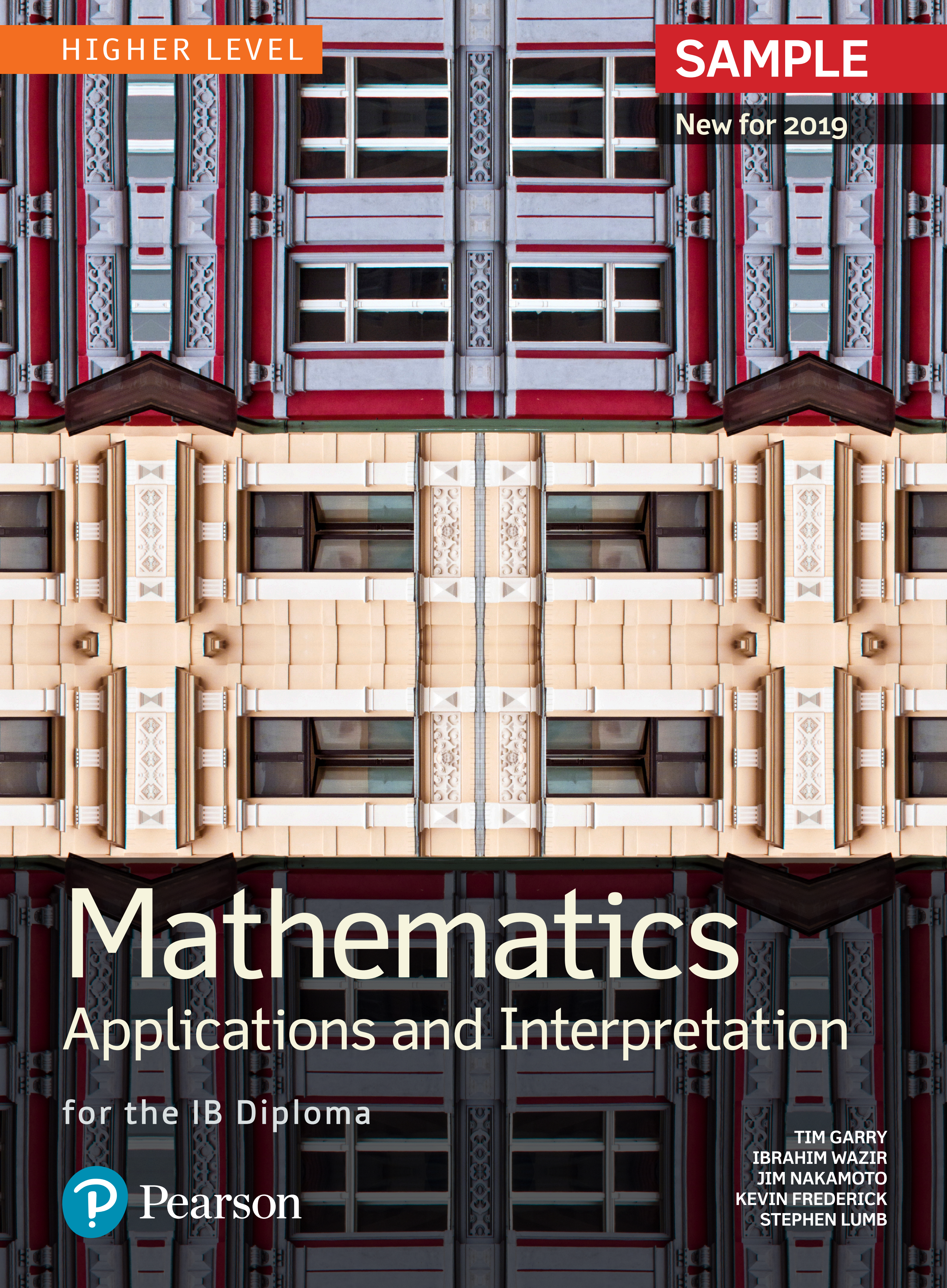


HIGHER LEVEL

SAMPLE

New for 2019

Mathematics
Applications and Interpretation
for the IB Diploma

TIM GARRY
IBRAHIM WAZIR
JIM NAKAMOTO
KEVIN FREDERICK
STEPHEN LUMB

Pearson

Contents

7

Matrix algebra

gettyimages
Vectorig

7 Matrix algebra

Learning objectives

By the end of this chapter, you should be familiar with...

- a matrix, its order, and elements; identity and zero matrices
- the algebra of matrices: equality, addition, subtraction, and multiplication by a scalar
- multiplying matrices manually and using technology
- calculating the determinant of a 2×2 and a 3×3 square matrix
- the inverse of a 2×2 matrix and using technology to find the inverse of $n \times n$ matrices
- the conditions for the existence of the inverse of a matrix
- the solution of systems of linear equations using inverse matrices (a maximum of three equations in three unknowns)
- eigenvectors and eigenvalues and how to find them for 2×2 matrices
- characteristic polynomials for 2×2 matrices
- diagonalizing 2×2 matrices and applying to powers of such matrices
- geometric transformations of points in two dimensions using matrices: reflections, horizontal and vertical dilations, translations, and rotations
- applications of transformations to fractals.

Matrices have been, and remain, significant mathematical tools. Uses of matrices span several areas, from simply solving systems of simultaneous linear equations to describing atomic structure, designing computer game graphics, analysing relationships, coding, and operations research. If you have ever used a spreadsheet program, or have ever created a table, then you have used a matrix. Matrices make the presentation of data understandable and help make calculations easy to perform. For example, your teacher's grade book may look something like this.

