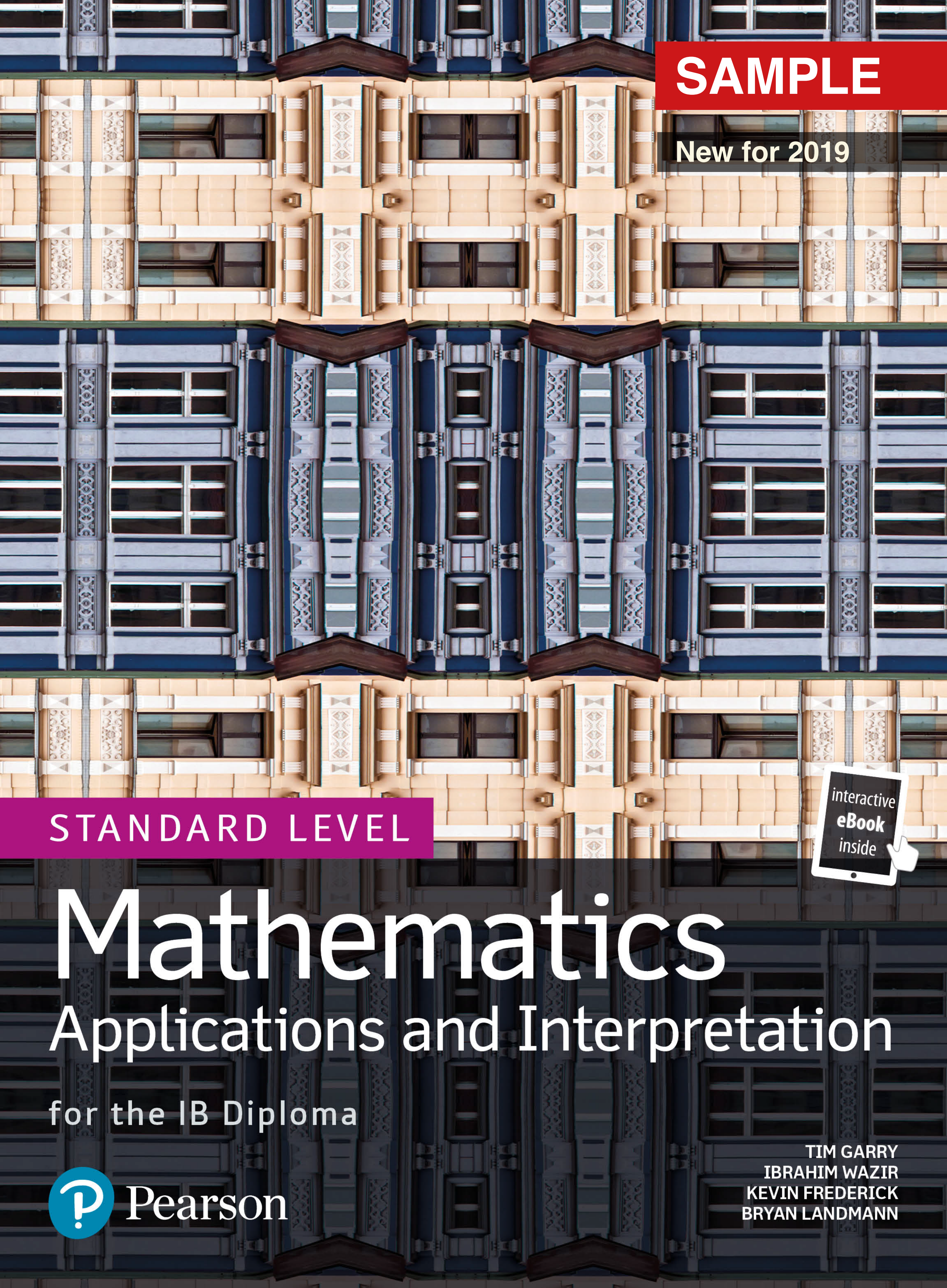

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Mathematics
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BRYAN LANDMANN

Pearson

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6 Modelling real-life phenomena

Learning objectives

By the end of this chapter, you should be familiar with…

- modelling linear, linear piecewise, quadratic, cubic, exponential, direct/ inverse variation, and trigonometric phenomena
- developing and fitting models (recognising the context, choosing an appropriate model, determining a reasonable domain and range, using technology to find parameters)
- testing and reflecting upon models (commenting on appropriateness and reasonableness of a model, justifying the choice of a model)
- using models (reading, interpreting, and making predictions, avoiding extrapolation).

Mathematical models help us to describe the world around us. In this chapter, we will look at several different kinds of mathematical models. We will examine how to choose, develop, test, apply, and extend a model. Here are some examples of the different kinds of models we see in the world around us:

- On a long flight, the airspeed of a plane is constant, so the distance remaining to the destination can be described by a **linear** model.
- In a situation where the price to manufacture x units of some product decreases **linearly**, the revenue from selling x units can be described by a **quadratic** model.
- The volume of a balloon relative to its diameter can be described by a **cubic** model.
- The spread of algae in a polluted lake can be described by an **exponential** model.
- A DJ charges a fixed amount to provide music for a party. The cost is spread equally among everyone who attends the party. The cost per person can be described by an **inverse variation** model.
- The price of electricity is often billed per kilowatt-hour (kWh), so the cost of powering the lights of a stadium relative to the time the lights are on can be described by a **direct variation** model.
- The height of a person above the ground on a Ferris wheel can be described by a **trigonometric** model.

The process of mathematical modelling is a design process and is illustrated in Figure 6.1.

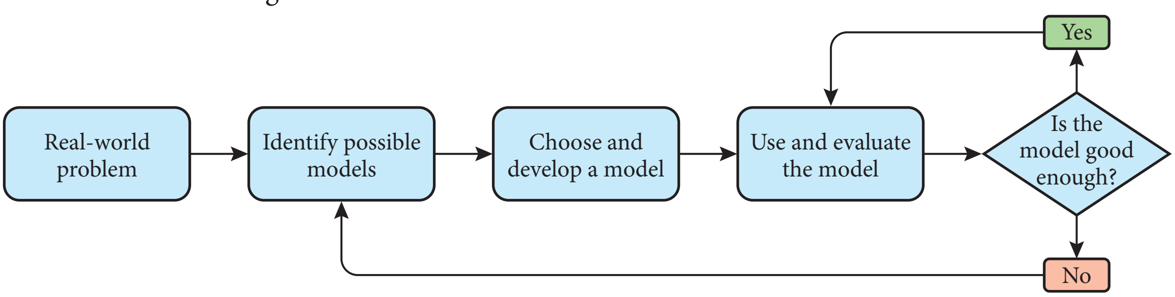


Figure 6.1 Mathematical modelling process

In this chapter we will learn how to develop models both by hand and by using technology. We will discuss how to test a model and then how to reflect upon or analyse the validity of the model. Finally, we will talk about how to use and, if needed, extend or revise a model.

It is important to remember that all models are simplifications of reality. We use models to tell us something about the way a system behaves and to make predictions. The goal of a model is to simplify and approximate a real system so that we can learn, predict, and analyse the behaviour. Part of our job is to use models wisely and be sure we understand the limitations and assumptions of any model we use.

