

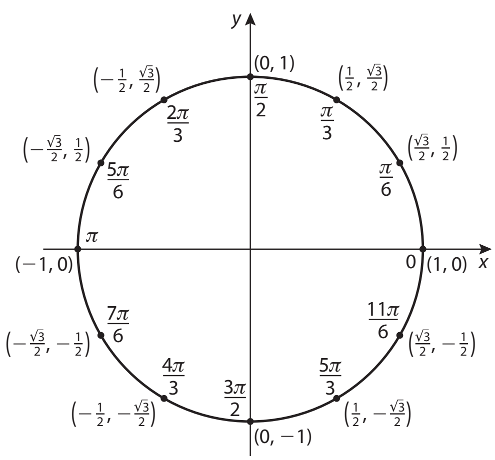


Figure 6.15 Arc lengths that are multiples of $\frac{\pi}{6}$ divide the unit circle into twelve equally spaced points.

You will find it very helpful to memorise the exact values of sine and cosine for numbers that are multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$. Use the unit circle diagrams shown in Figures 6.14 and 6.15 as a guide to help you do this and to

The following four identities follow directly from the definitions for the trigonometric functions.

$\tan t = \frac{\sin t}{\cos t}$ $\csc t = \frac{1}{\sin t}$

$\sec t = \frac{1}{\cos t}$ $\cot t = \frac{\cos t}{\sin t}$

visualise the location of the terminal points of different arc lengths. With the symmetry of the unit circle and a point's location in the coordinate plane telling us the sign of x and y (see Figure 6.9), we only need to remember the sine and cosine of common values of t in the first quadrant and on the positive x- and y-axes. These are organized in Table 6.1.

Table 6.1 The trigonometric functions evaluated for special values of t.

t	$\sin t$	$\cos t$	$\tan t$	$\csc t$	$\sec t$	$\cot t$
0	0	1	0	undefined	1	undefined
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{2}$	1	0	undefined	1	undefined	0

Memorize the values of $\sin t$ and $\cos t$ for the values of t in the red box in Table 6.1. These values can be used to derive the of all six trigonometric functions for any multiple of $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$ or $\frac{\pi}{2}$.

If t is not a multiple of $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$ or $\frac{\pi}{2}$, then the approximate values of the trigonometric functions for that number can be found using your GDC.

If s and t are coterminal arcs (i.e. terminate at the same point), then the trigonometric functions of s are equal to those of t.

For example, the arcs $s = \frac{3\pi}{2}$ and $t = -\frac{\pi}{2}$ are coterminal (Figure 6.16).

Thus, $\sin \frac{3\pi}{2} = \sin\left(-\frac{\pi}{2}\right)$, $\tan\frac{3\pi}{2} = \tan\left(-\frac{\pi}{2}\right)$, etc.

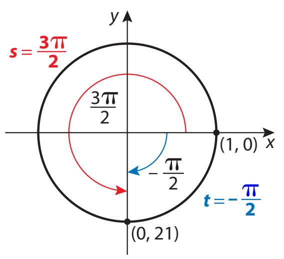
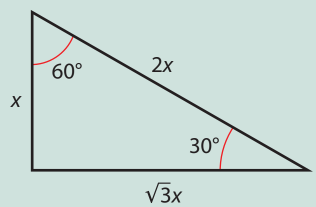


Figure 6.16 coterminal angles $\frac{3\pi}{2}$ and $-\frac{\pi}{2}$

Exercise 6.2

1. **(a)** By knowing the ratios of sides in any triangle with angles measuring 30°, 60° and 90° (see figure), find the coordinates of the points on the unit circle where an arc of length $t = \frac{\pi}{6}$ and $t = \frac{\pi}{3}$ terminate in the first quadrant.

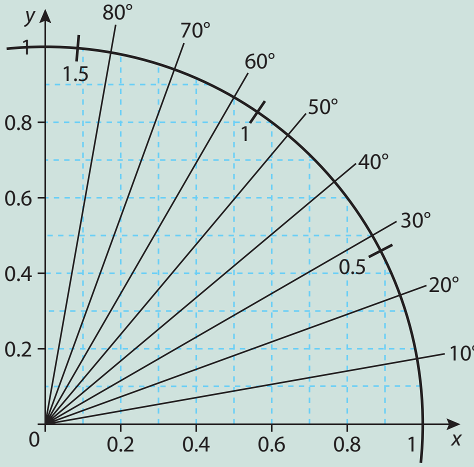


   (b) Using the result from (a) and applying symmetry about the unit circle, find the coordinates of the points on the unit circle corresponding to arcs whose lengths are

   $\frac{2\pi}{3}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{11\pi}{6},$

   Draw a large unit circle and label all of these points with their coordinates and the measure of the arc that terminates at each point.