Student	Quiz 1	Quiz 2	Test 1	Test 2	Homework	Grade
Tim	70	80	86	82	95	A
Maher	89	56	80	60	55	C
⋮	⋮	⋮	⋮	⋮	⋮	⋮

If we want to know Tim's grade on Test 2, we simply follow along the row 'Tim' to the column 'Test 2' and find that he achieved a mark of 82. Take a look at the matrix below about the number of cameras sold at shops in four cities.

	Venice	Rome	Budapest	Prague
Digital compact	153	98	74	56
Digital standard	211	120	57	29
DSLR	82	31	12	5
Other	308	242	183	107

If we want to know how many digital standard cameras were sold in the Budapest shop, we follow along the row 'Digital standard' to the column 'Budapest' and find that 57 digital standard cameras were sold.

7.1 Matrix definitions and operations

What is a matrix?

A matrix is a rectangular array of elements. The elements can be symbolic expressions or numbers.

Matrix **A** is denoted by

$$\mathbf{A} = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{n \text{ columns}} \begin{matrix} \leftarrow \\ \leftarrow \\ \vdots \\ \leftarrow \end{matrix} \Bigg\} m \text{ rows}$$

Row i of **A** has n elements and is $(a_{i1} \quad a_{i2} \quad \cdots \quad a_{in})$

Column j of **A** has m elements and is $\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}$

The number of rows and columns of the matrix defines its size (order). So, a matrix that has m rows and n columns is said to have an $m \times n$ (m by n) order. A matrix **A** with $m \times n$ order is sometimes denoted as $[\mathbf{A}]_{m \times n}$ or $[\mathbf{A}]_{mn}$ to show that **A** is a matrix with m rows and n columns. (Sometimes $[a_{ij}]$ is used to represent a matrix.) The camera sales matrix has a 4×4 order. When $m = n$, the matrix is said to be a square matrix with order n, so the camera sales matrix is a square matrix of order 4.

Every entry in a matrix is called an **entry** or **element** of the matrix and is denoted by a_{ij}, where i is the row number and j is the column number of that element. The ordered pair (i, j) is also called the **address** of the element. So, in the grade book matrix example, the entry (2, 4) is 60, the student Maher's grade on Test 2, while (2, 4) in the camera sales matrix example is 29, the number of digital standard cameras sold in the Prague shop.

7 Matrix algebra

Vectors

A vector is a matrix that has only one row or one column. There are two types of vector: row vectors and column vectors.

Row vector

If a matrix has one row, it is called a row vector.

$\mathbf{B} = (b_1 \quad b_2 \quad \dots \quad b_m)$ is a row vector with **dimension** m.

$\mathbf{B} = (1 \quad 2)$ could represent the position of a point in a plane and is an example of a row vector of dimension 2.

Column vector

If a matrix has one column, it is called a column vector.

$\mathbf{C} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$ is a column vector with dimension n.

$\mathbf{C} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ again could represent the position of a point in a plane and is an example of a column vector of dimension 2.

Vectors can be represented by row or column matrices.

Submatrix

If some row(s) and/or column(s) of a matrix **A** are deleted, the remaining matrix is called a **submatrix** of **A**.

For example, if we are interested in the sales of only the three main types of camera and only in Italian cities, we can represent them with the following submatrix of the original matrix

$$\begin{pmatrix} 153 & 98 \\ 211 & 120 \\ 82 & 31 \end{pmatrix}$$

Submatrix

$$\begin{pmatrix} \mathbf{153} & \mathbf{98} & 74 & 56 \\ \mathbf{211} & \mathbf{120} & 57 & 29 \\ \mathbf{82} & \mathbf{31} & 12 & 5 \\ 308 & 242 & 183 & 107 \end{pmatrix}$$

Original matrix

Zero matrix

A matrix for which all entries are equal to zero, ($a_{ij} = 0$ for all i and j)

Some zero matrix examples: $(0 \quad 0)$ $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Diagonal

In a square matrix, the entries $a_{11}, a_{22}, \dots, a_{nn}$ are called the **diagonal elements** of the matrix. Sometimes the diagonal of the matrix is also called the **principal** or **main diagonal** of the matrix.

What is the diagonal in our camera sales matrix? Here $a_{11} = 153$, $a_{22} = 120$, $a_{33} = 12$, and $a_{44} = 107$

$$\begin{pmatrix} 153 & 0 & 0 & 0 \\ 0 & 120 & 0 & 0 \\ 0 & 0 & 12 & 0 \\ 0 & 0 & 0 & 107 \end{pmatrix}$$

Triangular matrix

You can use a matrix to show distances between different cities.

	Graz	Salzburg	Innsbruck	Linz
Vienna	191	298	478	185
Graz		282	461	220
Salzburg			188	135
Innsbruck				320

Table 7.1 Distance (in km) between four Austrian cities.

The data in Table 7.1 can be represented by a triangular matrix. It is an upper triangular matrix, in this case.

$$\begin{pmatrix} 191 & 298 & 478 & 185 \\ 0 & 282 & 461 & 220 \\ 0 & 0 & 188 & 135 \\ 0 & 0 & 0 & 320 \end{pmatrix}$$

In a triangular matrix, the entries on one side of its diagonal are all zero.