6.1 Linear models

Linear models are used to describe situations where one quantity (the **dependent variable**) increases at a fixed rate relative to another quantity (the **independent variable**).

Developing and testing a linear model

Suppose that you need a plumbing repair in your home. You call a plumber to ask about how much it will cost. Of course, the plumber cannot give you an exact cost but does give you the estimates shown in Table 6.1.

Time required (hours)	Cost of repair (€)
1	110
2	185
3	260

Table 6.1 Repair cost estimates

Since we know that the cost must depend on the time required, **cost** is the dependent variable and **time required** is the independent variable.

To decide if this situation can be described by a linear model, we need to see if the rate of change is constant. To do this, check the gradient (in this context, the cost per hour) between two or more pairs of points:

$$\frac{185 - 110}{2 - 1} = 75 \quad \text{and} \quad \frac{260 - 185}{3 - 2} = 75$$

Since the cost per hour is constant, a linear model is appropriate for this situation. In addition, we have discovered that the cost per hour is €75. However, there seems to be another part to the cost. We can use slope–intercept form to find a linear model. We will use y for the dependent variable, cost, and x for the independent variable, time:

$$y - y_1 = m(x - x_1) \Rightarrow y - 110 = 75(x - 1) \Rightarrow y = 75x + 35$$

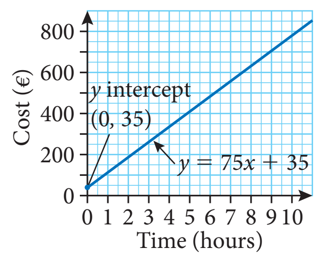


Figure 6.2 The graph of a linear model of the cost of hiring a particular plumber to come to your home

We can see that the plumber's hourly rate is €75, and he adds a fixed amount of €35. This is probably to compensate him for travelling to your home! Finally, it's a good idea to test the model to make sure it describes the situation:

For 1 hour: $y = 75(1) + 35 = 110$

For 2 hours: $y = 75(2) + 35 = 185$

This matches the estimates given so we can be confident that our model is appropriate. Now we can see what we might have to pay if it takes the plumber a whole 8-hour day to fix our problem:

For 8 hours: $y = 75(8) + 35 = €635$

Notice also that graphs of linear functions are always lines.

Remember that in a graph the independent variable is usually placed on the horizontal axis, and the dependent variable is placed on the vertical axis.

Linear models describe situations where the rate of change (gradient) is constant. The graph of a linear function is a line.

Extending and revising models

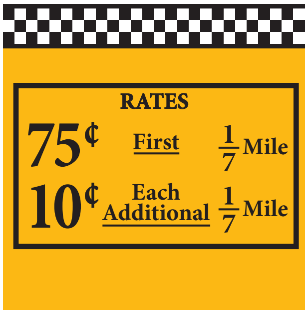


Figure 6.3 New York City tax rates decal from the late 1970s. Linear models are very well suited to this sort of situation

Sometimes we can make an initial simple model but need to revise it to be more useful. Here is an example.

Consider the taxi costs shown in Figure 6.3. Because the cost increases at a fixed rate relative to the distance driven (10 cents for each $\frac{1}{7}$ miles driven), this is a good candidate for a linear model. Since the cost depends on the number of miles driven, we will make the cost the dependent variable and the distance driven the independent variable. It often helps to make a table showing the independent and dependent variables:

Miles driven	Calculation	Cost ($)
$\frac{1}{7}$	$0.75 + 0$	0.75
$\frac{2}{7}$	$0.75 + 0.10(1)$	0.85
$\frac{3}{7}$	$0.75 + 0.10(2)$	0.95
$\frac{4}{7}$	$0.75 + 0.10(3)$	1.05

Table 6.2 Independent and dependent variables

By explicitly showing our calculations, we get a good idea of how to develop our model. From the table, it appears that an appropriate model is Cost $= 0.75 + 0.10x$. But be careful! In this case, what is x? Notice that the number we are multiplying by 0.10 is not the number of miles driven – it is the number of $\frac{1}{7}$ miles after the first $\frac{1}{7}$ mile!

Since most people don't think in terms of $\frac{1}{7}$ miles, it would be useful if our independent variable was simply distance in miles. Let's try to revise our table:

Miles driven	The number of $\frac{1}{7}$ miles after the first $\frac{1}{7}$ mile	Calculation	Cost ($)
$\frac{1}{7}$	$\frac{1}{7} - \frac{1}{7} = 0$	$0.75 + 0.10(0)$	0.75
$\frac{2}{7}$	$7\left(\frac{2}{7} - \frac{1}{7}\right) = 1$	$0.75 + 0.10(1)$	0.85
$\frac{3}{7}$	$7\left(\frac{3}{7} - \frac{1}{7}\right) = 2$	$0.75 + 0.10(2)$	0.95
$\frac{4}{7}$	$7\left(\frac{4}{7} - \frac{1}{7}\right) = 3$	$0.75 + 0.10(3)$	1.05
m	$7\left(m - \frac{1}{7}\right)$	$0.75 + 0.10x$	C

Table 6.3 Revised table

To change 'miles driven' into 'the number of $\frac{1}{7}$ miles after the first $\frac{1}{7}$ mile,' we need to subtract $\frac{1}{7}$ (representing the first $\frac{1}{7}$ mile) and then multiply by 7 (so that each $\frac{1}{7}$ mile is counted as one unit).

So now we know that $C = 0.75 + 0.10x$ **and** $x = 7\left(m - \frac{1}{7}\right)$ where m is the number of miles.

By substituting the equation for x into the equation for C, we get

$$C = 0.75 + 0.10\left(7\left(m - \frac{1}{7}\right)\right) \Rightarrow C = 0.7m + 0.65$$

A graph of this function is shown in Figure 6.4.

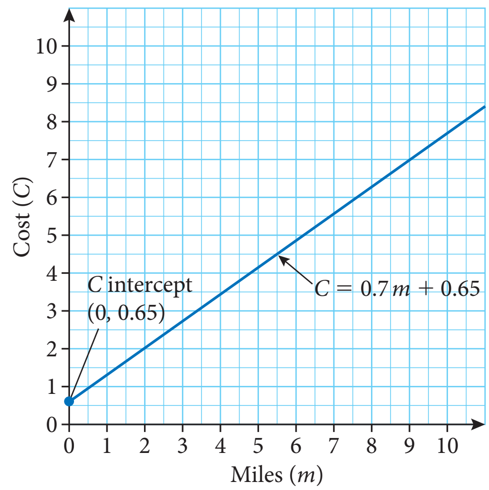


Figure 6.4 A graph of the linear model for the cost of hiring a New York taxi in the 1970s

Models don't always capture reality perfectly

If we look at the graph of our model from the taxi cost example, we see that the model suggests that for a journey of 0 miles, we will pay \$0.65. However, we know that we will always pay at least \$0.75. What went wrong?