2. The figure of quadrant I of the unit circle shown right indicates angles in intervals of 100 and also indicates angles in radian measure of 0.5, 1 and 1.5. Use the figure and the definitions of the sine and cosine functions to approximate the function values to one decimal place in questions (a) - (f). Check your answers with your GDC (be sure to be in the correct angle measure mode).



(a) cos 50° **(b)** sin 80° **(c)** cos 1

(d) sin 0.5 **(e)** tan 70° **(f)** cos 1.5

(g) sin 20° **(h)** tan 1

3. t is the length of an arc on the unit circle starting from (1, 0).

(i) State the quadrant in which the terminal point of the arc lies.

(ii) Find the coordinates of the terminal point (x, y) on the unit circle. Give exact values for x and y if possible, otherwise approximate to three significant figures.

(a) $t = \frac{\pi}{6}$ **(b)** $t = \frac{5\pi}{3}$ **(c)** $t = \frac{7\pi}{4}$

(d) $t = \frac{3\pi}{2}$ **(e)** $t = 2$ **(f)** $t = -\frac{\pi}{4}$

(g) $t = -1$ **(h)** $t = -\frac{5\pi}{4}$ **(i)** $t = 3.52$

4. State the exact value (if possible) of the sine, cosine and tangent of the given real number.

(a) $\frac{\pi}{3}$ **(b)** $\frac{5\pi}{6}$ **(c)** $-\frac{3\pi}{4}$

(d) $\frac{\pi}{2}$ **(e)** $-\frac{4\pi}{3}$ **(f)** 3π

(g) $\frac{3\pi}{2}$ **(h)** $-\frac{7\pi}{6}$ **(i)** 1.25π

5. Use the periodic properties of the sine and cosine functions to find the exact value of $\sin x$ and $\cos x$ for each value of x:

(a) $x = \frac{13\pi}{6}$ **(b)** $x = \frac{10\pi}{3}$

(d) $x = \frac{15\pi}{4}$ **(e)** $x = \frac{17\pi}{6}$

6. Find the exact function values, if possible. Do not use your GDC.

(a) $\cos \frac{5\pi}{6}$ (b) $\sin 315°$ (c) $\tan \frac{3\pi}{2}$

(d) $\sec \frac{5\pi}{3}$ (e) $\csc 240°$

7. Find the exact function values, if possible. Otherwise, find the approximate value accurate to three significant figures.

(a) $\sin 2.5$ (b) $\cot 120°$ (c) $\cos \frac{5\pi}{4}$

(d) $\sec 6$ (e) $\tan \pi$

8. Specify in which quadrant(s) an angle θ in standard position could be given the stated conditions.

(a) $\sin \theta > 0$

(b) $\sin \theta > 0$ and $\cos \theta < 0$

(c) $\sin \theta < 0$ and $\tan \theta > 0$

(d) $\cos \theta < 0$ and $\tan \theta < 0$

(e) $\cos \theta > 0$

(f) $\sec \theta > 0$ and $\tan \theta > 0$

(g) $\cos \theta > 0$ and $\csc \theta < 0$

(h) $\cot \theta < 0$

6.3 Graphs of trigonometric functions

```
sin(2.53)
      .5741721484
sin(2.53+2π)
      .5741721484
sin(2.53+4π)
      .5741721484
```

The period of $y = \sin x$ is 2π.

From the previous section we know that trigonometric functions are periodic, i.e. their values repeat in a regular manner. The graphs of the trigonometric functions should provide a picture of this periodic behaviour. In this section we will graph the sine, cosine and tangent functions and transformations of the sine and cosine functions.

Graphs of the sine and cosine functions

Since the period of the sine function is 2π, we know that two values of t (domain) that differ by 2π will produce the same value for y (range). This means that any portion of the graph of $y = \sin t$ with a t-interval of length 2π (called one period or cycle of the graph) will repeat. Remember that the domain of the sine function is all real numbers, so one period of the graph of $y = \sin t$ will repeat indefinitely in the positive and negative directions.

Therefore, in order to construct a complete graph of $y = \sin t$, we need to graph just one period of the function; for example, from $t = 0$ to $t = 2\pi$, and then repeat the pattern in both directions.

We know from section 6.2 that $\sin t$ is the y-coordinate of the terminal point on the unit circle corresponding to the real number t (Figure 6.17). In order to generate one period of the graph of $y = \sin t$, we need to record the y-coordinates of a point on the unit circle and the corresponding value of t as the point travels anticlockwise one revolution, starting from the point (1, 0). These values are then plotted on a graph with t on the horizontal axis and y (i.e. $\sin t$) on the vertical axis. Figure 6.18 illustrates this process in a sequence of diagrams.