A triangular matrix is a square matrix with order n for which $a_{ij} = 0$ when $i > j$ (upper triangular) or alternatively when $i < j$ (lower triangular).

Another way of representing the distance data is given by the following matrix.

	Vienna	Graz	Salzburg	Innsbruck	Linz
Vienna	0	191	298	478	185
Graz	191	0	282	461	220
Salzburg	298	282	0	188	135
Innsbruck	478	461	188	0	320
Linz	185	220	135	320	0

Again, the data in the table can be represented by a matrix called a **symmetric** matrix. In such matrices, $a_{ij} = a_{ji}$ for all i and j. All symmetric matrices are square.

$$\begin{pmatrix} 0 & 191 & 298 & 478 & 185 \\ 191 & 0 & 282 & 461 & 220 \\ 298 & 282 & 0 & 188 & 135 \\ 478 & 461 & 188 & 0 & 320 \\ 185 & 220 & 135 & 320 & 0 \end{pmatrix}$$

Matrix operations

Equal matrices

Two matrices **A** and **B** are equal if the size of **A** and **B** is the same (number of rows and columns are the same for **A** and **B**) and $a_{ij} = b_{ij}$ for all i and j.

For example, $\begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix}$ and $\begin{pmatrix} 2 & x \\ x^2 - 4 & 7 \end{pmatrix}$ are equal only if $x = 3$ and $x^2 - 4 = 5$

which can only be true if $x = 3$

Adding and subtracting matrices

We can add two matrices **A** and **B** only if they are the same size. If **C** is the sum of the two matrices, then $\mathbf{C} = \mathbf{A} + \mathbf{B}$ where $c_{ij} = a_{ij} + b_{ij}$, so we add corresponding terms, one by one.

For example

$$\begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix} + \begin{pmatrix} x & y \\ a & b \end{pmatrix} = \begin{pmatrix} 2 + x & 3 + y \\ 5 + a & 7 + b \end{pmatrix}$$

We carry out subtraction in a similar way

$$\begin{pmatrix} 2 & 3 & 1 \\ 5 & 7 & 0 \end{pmatrix} - \begin{pmatrix} x & y & 8 \\ a & b & 2 \end{pmatrix} = \begin{pmatrix} 2-x & 3-y & -7 \\ 5-a & 7-b & -2 \end{pmatrix}$$

The operations of addition and subtraction of matrices obey all rules of algebraic addition and subtraction.

Multiplying a matrix by a scalar

A scalar is any object that is not a matrix. You multiply each term of the matrix by the scalar.

A is an $m \times n$ matrix, and c is a scalar. The scalar product of c and **A** is another matrix **B** $= c$**A**, such that every entry b_{ij} of **B** is a multiple of its corresponding entry in **A**. So, for every entry in **B**, we have $b_{ij} = c \times a_{ij}$

It is often convenient to rewrite the scalar multiple c**A** by factoring c out of every entry in the matrix. For instance, in the matrix below, the scalar $\frac{1}{2}$ has been factored out of the matrix.

$$\begin{pmatrix} \frac{1}{2} & -\frac{3}{2} \\ \frac{5}{2} & \frac{1}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -3 \\ 5 & 1 \end{pmatrix}$$

Matrix multiplication

At first glance, the following definition may seem unusual. You will see later, however, that this definition of the product of two matrices has many practical applications.

A $= [a_{ij}]$ is an $m \times n$ matrix and **B** $= [b_{ij}]$ is an $n \times p$ matrix. The product **AB** is an $m \times p$ matrix **AB** $= [c_{ij}]$ where

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \ldots + a_{in} b_{nj}$$

for each $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$

For the product of two matrices to be defined, the number of columns in the first matrix must be the same as the number of rows in the second matrix.

A	B	= AB
$m \times n$	$n \times p$	$m \times p$

n and n: equal

m and p: order of AB

This definition means that each entry with an address ij in the product **AB** is obtained by multiplying the entries in the ***i*th row** of **A** by the corresponding entries in the ***j*th column** of **B** and then adding the results:

$$c_{ij} = \begin{pmatrix} a_{i1} & a_{i2} & \ldots & a_{in} \end{pmatrix} \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{pmatrix} = a_{i1} b_{1j} + a_{i2} b_{2j} + \ldots + a_{in} b_{nj}$$

Example 7.1

Find **C** = **AB** when **A** $= \begin{pmatrix} 3 & -5 & 2 \\ 2 & 1 & 7 \end{pmatrix}$ and **B** $= \begin{pmatrix} 3 & -2 & 1 & 5 \\ 5 & 8 & -4 & 0 \\ -9 & 10 & 5 & 3 \end{pmatrix}$

Solution

A is a 2×3 matrix, **B** is a 3×4 matrix, so the product will be a 2×4 matrix. Every entry in the product is the result of multiplying the entries in the rows of **A** and columns of **B**. For example

$$c_{12} = \sum_{k=1}^{3} a_{1k} b_{k2} = (a_{11} \quad a_{12} \quad a_{13}) \begin{pmatrix} b_{12} \\ b_{22} \\ b_{32} \end{pmatrix} = (3 \quad -5 \quad 2) \begin{pmatrix} -2 \\ 8 \\ 10 \end{pmatrix}$$

$$= 3 \times (-2) - 5 \times 8 + 2 \times 10 = -26$$

and

$$c_{23} = \sum_{k=1}^{3} a_{2k} b_{k3} = (a_{21} \quad a_{22} \quad a_{23}) \begin{pmatrix} b_{13} \\ b_{23} \\ b_{33} \end{pmatrix} = (2 \quad 1 \quad 7) \begin{pmatrix} 1 \\ -4 \\ 5 \end{pmatrix}$$

$$= 2 \times 1 + 1 \times (-4) + 7 \times 5 = 33$$

Repeat the operation for each entry in the solution matrix to get:

$$\mathbf{C} = \mathbf{AB} = \begin{pmatrix} -34 & -26 & 33 & 21 \\ -52 & 74 & 33 & 31 \end{pmatrix}$$