The problem here is that the model assumes that the incremental cost (represented by the gradient of the function) is continuous – that is, that the taxi will charge us for any increment of a mile. However, we know that the taxi will charge for each $\frac{1}{7}$ of a mile. To make this point clear, consider what happens if we drive 0.5 miles. The model suggests that the cost would be

$$C = 0.7(0.5) + 0.65 = \$1.00$$

But we know that 0.5 miles is more than $\frac{3}{7}$ of a mile and less than $\frac{4}{7}$ so we would actually get charged

$$C = 0.7\left(\frac{4}{7}\right) + 0.65 = \$1.05$$

That is, our model works as long as we round the miles up to the nearest $\frac{1}{7}$ of a mile. As stated in the introduction, all models are a simplification of reality. This model can help us see how the cost of the ride relates to the length of the ride, but we need to be careful in using it to predict exact costs.

Example 6.1

The I&T Fitness Centre charges a one-time joining fee of \$100 and then charges \$2 per visit.

(a) Develop a model for the total cost after v visits.

(b) Use your model to find the cost of 20 visits.

(c) Draw a graph of your model.

Solution

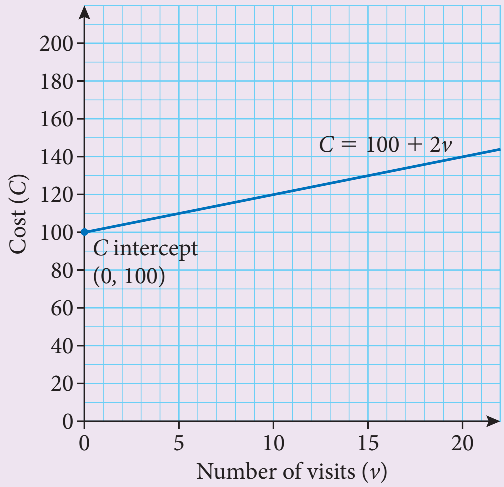
(a) Since the rate is already given ($2 per visit), plus a one-time cost of $100, we can write the model directly:

$$C = 100 + 2v$$

where C is the total cost and v is the number of visits.

(b) Using our model, 20 visits would cost $C = 100 + 2(20) = \$140$

(c)



We have drawn the graph with a continuous line, even though it is only possible to visit a whole number of times – e.g., we can't visit 1.5 times. For convenience, we often draw the graph as a continuous line even though that may not strictly represent reality.

Example 6.2

A visitor to the I&T Fitness Centre decides not to buy a membership and instead just pays the daily rate. The first visit costs her $12. At the end of the month, she notices she has visited 5 more times and paid a total of $60 for those 5 visits.

(a) Develop a linear model for the total cost after v visits.

(b) Use your model to predict the cost of 20 visits.

(c) After how many visits is it better to buy the membership described in Example 6.1?

Solution

(a) One visit costs $12, and 5 visits cost $60. To be sure there are no extra fees, check that the cost per visit for 5 visits is the same as the cost for 1 visit. For 5 visits, the cost per visit is $\frac{60}{5} = \$12$

Therefore, the rate of change is constant, and we can develop the model:

$C = 12v$ where C is the total cost and v is the number of visits.

(b) Using our model, 20 visits would cost $C = 12(20) = \$240$

This is much more expensive than the membership plan in Example 6.1!

(c) To find out at which point the membership plan in Example 6.1 becomes the better option, we first find at which point the two plans are equal. We can do this algebraically or graphically.

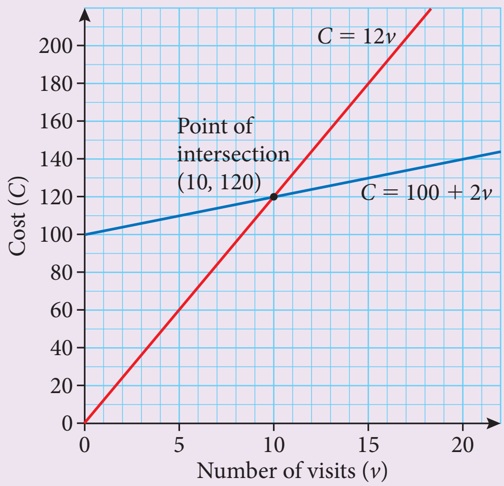
Algebraically, we are looking for the number of visits (v) that produces the same cost. Therefore, we can write $C = 100 + 2v = 12v$ and solve for v:

$$100 + 2v = 12v \Rightarrow v = 10$$

Therefore, the two plans are the same for 10 visits. Since we know that the membership plan costs only \$2 per visit, we know that it will be cheaper for any number of visits more than 10.

We can graph both models and look for the intersection.

The intersection point of the two graphs tells us that the two plans will cost the same (\$120) at 10 visits. We can see clearly that the membership plan increases much more slowly: \$2 per visit instead of \$12 per visit, so it is cheaper after 10 visits.



Interpreting and evaluating linear models

It is important to be able to recognise the structure of a linear model and interpret its meaning. Also, we must be careful to recognise limitations of linear models.

Example 6.3

The number of items that a clothing store sells can be modelled by the function $N = 1000 - 5p$, where N is the number of items sold and p is the price of the jeans in euros.

(a) Use the model to predict the number of jeans sold when the jeans are priced at €100.

(b) Interpret the value of the gradient and N-intercept in context.

(c) Interpret the value of the p-intercept in context.

(d) Use the model to predict the number of jeans sold when the jeans are priced at €500. Give a reason why this prediction is not reasonable.

Solution

(a) The model predicts that the number of jeans sold when the jeans are priced at 100 euros is: $N = 1000 - 5(100) = 500$ jeans.

(b) The gradient of -5 represents that for each euro that the price increases, the number of jeans sold decreases by 5 jeans. The N-intercept of (0, 1000) theoretically represents that the number of jeans sold when the jeans are free will be 1000 jeans – probably not realistic!

(c) The p-intercept occurs when $N = 0$. Therefore, we must solve $0 = 1000 - 5p$ to obtain $1000 = 5p \Rightarrow p = 200$

This tells us that when the price of jeans is 200 euros, the number of jeans sold will be 0.

(d) The model predicts that the number of jeans sold when the jeans are priced at €500 will be: $N = 1000 - 5(500) = -1500$ jeans

This is not reasonable because this suggests that customers will be giving back their jeans!

Example 6.3 shows that linear models often become nonsensical for certain extreme values of the independent variable. For this reason, it's best to give a limitation on the domain of the model to avoid these silly results. In the model above, it is sensible to limit the domain of the model to $1 \leqslant p \leqslant 200$

The upper bound of $p = 200$ is when the model predicts that N will be zero.

Piecewise linear models

Sometimes a real-life situation is not modelled by a single linear function, but is linear in parts. Consider the following example:

Example 6.4

A phone company charges a rate of \$0.24 for the first minute of a call, then \$0.12 per minute for the next 9 minutes, then \$0.06 per minute thereafter. Calls are charged on a per-second basis.

(a) Develop a piecewise linear model for the cost C of a call lasting t seconds.

(b) Use your model to calculate the cost of calls lasting:
(i) 45 seconds (ii) 4 minutes (iii) 15 minutes.