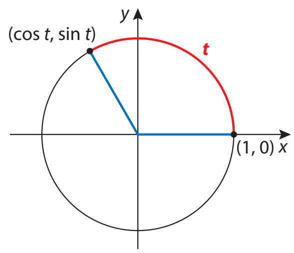


Figure 6.17 Coordinates of the terminal point of arc t gives the values of $\cos t$ and $\sin t$.

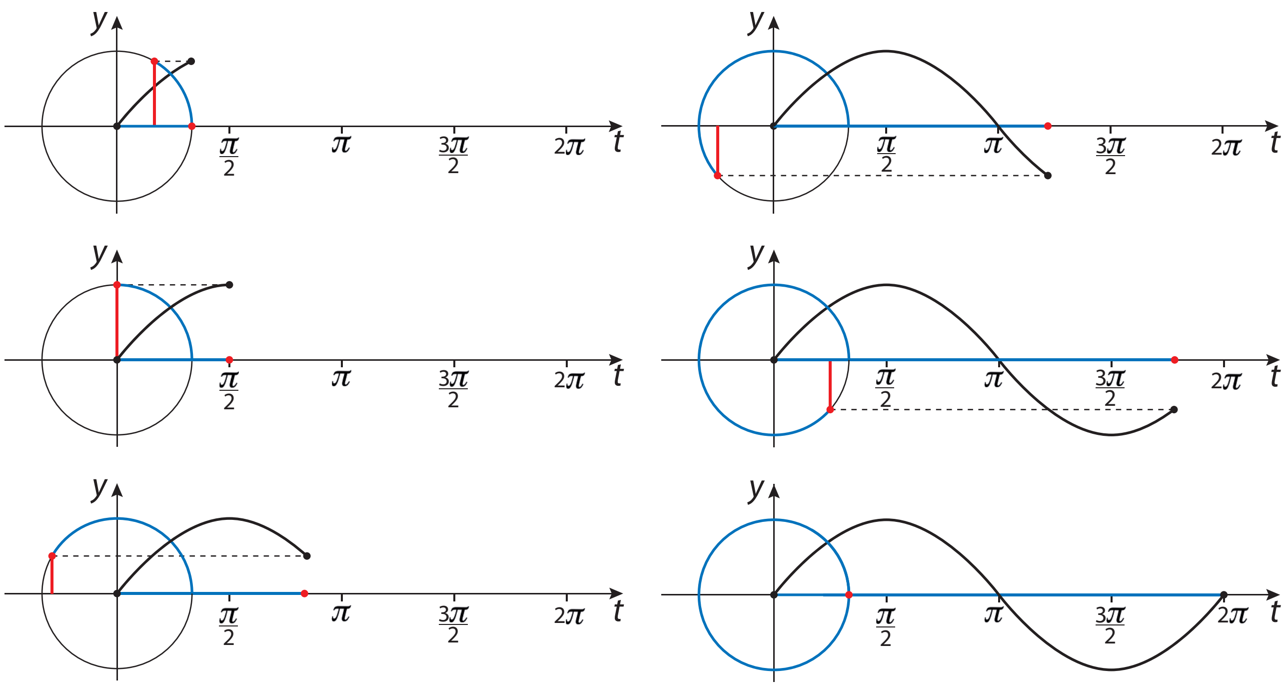


Figure 6.18 Graph of the sine function for $0 \le t \le 2\pi$ generated from a point travelling along the unit circle.

As the point $(\cos t, \sin t)$ travels along the unit circle, the x-coordinate (i.e. $\cos t$) goes through the same cycle of values as does the y-coordinate ($\sin t$). The only difference is that the x-coordinate begins at a different value in the cycle – when $t = 0$, $y = 0$, but $x = 1$. The result is that the graph of $y = \cos t$ is the exact same shape as $y = \sin t$ but it has been shifted to the left $\frac{\pi}{2}$ units. The graph of $y = \cos t$ for $0 \le t \le 2\pi$ is shown in Figure 6.20.