We can also use our GDC to find the product.

```
[A][B]
[[-34 -26 33 21…
 [-52 -74 33 31…
```

Here are some examples of matrix multiplication. Multiplying a 2×3 matrix by a 3×2 matrix results in a 2×2 product matrix.

$$\begin{pmatrix} 5 & 0 & 3 \\ -2 & 1 & 2 \end{pmatrix} \begin{pmatrix} -2 & 4 \\ 1 & -1 \\ 3 & -2 \end{pmatrix} = \begin{pmatrix} -1 & 14 \\ 11 & -13 \end{pmatrix}$$

$$2 \times 3 \qquad 3 \times 2 \qquad 2 \times 2$$

When matrices are the same size, the product is the same size.

$$\begin{pmatrix} 4 & -5 \\ 1 & 7 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & -5 \\ 1 & -7 \end{pmatrix}$$

$$2 \times 2 \qquad 2 \times 2 \qquad 2 \times 2$$

$$\begin{pmatrix} 5 & 0 & 3 \\ -2 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} -\frac{1}{7} & -\frac{3}{7} & \frac{3}{7} \\ -\frac{10}{7} & -\frac{9}{7} & \frac{16}{7} \\ \frac{4}{7} & \frac{5}{7} & -\frac{5}{7} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$3 \times 3 \qquad 3 \times 3 \qquad 3 \times 3$$

When a matrix of order 2 is multiplied by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, the product is the original matrix. The matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is called the **identity** matrix of order 2.

The **identity matrix** of order n is a diagonal matrix where $a_{ii} = 1$

Two further identity matrices are $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Sometimes, the identity matrix is denoted simply by I, or by I_n, where n is the order. So, the identity matrix with three rows and columns is I_3, and the identity matrix with four rows and columns is I_4.

Example 7.2

Let $\mathbf{A} = (2 \quad -1 \quad 3)$ and $\mathbf{B} = \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix}$

Work out

(a) **AB** (b) **BA**

Solution

(a) $(2 \quad -1 \quad 3)\begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix} = 2 \times 2 + (-1) \times 5 + 3 \times 4 = 11$

(b) $\begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix}(2 \quad -1 \quad 3) = \begin{pmatrix} 2 \times 2 & 2 \times (-1) & 2 \times 3 \\ 5 \times 2 & 5 \times (-1) & 5 \times 3 \\ 4 \times 2 & 4 \times (-1) & 4 \times 3 \end{pmatrix} = \begin{pmatrix} 4 & -2 & 6 \\ 10 & -5 & 15 \\ 8 & -4 & 12 \end{pmatrix}$

Note that the order of multiplication affects the product. Matrix multiplication, in general, is **not commutative**. It is usually not true that **AB** = **BA**

Let $\mathbf{A} = \begin{pmatrix} 3 & 6 \\ 5 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -2 & 3 \\ 1 & 5 \end{pmatrix}$, then $\mathbf{AB} = \begin{pmatrix} 3 & 6 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} -2 & 3 \\ -1 & 5 \end{pmatrix} = \begin{pmatrix} -0 & 39 \\ -8 & 25 \end{pmatrix}$

but $\mathbf{BA} = \begin{pmatrix} -2 & 3 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} 3 & 6 \\ 5 & 2 \end{pmatrix} = \begin{pmatrix} 9 & -6 \\ 28 & 16 \end{pmatrix} \Rightarrow \mathbf{AB} \neq \mathbf{BA}$

However, there are some special cases where matrix multiplication is commutative. For example

$\mathbf{A} = \begin{pmatrix} 3 & 6 \\ 5 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & 6 \\ 5 & 1 \end{pmatrix}$, then $\mathbf{AB} = \begin{pmatrix} 3 & 6 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} 2 & 6 \\ 5 & 1 \end{pmatrix} = \begin{pmatrix} 36 & 24 \\ 20 & 32 \end{pmatrix}$ and

$\mathbf{BA} = \begin{pmatrix} 2 & 6 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} 3 & 6 \\ 5 & 2 \end{pmatrix} = \begin{pmatrix} 36 & 24 \\ 20 & 32 \end{pmatrix} \Rightarrow \mathbf{AB} = \mathbf{BA}$

Multiplying by an identity matrix is also commutative.

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

Example 7.3

Use the information given in the table to set up a matrix to find the camera sales in each city.