Solution

(a) For $0 < t \leqslant 60$, the cost is \$0.24 per minute, or $\frac{0.24}{60} = \$0.004$ per second. Then, for $60 < t \leqslant 600$, the cost is \$0.12 per minute or $\frac{0.12}{60} = \$0.002$ per second. However, we must also add in the cost for the first minute, and not charge twice for the first minute, so in total the cost will be $0.24 + 0.002(t - 60)$ for a call of t seconds. Likewise, for calls more than 10 minutes, we have the cost of the first minute, plus the cost of the next 9 minutes, plus the remaining cost of $\frac{0.06}{60} = 0.001$ per minute, which gives us $0.24 + 0.12(9) + 0.001(t - 600)$, which simplifies to $1.32 + 0.001(t - 600)$ for a call of t seconds.

We express this mathematically with the following notation:

$$C = \begin{cases} 0.004t, & 0 < t \leqslant 60 \\ 0.24 + 0.002(t - 60), & 60 < t \leqslant 600 \\ 1.32 + 0.001(t - 600), & t > 600 \end{cases}$$

In this notation, we define the value of C through a **piecewise function**. To evaluate the function, we simply select the piece of the function that applies to the value of t we want, as in part (b).

(b) For each value of t, we simply select the appropriate piece of the function.

(i) Since $t = 45$ is between 0 and 60 seconds, we use $C = 0.004t$ to give us $C = 0.004(45) = \$0.18$

(ii) Since 4 minutes is 240 seconds, and $t = 240$ is between 60 and 600 seconds, we use $C = 0.24 + 0.002(t - 60)$ to give us $C = 0.24 + 0.002(240 - 60) = \$0.60$

(iii) Since 15 minutes is $15 \times 60 = 900$ seconds, and $t = 900$ is more than 600 seconds, we use $C = 1.32 + 0.001(t - 600)$ to give us $C = 1.32 + 0.001(900 - 600) = \$1.62$

Many calculators can graph piecewise functions, as shown in Figure 6.5.

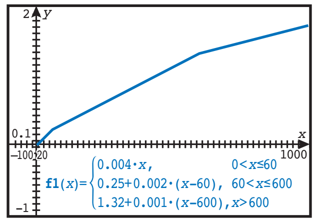


Figure 6.5 GDC piecewise function

Exercise 6.1

1. A plane is currently 3000 km from its destination, travelling at a constant speed of 900 km h^{-1}.

(a) Develop a linear model for the distance d remaining after t hours of travel.

(b) Interpret the d-intercept of your model in context.

(c) Interpret the t-intercept of your model in context.

(d) State a reasonable domain and range for your model.

2. A plane is currently 5000 km from its destination. 1.5 hours later, it is 3800 km from its destination.

(a) Develop a linear model for the distance d remaining after t hours of travel.

(b) Interpret the gradient of your model in context.

(c) Interpret the d-intercept of your model in context.

(d) Interpret the t-intercept of your model in context.

(e) State a reasonable domain and range for your model.

3. The table shows a comparison between EU and USA shoe sizes.

USA (Men's)	7	8	9
EU	40	41	42

(a) Develop a linear model to find the EU shoe size given the USA shoe size.

(b) Use your model to predict the EU shoe size for a USA Men's shoe size of 12.

(c) Use your model to predict the USA Men's shoe size for an EU shoe size of 44.

(d) Interpret the gradient of your model in context.

(e) Given that USA Men's shoe sizes typically run from 6 to 16, calculate a reasonable domain and range for your model.

4. Julie has collected data on how long it takes her to read books, based on the number of pages. The data she collected are shown in the table.

Number of pages	340	290	500
Time to read (minutes)	490	420	714

(a) Develop a linear model for the time required to read n pages.

(b) Use your model to predict the time required to read 1000 pages. Give your answer to the nearest 10 minutes.

(c) Interpret the gradient and y-intercept of your model in context.

(d) State a reasonable domain and range for your model.

5. Given that 68°F = 20°C and 212°F = 100°C:

(a) Develop a linear model for degrees Fahrenheit (F) in terms of degrees Celsius (C).

(b) Explain, in context, what the gradient of your model represents.

(c) Interpret the F-intercept of your model in context.

(d) Interpret the C-intercept of your model in context.

(e) Use your model to convert 10°C into °F.

(f) Use your model to find the numerical value in °C that is the same in °F.

(g) Given that absolute zero (the lowest possible temperature) is −273°C, calculate a reasonable domain and range for your model.

6. The DJ, IB Cool, charges a flat fee of \$150 per party plus \$75 per hour. The DJ, MC Numbers, charges \$120 per party plus \$80 per hour.

(a) Find linear models for each DJ as a function of the length of the party in hours.

(b) For parties longer than n hours, IB Cool is less expensive. Find the value of n.

7. A cyclist pedals at the rate of 300 m min^{-1} for 20 minutes, then slows down to 150 m min^{-1} for 16 minutes, then races at 400 m min^{-1} for 4 minutes.

(a) Find the distance travelled after:

(i) 20 minutes (ii) 36 minutes (iii) 40 minutes.

(b) Write a piecewise linear function for the distance $D(t)$ in terms of the time t in minutes.

(c) Find the distance travelled after:

(i) 30 minutes (ii) 38 minutes.

(d) When has the cyclist travelled:

(i) 8 km (ii) 9 km?

8. The Athabasca Glacier in Alberta, Canada, has been slowly shrinking for many years. Photos from 1844 show the glacier about 2 km longer than it was in 2018.

   (a) Calculate the average rate at which the glacier is shrinking.

   (b) In 1844, the glacier was 8 km long. Assuming the present rate continues, in what year will the glacier be gone?

   In the year 1900, the end of the glacier was about 300 km from the US–Canada border; as it shrinks it moves farther from the border.

   (c) Write a function for the distance d in km between the end of the glacier and the US–Canada border, in terms of time t, in years, since 1900.

   (d) Assuming that the rate of change has been constant for many years, at what time did the glacier reach the US–Canada border?

9. Two plastic cup factories, Cups R Us and Cupomatic, can produce cups printed with the image of your choice. At Cups R Us, the mandatory setup and design cost is ZAR350 and the cost per cup is ZAR8.50.

   (a) Develop a linear model for the cost, C, of an order at Cups R Us based on the number of cups, n.

   (b) Write down a reasonable domain and range for your model.

   (c) Use your model to calculate the cost of an order of:
   (i) 100 cups **(ii)** 200 cups **(iii)** 400 cups.

   (d) Calculate the average cost per cup for:
   (i) 100 cups **(ii)** 200 cups **(iii)** 400 cups.

   (e) Hence, give a reason why, in general, it is more cost-effective to order more cups.

   Cupomatic charges ZAR2150 for 200 cups and ZAR3750 for 400 cups.

   (f) Develop a linear model for the cost, D, of an order at Cupomatic based on the number of cups, n.

   (g) Write down a reasonable domain and range for your model.

   (h) Interpret the gradient of your model in context.

   (i) Use your linear model to predict the cost of 600 cups.

   (j) For orders of more than x cups, it is more cost-effective to order from Cupomatic. Find the value of x.

10. Continuing from the previous question, Cups R Us will waive the setup and design cost if the order is at least 500 cups.

   (a) Develop a piecewise model for the cost, C, of an order at Cups R Us based on the number of cups, n.

   (b) The main competitor of Cups R Us is Cupomatic. You are given that the model for the cost, D, of ordering n cups from Cupomatic is $D = 8n + 550$. It is less expensive to order from Cupomatic if x, the number of cups ordered, is in the intervals $a \leqslant x < b$ or $x > k$. Find the values of a, b, and k.