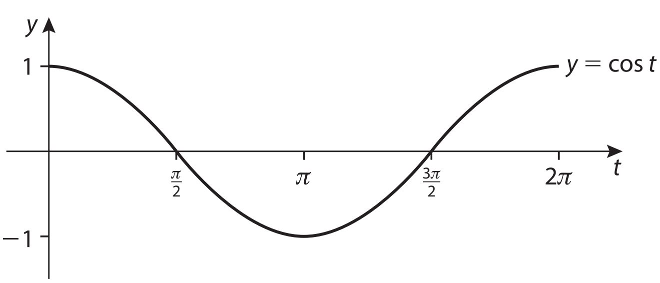
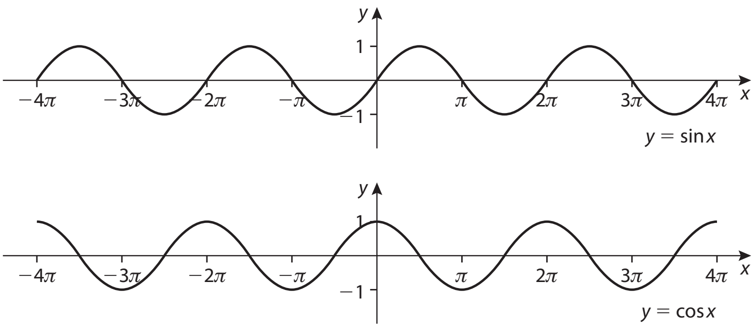


Figure 6.19 Graph of $y = \cos t$ for $0 \le t \le 2\pi$.

The convention is to use the letter x to denote the variable in the domain of the function. Hence from here on, we will use the letter x rather than t and write the trigonometric functions as $y = \sin x$, $y = \cos x$ and $y = \tan x$.

Figure 6.20 $y = \sin x$ and $y = \cos x$, $0 \leq x \leq 4\pi$.



A function is odd if, for each x in the domain of f, $f(-x) = -f(x)$. The graph of an odd function is symmetric with respect to the origin (rotational symmetry). A function is even if, for each x in the domain of f, $f(-x) = f(x)$. The graph of an even function is symmetric with respect to the x-axis (line symmetry).

Aside from their periodic behaviour, these graphs reveal further properties of the functions $y = \sin x$ and $y = \cos x$. Note that the sine function has a maximum value of $y = 1$ for all $x = \frac{\pi}{2} + k \cdot 2\pi, k \in \mathbb{Z}$, and has a minimum value of $y = -1$ for all $x = -\frac{\pi}{2} + k \cdot 2\pi, k \in \mathbb{Z}$. The cosine function has a maximum value of $y = 1$ for all $x = k \cdot 2\pi, k \in \mathbb{Z}$, and has a minimum value of $y = -1$ for all $x = \pi + k \cdot 2\pi, k \in \mathbb{Z}$. This also confirms that both functions have a domain of all real numbers and a range of $-1 \leqslant y \leqslant 1$.

Closer inspection of the graphs, in Figure 6.20, shows that the graph of $y = \sin x$ has rotational symmetry about the origin, that is, it can be rotated 180° about the origin and it remains the same. This graph symmetry can be expressed with the identity: $\sin(-x) = -\sin x$. For example, $\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$ and $-\left[\sin\left(\frac{\pi}{6}\right)\right] = -\left[\frac{1}{2}\right] = -\frac{1}{2}$. A function that is symmetric about the origin is called an **odd function**. The graph of $y = \cos x$ has line symmetry over the x-axis – that is, it remains the same when reflected over the x-axis. This graph symmetry can be expressed with the identity: $\cos(-x) = \cos x$. For example, $\cos\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ and $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$.

A function that is symmetric about the y-axis is called an **even function**.

The sine function is odd because $\sin(-x) = -\sin(x)$, and the cosine function is even because $\cos(-x) = \cos(x)$.

Mathematics

Analysis and Approaches

HIGHER LEVEL

For the IB Diploma

Mathematics Analysis and Approaches for the IB Diploma Higher Level is a comprehensive textbook covering the 2019 curriculum.
The book also includes the following features:

- written by an expert authoring team
- additional integrated digital content including GeoGebra applets created specifically for the course
- worked examples to help you tackle questions
- practice questions to help you prepare for the exam
- rich and wide-ranging chapter on Mathematics in Theory of Knowledge
- guidance on internal assessment

Ibrahim Wazir taught Mathematics at Vienna International School, the American International School in Vienna and at Webster University, Vienna. He has been an IB Mathematics Assistant Examiner, Chief Examiner and a member of the Curriculum Committee. He has also run IB workshops for new and experienced teachers. Ibrahim founded the International Schools Math Teachers Foundation (ISMTF) and inScholastics, an organisation that runs workshops for students and teachers in Vienna.

Tim Garry has taught Mathematics from middle school to university since 1984 and has been Head of Maths at three international schools. He has been an IB examiner for external and internal assessment, a contributor to IB curriculum reviews, a workshop leader, a member of the HL Mathematics exam writing team and he also edits an IB Maths website for teachers. Tim frequently delivers workshops and conference presentations and has co-authored with Ibrahim two previous editions of Mathematics SL and HL textbooks. He currently teachers at the International School of Aberdeen, Scotland.

www.pearsonbaccalaureate.com