	Venice	Rome	Budapest	Prague
Digital compact	153	98	74	56
Digital standard	211	120	57	29
DSLR	82	31	12	5
Other	308	242	183	107

The average selling price for each type of camera is as follows:

Digital compact €1200; Digital standard €1100; DSLR €900; Other €600

Solution

We set up a matrix multiplication in which the individual camera sales are multiplied by the corresponding price. Since the rows represent the sales of the different types of camera, create a row matrix of the different prices and perform the multiplication.

$$\begin{pmatrix} 1200 & 1100 & 900 & 600 \end{pmatrix}\begin{pmatrix} 153 & 98 & 74 & 56 \\ 211 & 120 & 57 & 29 \\ 82 & 31 & 12 & 5 \\ 308 & 242 & 183 & 107 \end{pmatrix}$$

$$= \begin{pmatrix} 674\,300 & 422\,700 & 272\,100 & 167\,800 \end{pmatrix}$$

So, the sales from each city are

	Venice	Rome	Budapest	Prague
Sales	674 300	422 700	272 100	167 800

Remember that we are multiplying a 1×4 matrix with a 4×4 matrix and hence we get a 1×4 matrix.

Exercise 7.1

1. Consider the matrices

$$\mathbf{A} = \begin{pmatrix} -2 & x \\ y-1 & 3 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} x+1 & -3 \\ 4 & y-2 \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} 1 & 2x & -1 \\ 2 & 3 & 0 \end{pmatrix} \qquad \mathbf{D} = \begin{pmatrix} 1 & 2 \\ 2x & 3 \\ -1 & 0 \end{pmatrix}$$

(a) Evaluate

(i) $\mathbf{A} + \mathbf{B}$ **(ii)** $3\mathbf{A} - \mathbf{B}$ **(iii)** $\mathbf{A} + \mathbf{C}$

(b) Find x and y such that $\mathbf{A} = \mathbf{B}$

(c) Find x and y such that $\mathbf{A} + \mathbf{B}$ is a diagonal matrix.

(d) Find **AB** and **BA**

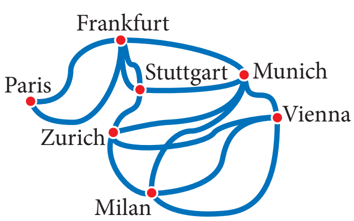
(e) Find x and y such that $\mathbf{C} = \mathbf{D}$

2. Solve for the variables:

(a) $\begin{pmatrix} 3 & 0 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ -12 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & p \\ 3 & q \end{pmatrix}\begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 18 \\ -8 \end{pmatrix}$

(c) $\begin{pmatrix} 3 & -6 \\ 5 & 7 \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 6 & -4 \end{pmatrix}$



3. The diagram shows the major highways connecting some European cities: Vienna (V), Munich (M), Frankfurt (F), Stuttgart (S), Zurich (Z), Milan (L), and Paris (P).

The partially completed matrix below shows the number of direct routes between these cities.

(a) Use the diagram to copy and complete the matrix.

	V	M	F	S	Z	L	P
V	0	1	0	0	1	2	0
M							
F							
S							
Z							
L							
P							

(b) Multiply the matrix from part **(a)** by itself and interpret what it signifies.

4. Consider the matrices

$$\mathbf{A} = \begin{pmatrix} 2 & 5 & 1 \\ 0 & -3 & 2 \\ 7 & 0 & -1 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} m & -2 \\ 3m & -1 \\ 2 & 3 \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} x-1 & 5 & y \\ 0 & -x & y+1 \\ 2x+y & x-3y & 2y-x \end{pmatrix}$$

(a) Find $\mathbf{A} + \mathbf{C}$

(b) Find $\mathbf{AB}$

(c) Find $\mathbf{BA}$

(d) Solve for x and y if $\mathbf{A} = \mathbf{C}$

(e) Find $\mathbf{B} + \mathbf{C}$

(f) Solve for m if $3\mathbf{B} + 2\begin{pmatrix} -1 & m^2 \\ -5 & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 7 & 12 \\ 17 & 1 \\ 2m+2 & 7 \end{pmatrix}$

5. Find a, b, and c so that the following equation is true.

$$2\begin{pmatrix} a-1 & b \\ c+2 & 3 \end{pmatrix} + \begin{pmatrix} 3 & -1 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} -5 & 5 \\ 8 & c+9 \end{pmatrix}$$

6. Find x and y so that the following equation is true.

$$\begin{pmatrix} 2 & -3 \\ -5 & 7 \end{pmatrix}\begin{pmatrix} x-11 & 1-x \\ -5 & x+2y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

7. Find m and n so that the following equation is true.

$$\begin{pmatrix} m^2-1 & m+2 \\ 5 & -2 \end{pmatrix} = \begin{pmatrix} 3 & n+1 \\ 5 & n-5 \end{pmatrix}$$

8. There are two shops in your area. Your shopping list consists of 2 kg of tomatoes, 500 g of meat, and 3 litres of milk. Prices differ between the different shops, and it is difficult to switch between shops to make certain you are paying the least amount of money. A better strategy is to check where you pay less on average. The prices of the different items are given in the table. Which shop should you go to?

Product	Price in shop A	Price in shop B
Tomatoes	€1.66/kg	€1.58/kg
Meat	€2.55/100 g	€2.6/100 g
Milk	€0.90/litre	€0.95/litre

9. Consider the matrices

$$\mathbf{A} = \begin{pmatrix} 2 & 0 \\ -5 & 1 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 3 & -1 \\ 1 & 4 \end{pmatrix} \qquad \mathbf{C} = \begin{pmatrix} -3 & 5 \\ 2 & 7 \end{pmatrix}$$

(a) Find $\mathbf{A} + (\mathbf{B} + \mathbf{C})$ and $(\mathbf{A} + \mathbf{B}) + \mathbf{C}$

(b) Make a conjecture about the addition of 2×2 matrices observed in part **(a)** and prove it.