11. As of 2018, taxi cab tariffs for working hours in London, England, are as follows.

- For the first 234.8 metres or 50.4 seconds (whichever is reached first) there is a minimum charge of 2.60 GBP.
- For each additional 117.4 metres or 25.2 seconds (whichever is reached first), or part thereof, if the distance travelled is less than 9656.1 metres there is a charge of 0.20 GBP.
- Once the distance has reached 9656.1 metres then there is a charge of 0.20 GBP for each additional 86.9 metres or 18.7 seconds (whichever is reached first), or part thereof.

(a) Develop a piecewise linear model for the cost, C, of a taxi ride based on the distance travelled, m, in metres.

(b) Find the cost of 0.2 km, 5 km, and 15 km rides.

(c) Develop a piecewise linear model for the cost, D, of a taxi ride based on the time taken, t, in seconds, ignoring distance.

(d) Find the cost of 0.5 minute, 5 minute, and 15 minute rides.

(e) Given that the actual taxi fare is always the greater of the two models, find:

(i) the cost of a ride that takes 10 minutes to go 4 km

(ii) the cost of a ride that takes 5 minutes to go 4 km.

6.2 Quadratic models

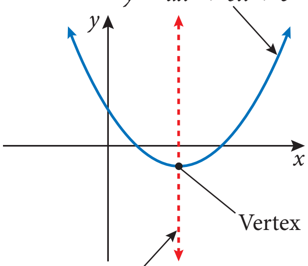


Figure 6.6 If $a > 0$ then the parabola is concave up

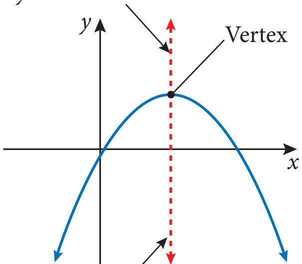


Figure 6.7 If $a < 0$ then the parabola is concave down

Quadratic models appear frequently in real-world situations involving area, economics, projectile motion, and falling objects, among others. In this section we will look primarily at quadratic models of the form $y = ax^2 + bx + c$

Quadratic models are used in situations where the rate of change in the dependent variable changes linearly with respect to the independent variable. For example:

- Falling objects due to gravity, projectile motion: the acceleration is constant, the velocity follows a linear model, and the displacement (position) follows a quadratic model.
- Revenue models: the number of items sold of some item based on the price follows a linear model; the revenue from selling the number of items follows a quadratic model.

Quadratic models also have some geometric properties that make them well-suited to designing satellite dishes and to modelling bridge spans.

Before we begin, we will restate key properties of quadratic functions that you have seen before.

For a quadratic function of the form $y = ax^2 + bx + c$, where $a \neq 0$

- The graph of a quadratic function is roughly U-shaped and is called a **parabola**.
- Concavity: The graph of the function is **concave up** if and only if $a > 0$
  It is **concave down** if and only if $a < 0$
- Symmetry: The graph of the function is symmetrical about the vertical line with equation $x = -\frac{b}{2a}$
  This line is called the **axis of symmetry**.
- Maximum/minimum: The function has a **maximum** (when concave down) or **minimum** (when concave up) where the graph intersects the axis of symmetry. This point is called the **vertex** of the parabola. The x coordinate of the vertex is therefore given by $x = -\frac{b}{2a}$
- The ***y*-intercept** of the graph is found at $(0, c)$
- The ***x*-intercepts** of the graph, also called the **zeros** of the function, can be found with the quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
  Remember that a graph may have zero, one, or two x-intercepts.

Example 6.5

A ball is thrown upwards from the top of a building. The height of the ball from ground level can be modelled by the function $h(t) = -4.9t^2 + 11t + 50$ where $h(t)$ is the height of the ball in metres and t is the time in seconds after the ball was thrown.

(a) Sketch a graph of the function.

(b) Write down the height of the building.

(c) Find the time when the ball reaches its maximum height.

(d) Find the maximum height reached by the ball.

(e) Find the time when the ball hits the ground.

(f) Describe a reasonable domain and range for this model.

Solution

We can solve this problem either by using an algebraic approach ('by hand') or by using our GDC to analyse the graph.

Algebraic approach

(a) We know the ball travels upwards before coming back down and we know the parabola is concave down since $a < 0$

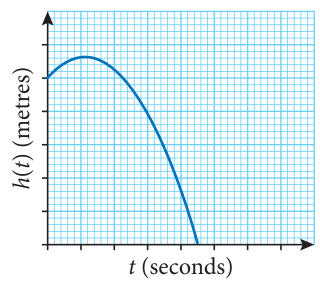


Figure 6.8 Solution to Example 6.5 (a)

(b) The height of the building is the initial height of the object, which is the height of the ball when $t = 0$, in other words, the y intercept. Therefore, the height of the building is 50 m.

(c) The time when the ball reaches its maximum height is the t-coordinate of the vertex of the parabola. We start by finding the t coordinate, which is also the location of the axis of symmetry:

$$t = -\frac{b}{2a} = -\frac{11}{2(-4.9)} = 1.12 \text{ s (3 s.f.)}$$

(d) The maximum height reached by the ball is found by substituting the t value from (c) back into the function:

$$h(1.12) = -4.9(1.12)^2 + 11(1.12) + 50 = 56.2\text{ m (3 s.f.)}$$

(e) The ball hits the ground when height is zero. Therefore, we must solve the equation $0 = -4.9t^2 + 11t + 50$

We can use the quadratic formula to solve this:

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-11 \pm \sqrt{11^2 - 4(-4.9)(50)}}{2(-4.9)} = 4.51\text{ s (3 s.f.)}$$

(f) We should limit the domain to non-negative values. Also, the model doesn't make sense after the ball hits the ground since the graph suggests that the ball goes underground. Therefore a reasonable domain for this model is $0 \leqslant t \leqslant 4.51$

The range indicates the possible heights of the ball. Here, $0 \leqslant h(t) \leqslant 56.2$ makes sense since we know the maximum height of the ball and we presume the ball does not go underground.

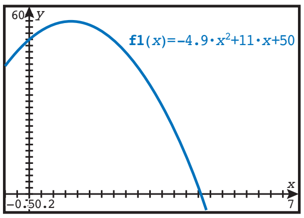


Figure 6.9 GDC approach (a)

GDC approach

(a) Here we use the GDC to obtain a graph, as shown in Figure 6.9. We need to be careful to adjust the viewing window appropriately.

On our calculator, we need to use x in place of t. It is important to note that this graph is height versus time – it is not the trajectory (path) of the object. The horizontal axis represents time, not distance.

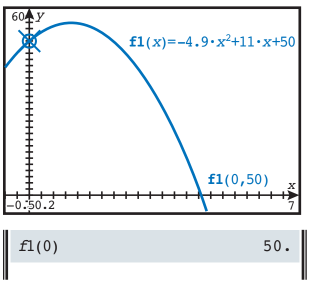


Figure 6.10 GDC approach (b)

(b) The Graph Trace feature can be used to evaluate the function at $t = 0$ to find the initial height, as shown in Figure 6.10. We could also evaluate the function directly.

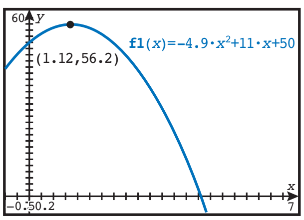


Figure 6.11 GDC approach (c)

(c) We can use our GDC to find the maximum height, as in Figure 6.11.

We see that the x coordinate is 1.12, so the ball reaches a maximum height at 1.12 s.

(d) Using the same point from part (c), we see that the y coordinate is 56.2, so the balls, maximum height is 56.2 m.

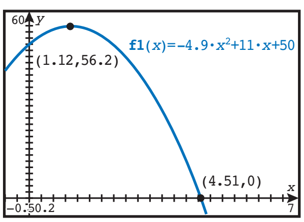


Figure 6.12 GDC approach (e)

(e) We can use our GDC to find the positive zero of the function, as in Figure 6.12. The ball hits the ground after 4.51 s.

(f) We use the same logic as in the algebraic approach to conclude that a reasonable domain is $0 \leqslant t \leqslant 4.51$ and a reasonable range is $0 \leqslant h(t) \leqslant 56.2$

In the next few examples we will look at building a quadratic function.

Example 6.6

The number of jeans sold by a clothing store can be modelled by the function $N = 1000 - 5p$, where N is the number of items sold and p is price of the jeans in euros.

(a) Develop a model for the revenue earned from selling these jeans based on the selling price p.

(b) Use your model to find the price the jeans should be sold at in order to maximise revenue.

(c) Find the maximum revenue predicted by your model.

(d) Give a reasonable domain and range for your model.

Solution

(a) In general, revenue = selling price × number of units sold. Therefore, we can develop a model for revenue by multiplying the selling price, p, by the expression for the number of jeans sold:

$$R = (p)(1000 - 5p) \quad \Rightarrow \quad R = -5p^2 + 1000p$$

(b) Since this is a quadratic model with a concave-down graph ($a < 0$), we know there will be a maximum at the vertex. In this case, the p coordinate of the vertex is the selling price and the R coordinate is the revenue. To find the price, we use the formula for the p coordinate of the vertex:

$$p = -\frac{b}{2a} = -\frac{1000}{2(-5)} = 100 \text{ euros}$$

(c) To find the maximum revenue, we need to evaluate the model for 100 jeans (from part (b)):

$$R = -5(100)^2 + 1000(100) = 50\,000 \text{ euros}$$

(d) Clearly, our model doesn't make much sense if the price p is less than zero. But what about a maximum price? Since we know that this is a concave-down quadratic function with a vertex above the p axis, we know there will be two p intercepts. Solve for them algebraically:

$$0 = -5p^2 + 1000p \quad \Rightarrow \quad 0 = (-5p)(p - 200) \quad \Rightarrow \quad p = 0,\ p = 200$$

Therefore the p intercepts are 0 and 200. So $0 \leqslant p \leqslant 200$ is a reasonable domain. The range is given by the minimum and maximum values of the function on this domain: $0 \leqslant R \leqslant 50\,000$

It's often helpful to generate a graph of a model to get a picture of it. In this case, we can clearly see the maximum revenue and can use our GDC to verify our results in (c) and (d) , as shown in Figure 6.13.

Note that 5E+4 for the y coordinate of the vertex is the GDC's version of scientific notation. We should read it like this: $5\text{E}{+}4 = 5 \times 10^4 = 50\,000$

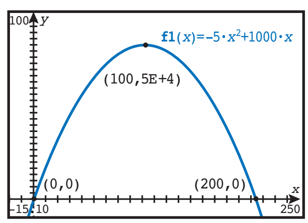
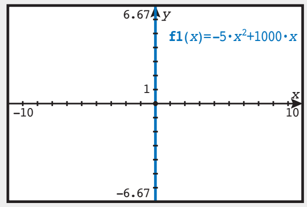


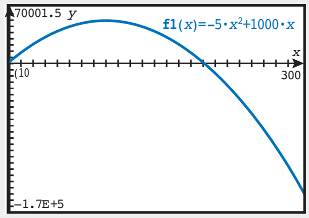
Figure 6.13 ?The graph of the model for revenue vs price of jeans

When using a GDC to analyse a model graphically, it's important to take care when setting the viewing window. The default view for many GDCs is $-10 \leqslant x \leqslant 10$ and $-10 \leqslant y \leqslant 10$ or smaller. This view can be very misleading. For example, the model in Example 6.5 looks like this with the default viewing window.



The graph is there but with the current settings it is almost indistinguishable from the y axis. We can use our knowledge of the function to choose more appropriate settings.

After setting the x-axis to a suitable domain, in this case $0 \leqslant x \leqslant 300$, and using the Zoom Fit feature to scale the y-axis to fit the function, we get the image shown.



The negative part of the graph isn't very useful, but you can then use Zoom Box to examine the function more accurately.

Example 6.7

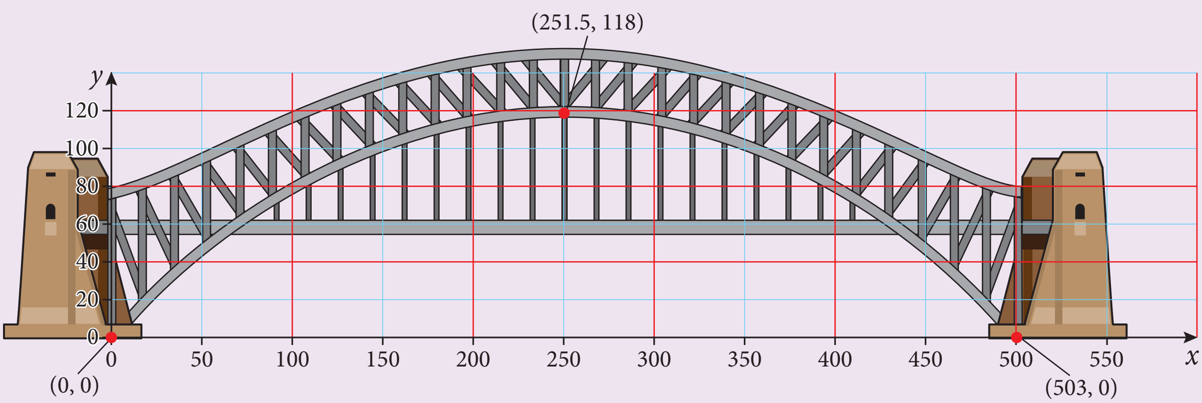
The Sydney Harbour Bridge is supported by two spans that can be modelled by quadratic functions.

The lower span is approximately 503 m wide and 118 m tall at its highest point.

Develop a quadratic model for the lower span such that one end of the span is positioned at (0, 0).

Solution

We can start by drawing our axes and locating the vertex at the maximum point of the lower span. Since the vertex must be halfway along the length of the bridge, its x coordinate must be 251.5.