(c) Find $\mathbf{A}(\mathbf{BC})$ and $(\mathbf{AB})\mathbf{C}$

(d) Make a conjecture about the multiplication of 2×2 matrices observed in part **(c)** and prove it.

10. A company sells air conditioning units, electric heaters and humidifiers. Row matrix **A** represents the number of units sold of each appliance last year, and matrix **B** represents the profit margin for each unit. Find **AB** and describe what this product represents.

$$\mathbf{A} = (235 \quad 562 \quad 117) \qquad \mathbf{B} = \begin{pmatrix} €120 \\ €95 \\ €56 \end{pmatrix}$$

11. Find r and s such that $r\mathbf{A} + \mathbf{B} = \mathbf{A}$ is true, where

$$\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} -12 & -17 \\ s - 8 & -13 \end{pmatrix}$$

12. Let $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

(a) Find

(i) $\mathbf{A}^2$ **(ii)** $\mathbf{A}^3$ **(iii)** $\mathbf{A}^4$ **(iv)** $\mathbf{A}^n$

Let $\mathbf{B} = \begin{pmatrix} 3 & 3 \\ 0 & 3 \end{pmatrix}$

(b) Find

(i) $\mathbf{B}^2$ **(ii)** $\mathbf{B}^3$ **(iii)** $\mathbf{B}^4$ **(iv)** $\mathbf{B}^n$

13. Solve for x and y such that $\mathbf{AB} = \mathbf{BA}$ when

$$\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} x & 2 \\ y & 3 \end{pmatrix}$$

14. Solve for x and y such that $\mathbf{AB} = \mathbf{BA}$ when

$$\mathbf{A} = \begin{pmatrix} 3 & x \\ -2 & 1 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 5 & 2 \\ y & 1 \end{pmatrix}$$

15. Solve for x such that $\mathbf{AB} = \mathbf{BA}$ when

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ x & 2 & -3 \\ 1 & 0 & 4 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} -8 & x+3 & 12 \\ 23 & x-6 & -18 \\ 2 & -2 & 8 \end{pmatrix}$$

16. Solve for x and y such that $\mathbf{AB} = \mathbf{BA}$ when

$$\mathbf{A} = \begin{pmatrix} y & 2 & y+2 \\ x & 2 & -3 \\ 1 & y-1 & 4 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} -8 & x+3 & 12 \\ 23 & x-6 & -18 \\ 2 & -2 & 8 \end{pmatrix}$$

7.2 Applications to systems

There is a wide range of applications of matrices in solving systems of equations.

Recall from algebra that the equation of a straight line can take the form

$ax + by = c$ where a, b, and c are constants, and x and y are variables.

We say this is a linear equation in two variables. Similarly, the equation of a plane in three-dimensional space has the form

$ax + by + cz = d$ where a, b, c, and d are constants, and x, y, and z are variables.

We say that this is a linear equation in three variables.

A solution of a linear equation in n variables (in this case 2 or 3) is an ordered set of real numbers (x_0, y_0, z_0) so that the equation in question is satisfied when these values are substituted for the corresponding variables. For example, the equation

$x + 2y = 4$ is satisfied when $x = 2$ and $y = 1$

Some other solutions are: $x = -4$ and $y = 4$
$x = 0$ and $y = 2$
$x = -2$ and $y = 3$

The set of all solutions of a linear equation is its solution set, and when this set is found, the equation is said to have been solved. To describe the entire solution set we often use a **parametric representation**, as illustrated in the following examples.

Example 7.4

Solve the linear equation $x + 2y = 4$

Solution

To find the solution set of an equation in two variables, we solve for one variable in terms of the other. For instance, if we solve for x, we obtain

$$x = 4 - 2y$$

In this form, y is free, as it can take on any real value, while x is not free, since its value depends on that of y. To represent this solution set in general terms, we introduce a third variable, for example t, called a parameter, and by letting $y = t$ we represent the solution set as

$$x = 4 - 2t,\ y = t,\ t \text{ is any real number}$$

Particular solutions can then be obtained by assigning values to the parameter t. For instance, $t = 1$ yields the solution $x = 2$ and $y = 1$, and $t = 3$ yields the solution $x = -2$ and $y = 3$

Note that the solution set of a linear equation can be represented parametrically in several ways. For instance, in Example 7.4, if we solve for y in terms of x, the parametric representation would take the form:

$$x = m,\ y = 2 - \frac{1}{2}m,\ m \text{ is a real number}$$

Also, by choosing $m = 2$, one particular solution is $(x, y) = (2, 1)$, and when $m = -2$, another particular solution is $(-2, 3)$.

Example 7.5

Solve the linear equation $3x + 2y - z = 3$

Solution

Choosing x and y as the free variables, we solve for z.

$$z = 3x + 2y - 3$$

Letting $x = p$ and $y = q$, we obtain the parametric representation:

$$x = p,\ y = q,\ z = 3p + 2q - 3, \text{ where } p \text{ and } q \text{ are any real numbers}$$

A particular solution is $(x, y, z) = (1, 1, 2)$

Parametric representation is very important when we study vectors and lines later on in the book.