Then we can use the general quadratic model $y = ax^2 + bx + c$ and some algebra to find the model for the lower span. First, since the graph must pass through (0, 0), it must be true that

$$0 = a(0)^2 + b(0) + c \Rightarrow c = 0$$

Upon reflection, we could have deduced the value of c from recognising that the y intercept of the graph is at (0,0). So far, our model is therefore

$$y = ax^2 + bx$$

Next, since the graph must also pass through (251.5, 118) and (503, 0), we can generate a system of equations by substituting each coordinate pair into the model:

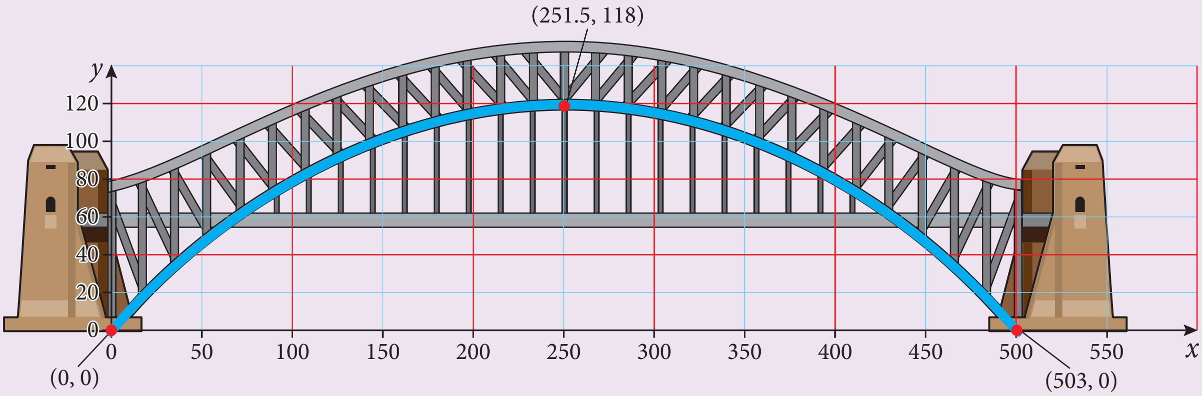
$$\begin{cases} 118 = a(251.5)^2 + b(251.5) \\ 0 = a(503)^2 + b(503) \end{cases} \Rightarrow \begin{cases} 63\,252.25a + 251.5b = 118 \\ 25\,3009a + 503b = 0 \end{cases}$$

We can solve this system (algebraically, or by using a GDC) to obtain:

$a = -0.001866,\ b = 0.9384$

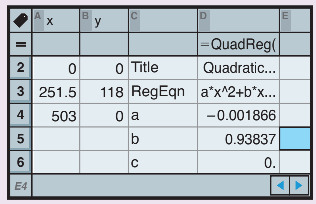
Therefore, our model is $y = -0.001866x^2 + 0.9384x$

If we graph this function with appropriate axes, we can see that it fits the lower span very well:

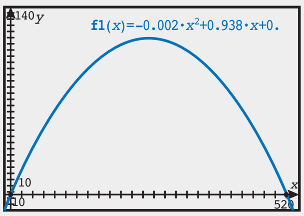


Note that most GDCs have a **quadratic regression** feature that can also find the model.

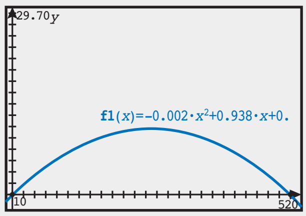
Notice that the values for a and b calculated by the GDC agree with our values.



Remember that the graph may appear differently depending on the viewing window chosen on your graph. For example, in the first screenshot, the scale of the units on the x and y axes is not 1:1, so the function appears taller than it should.



A graph can appear distorted if the x and y axes are not scaled in a 1:1 ratio



Using the Zoom Square feature changes the x and y axes to a 1:1 ratio

Using the GDC's Zoom Square function fixes this, as shown in the second screenshot.

Exercise 6.2

1. On Earth, the position of a falling object can be modelled by the function $h(t) = -4.9t^2 + v_0t + h_0$ where $h(t)$ is the height in metres after t seconds, v_0 is the initial velocity and h_0 is the initial height.

(a) Write a model for the height of a ball thrown upwards with an initial velocity of $5\ \text{m s}^{-1}$ from the roof of a 60-metre tall building.

(b) Use your model to find:

(i) the maximum height of the ball

(ii) the time until the ball hits the ground

(iii) the interval of time for which the ball is more than 50 metres above the ground.

2. A small manufacturing company makes and sells x machines each month. The monthly cost, C, in dollars, of making x machines is given by

$$C(x) = 0.35x^2 + 3200$$

The monthly revenue, R, in dollars, obtained by selling x machines is given by $R(x) = 180x - 0.55x^2$

Remember that profit = revenue − cost.

(a) Show that the company's monthly profit can be calculated using the quadratic function $P(x) = -0.9x^2 + 180x - 3200$

(b) The maximum profit occurs at the vertex of the function $P(x)$. How many machines should be made and sold each month for a maximum profit?

(c) If the company does maximise profit, what is the selling price of each machine?

(d) Find the smallest number of machines the company must make and sell each month in order to make a positive profit.

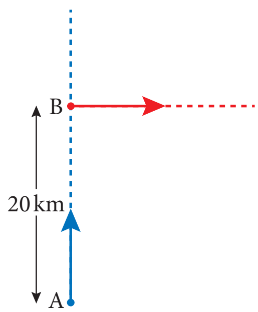


Figure 6.14 Diagram for question 3

3. The diagram shows two ships, A and B. At noon, ship A was 20 km due south of ship B. Ship A was moving north at 10 km h^{-1} and ship B was moving east at 4 km h^{-1}

Find the distance between the ships at:

(a) 13:00

(b) 14:00

Let $s(t)$ be the distance between the ships t hours after noon, for $0 \leqslant t \leqslant 4$

(c) Show that $s(t) = \sqrt{116t^2 - 400t + 400}$

(d) Sketch the graph of $s(t)$

(e) Due to poor weather, the captain of ship A can see another ship only if they are less than 9 km apart.

(i) Find the values of t during which ship A can see ship B.

(ii) Write down the times between which ship A can see ship B.

4. Worldwide grain production for the years 1965 to 2000 can be modelled by the function $G = -0.144t^2 + 6.88t + 266$, where G is the number of kilograms per person and t is years since 1965.

(a) Based on this model, what was the grain production in 1975?

(b) What was the maximum level of grain production and when did it occur?

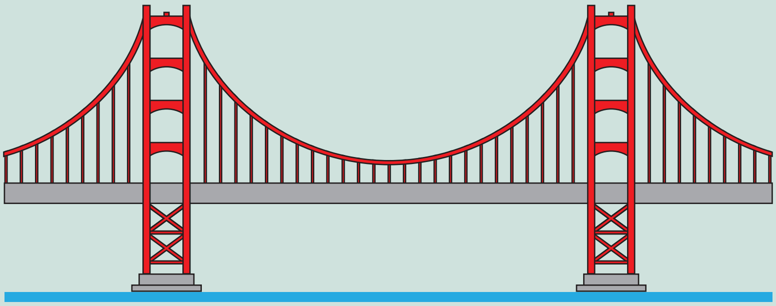
(c) The actual worldwide grain production in 2005 was 10 kg per person. What does this model predict for 2005?

(d) If worldwide grain production drops below 100 kg per person, major economic and health consequences are possible. Based on this model, when might this occur?