Systems of linear equations

A system of k equations in n variables is a set of k linear equations in the same n variables. For example

$$\begin{aligned} 2x + 3y &= 3 \\ x - y &= 4 \end{aligned}$$

is a system of two linear equations in two variables, while

$$\begin{aligned} x - 2y + 3z &= 9 \\ x - 3y &= 4 \end{aligned}$$

is a system with two equations and three variables, and

$$\begin{aligned} x - 2y + 3z &= 9 \\ x - 3y &= 4 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

is a system with three equations and three variables.

A solution of a system of equations is an ordered set of numbers $x_0, y_0, \ldots$ which satisfy every equation in the system. For example $(3, -1)$ is a solution of

$$\begin{aligned} 2x + 3y &= 3 \\ x - y &= 4 \end{aligned}$$

Both equations in the system are satisfied when $x = 3$ and $y = -1$ are substituted into the equations. However, $(0, 1)$ is not a solution of the system; it satisfies the first equation, but it does not satisfy the second.

In this chapter, we will use matrix methods to solve systems of equations.

Taking our example above, we can write the system of equations in matrix form:

$$\begin{cases} 2x + 3y = 3 \\ x - y = 4 \end{cases} \Rightarrow \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

The representation of the system of equations this way enables us to use matrix operations in solving systems of equations. This matrix equation can be written as

$$\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \Rightarrow \mathbf{AX} = \mathbf{C}$$

where $\mathbf{A}$ is the coefficient matrix, $\mathbf{X}$ is the variable matrix, and $\mathbf{C}$ is the constant matrix. However, to solve this equation, the inverse of a matrix has to be defined as the solution of the system in the form

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{C}$$

where $\mathbf{A}^{-1}$ is the inverse of the matrix $\mathbf{A}$.

7 Matrix algebra

Matrix inverse

To solve the equation $2x = 6$ for x, we need to multiply both sides of the equation by $\frac{1}{2}$:

$$\frac{1}{2} \times 2x = \frac{1}{2} \times 6 \Rightarrow x = 3 \qquad \text{This is so, because } \frac{1}{2} \times 2 = 2 \times \frac{1}{2} = 1$$

$\frac{1}{2}$ is the multiplicative inverse of 2. The inverse of a matrix is defined in a similar manner and plays a similar role in solving a matrix equation, such as $\mathbf{AX} = \mathbf{C}$

A square matrix **B** is the inverse of a square matrix **A** if $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$ where **I** is the identity matrix.

The notation $\mathbf{A}^{-1}$ is used to denote the inverse of a matrix **A**. Thus, $\mathbf{B} = \mathbf{A}^{-1}$

Note that only square matrices can have multiplicative inverses.

Example 7.6

Are the matrices $\mathbf{A} = \begin{pmatrix} 7 & 5 \\ 4 & 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 & -5 \\ -4 & 7 \end{pmatrix}$ multiplicative inverses?

Solution

$$\mathbf{AB} = \begin{pmatrix} 7 & 5 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 3 & -5 \\ -4 & 7 \end{pmatrix} = \begin{pmatrix} 21 - 20 & -35 + 35 \\ 12 - 12 & -20 + 21 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{BA} = \begin{pmatrix} 3 & -5 \\ -4 & 7 \end{pmatrix}\begin{pmatrix} 7 & 5 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} 21 - 20 & 15 - 15 \\ -28 + 28 & -20 + 21 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So **A** and **B** are multiplicative inverses.

We can also find the inverse using a GDC.

```
[A]⁻¹
            [[3  -5]
             [-4 7 ]]
[A]⁻¹[A]
              [[1 0]
               [0 1]]
```

We will now find the general form for the inverse of a matrix.

Let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and assume $\mathbf{A}^{-1} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$ and then solve the following matrix equation for $e, f, g,$ and h in terms of $a, b, c,$ and d.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Now we can set up two systems to solve for the required variables:

$$\begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\left.\begin{matrix} ae + bg = 1 \\ ce + dg = 0 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} dae + dbg = d \\ bce + bdg = 0 \end{matrix}\right\} \Rightarrow e = \frac{d}{ad - bc}, g = \frac{-c}{ad - bc}$$

$$\left.\begin{matrix} af + bh = 0 \\ cf + dh = 1 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} daf + dbh = 0 \\ bcf + bdh = b \end{matrix}\right\} \Rightarrow f = \frac{-b}{ad - bc}, h = \frac{a}{ad - bc}$$

In a matrix $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, if $ad - bc \neq 0$, then its inverse $\mathbf{A}^{-1} = \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$

or $\mathbf{A}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Example 7.7

Find the inverse of $\mathbf{A} = \begin{pmatrix} 4 & 7 \\ 3 & 5 \end{pmatrix}$

Solution

Here $a = 4$, $b = 7$, $c = 3$, and $d = 5$, so $ad - bc = -1$

Thus $\mathbf{A}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{-1}\begin{pmatrix} 5 & -7 \\ -3 & 4 \end{pmatrix} = \begin{pmatrix} -5 & 7 \\ 3 & -4 \end{pmatrix}$

```
[A]
                [[4  7]
                 [3  5]]
[A]⁻¹
               [[-5  7]
                [3  -4]]
```

The number $ad - bc$ is called the determinant of the 2×2 matrix

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The notation we will use for this number is **det A** or **|A|**, so we write this as:

$$\det \mathbf{A} = |\mathbf{A}| = ad - bc$$

The determinant plays an important role in determining whether or not a matrix has an inverse.

When the determinant is zero ($ad - bc = 0$), the matrix does not have an inverse. A matrix that does not have an inverse is called a **singular matrix**; a matrix that does have an inverse is called a **non-singular** matrix.

Example 7.8

Solve the system of equations using matrices.

$$2x + 3y = 3$$
$$x - y = 4$$

Solution

In matrix form, the system can be written as

$$\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

Write the equation in the form $\mathbf{X} = \mathbf{A}^{-1}\mathbf{C}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}^{-1}\begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

Find $\mathbf{A}^{-1}$, then substitute into the equation and simplify

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = -\frac{1}{5}\begin{pmatrix} -1 & -3 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = -\frac{1}{5}\begin{pmatrix} -15 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

```
[A]-1[C]
              [[3  ]
               [-1]]
```

In general, a system of equations can be written in matrix form as $\mathbf{AX} = \mathbf{B}$

There is a solution to the system when $\mathbf{A}$ is non-singular, which is $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$

If $\mathbf{B} = 0$, the system is **homogeneous**. A homogeneous system will always have a solution, called the **trivial** solution, $\mathbf{X} = 0$ when $\mathbf{A}$ is non-singular. When $\mathbf{A}$ is singular then the system has an infinite number of solutions.

We use a similar procedure to solve systems of equations in three variables. However, we will use a GDC to find the inverse of a 3×3 matrix . As in the case of a 2×2 matrix, the existence of an inverse for a 3×3 matrix depends on the value of its determinant.

There are two methods of calculating the determinant of a 3×3 matrix $\mathbf{A}$:

```
[A]
     [[5  1  -4]
      [2 -3  -5]
      [7  2  -6]]
det([A])
                17
```

Method 1

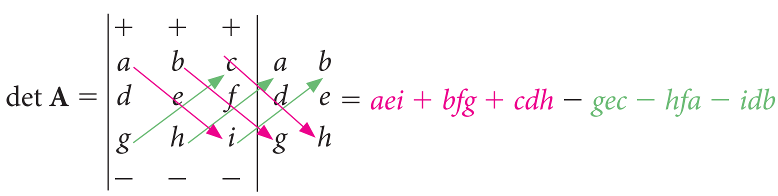
$$\mathbf{A} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \Rightarrow \det\mathbf{A} = a(ei - fh) - b(di - fg) + c(dh - eg)$$

For example, if $\mathbf{A} = \begin{pmatrix} 5 & 1 & -4 \\ 2 & -3 & -5 \\ 7 & 2 & -6 \end{pmatrix}$

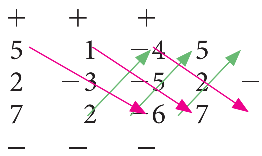
then $\det\mathbf{A} = 5(18 + 10) - 1(-12 + 35) - 4(4 + 21) = 17$

Method 2

Use a special set up as follows:



This is done by copying the first two columns and adding them to the end of the matrix, multiplying down the main diagonals and adding the products, and then multiplying up the second diagonals and subtracting them from the previous product as shown. For example:



$= 5 \cdot (-3)(-6) + 1 \cdot (-5) \cdot 7 + (-4) \cdot 2 \cdot 2 - 7(-3)(-4)$
$\quad - 2(-5) \cdot 5 - (-6) \cdot 2 \cdot 1$

$= 90 - 35 - 16 - 84 + 50 + 12 = 152 - 135 = 17$

This arrangement is a re-ordering of the calculations involved in the first method.

Example 7.9

Solve the system of equations

$$5x + y - 4z = 5$$
$$2x - 3y - 5z = 2$$
$$7x + 2y - 6z = 5$$

Solution

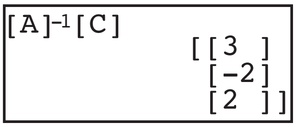
We write this system in matrix form

$$\begin{pmatrix} 5 & 1 & -4 \\ 2 & -3 & -5 \\ 7 & 2 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 5 \end{pmatrix}$$

Since det $\mathbf{A} = 17 \neq 0$, we can find the solution in the same way we did for the 2×2 matrix:

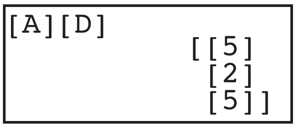
$$\begin{pmatrix} 5 & 1 & -4 \\ 2 & -3 & -5 \\ 7 & 2 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 & 1 & -4 \\ 2 & -3 & -5 \\ 7 & 2 & -6 \end{pmatrix}^{-1}\begin{pmatrix} 5 \\ 2 \\ 5 \end{pmatrix}$$



To check our work, using a GDC, we can store the answer matrix as D and then substitute the values into the system

$$\begin{pmatrix} 5 & 1 & -4 \\ 2 & -3 & -5 \\ 7 & 2 & -6 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 15 - 2 - 8 \\ 6 + 6 - 10 \\ 21 - 4 - 12 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 5 \end{pmatrix}$$



Mathematics
Applications and Interpretation

For the IB Diploma

HIGHER LEVEL

Mathematics Analysis and Approaches for the IB Diploma Higher Level is a comprehensive textbook covering the 2019 curriculum. The book also includes the following features:

- written by an expert authoring team
- additional integrated digital content including GeoGebra applets created specifically for the course
- worked examples to help you tackle questions
- practice questions to help you prepare for the exam
- rich and wide-ranging chapter on Mathematics in Theory of Knowledge
- guidance on internal assessment

www.pearsonbaccalaureate.com