(e) Give a reason why this model will not (we hope) continue to be true.

5. The density of water based on temperature follows a quadratic model. The maximum density of water is 1 g mm^{-1} at 4°C. At 80°C, the density is 0.97183 g mm^{-1}.
   (a) Find a quadratic model for density, D, in terms of temperature, T, in the form $D = a(T - h)^2 + k$ where a, h, and k are constants to be determined.
   (b) Use your model to find the density of water at 0°C to 5 significant figures.
   (c) At what temperature $t > 0$ does the density of water drop below 0.960 g mm^{-1}?

6. The main span of the Verrazzano-Narrows bridge in New York City has a central span that is about 1300 m wide.



The main cable supporting this span is about 150 m above the roadway at the top of each suspension tower, and about 6 m above the roadway at its lowest point in the centre.
   (a) Find a quadratic function to model the height h of the cable above the roadway in terms of the distance d from the left suspension tower.
   (b) Write down the domain and range of your function.
   (c) Calculate the height of the cable at a point 100 m from the left suspension tower.
   (d) At what distances from the left suspension tower is the cable less than 50 m above the roadway?

7. A farmer wants to fence two identical adjacent fields as shown in Figure 6.15.
   He has 1200 m of fencing to enclose the two identical regions.
   (a) Write down an expression for the total area, A, in terms of x.
   (b) Find the maximum total area for the two fields and the dimensions x and y.

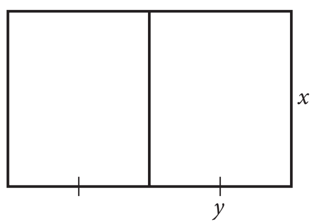


Figure 6.15 Diagram for question 7

8. A shop sells t-shirts for €16 each and ls 40 t-shirts per day. Analysing past sales shows that for every euro increase in price, the shop sells 2 fewer t-shirts per day. Let x be the price increase in euros.
   (a) Write an expression for the price of a single t-shirt in terms of x.
   (b) Write an expression for the number of t-shirts the shop can expect to sell per day in terms of x.

(c) For what value of x should the shop expect to sell no t-shirts?

(d) Hence write an expression for R, the expected revenue per day, in terms of x.

(e) Find the maximum value of R, and the price the shop should set in order to attain that maximum.

9. A decorative archway follows a parabolic curve. The inside height is 5 m and the inside width is 6 m. A large truck must pass through this archway. The truck is 4.3 m tall and 2.6 m wide.

(a) Show that the truck will not fit through the archway.

(b) Find the maximum height of a 2.6 m wide truck that can fit through the archway.

(c) Find the maximum width of a 4.3 m tall truck that can fit through the archway.

10. A tennis player hits a ball straight up. The height of the ball above the ground is described by the model $h(t) = -4.9t^2 + kt + 1$, where $h(t)$ is the height in metres at time t seconds after the ball is hit.

(a) Find the h intercept and interpret in context.

(b) Given that the tennis ball hits the ground after 3.2 seconds, find the value of k.

(c) The tennis ball is more than 10 m above the ground during the time $a < t < b$. Find the values of a and b.

(d) Find the average speed of the tennis ball for the first 0.5 seconds.

(e) Find the maximum height of the tennis ball and the time at which this occurs.

6.3 Cubic models

Cubic models are slightly more complex than quadratic models. We often encounter cubic models when we are dealing with quantities based on volume (such as optimising the volume or surface area of a package, or calculating forces from wind or water). Cubic models are also widely used in computer graphics, in everything from modelling the curves of the letters in this textbook to smoothing computer-generated effects and animation. In this section, we will examine cubic models of the form

$$f(x) = ax^3 + bx^2 + cx + d$$

Note that a simple cubic model, as in the next example, may have b, c, and d equal to zero. In that case, it can also be considered a direct variation model, which we will study later in this chapter.

Example 6.8

The maximum theoretical power that can be generated by a wind turbine can be modelled by the function

$$P = 0.297AdV^3$$

where P is the power in watts

A is the area swept by the turbine blades (swept area), in m^2

d is the air density, in kg m^{-3}

V is the wind speed in m s^{-1}

A certain wind turbine has a swept area of $80\ \text{m}^2$ and is located at sea level, where the air density is $1.225\ \text{kg m}^{-3}$

(a) Find the cubic model for this wind turbine.

(b) Use your model to calculate the maximum theoretical power generated when the wind speed is $10\ \text{m s}^{-1}$

(c) Given that wind speeds above $20\ \text{m s}^{-1}$ are strong enough to cause damage, give a reason why this turbine will not produce more than 300 000 watts.

(d) Determine a reasonable domain for this model.

Solution

(a) We substitute the known values to find the model for this particular turbine. Therefore, the cubic model for this turbine is

$$P = 0.297AdV^3 = 0.297(80)(1.225)V^3 \quad \Rightarrow \quad P = 29.1V^3$$

(b) The maximum theoretical power generated when the wind speed is $10\ \text{m s}^{-1}$ is

$$P = 29.1V^3 = 29.1(10)^3 = 29\,100 \text{ watts (3 s.f.)}$$

(c) Using the model, we can find the wind speed required to produce 300 000 watts.

$$P = 29.1V^3 \quad \Rightarrow \quad 300\,000 = 29.1V^3 \quad \Rightarrow \quad V = \sqrt[3]{10\,309} = 21.8\ \text{m s}^{-1}$$

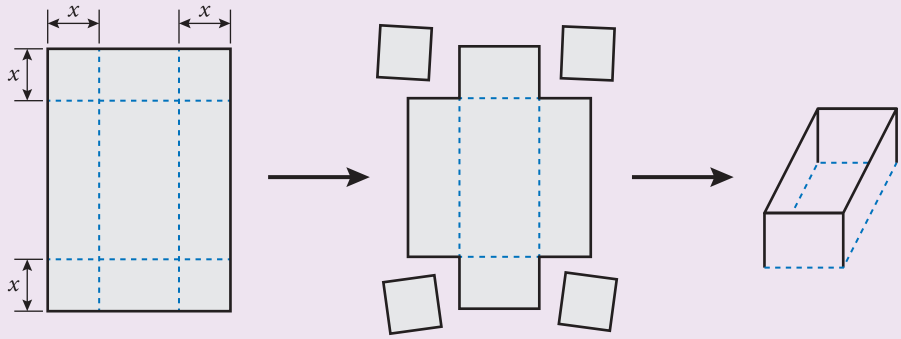
Since this is greater than the wind speed we are told will cause damage, it is not reasonable to expect this turbine to generate more than 300 000 watts.

(d) We are told that wind speeds above $20\ \text{m s}^{-1}$ are strong enough to cause damage, so that can be the upper limit for our domain. It doesn't make sense to predict power for negative wind speeds, so a reasonable domain is $0 \leqslant V \leqslant 20$

We often use models to find **optimal** solutions: that is, we are interested in maximising or minimising a quantity. Example 6.9 examines a classic problem.

Example 6.9

The dimensions of a piece of A4 paper, to the nearest centimetre, are 21×30 cm. It is possible to create an open box by cutting out square corners and folding the remaining flaps up, as shown in the diagram.



Mark 4 fold lines, equidistant from the edges of the paper

Cut out and discard corners

Fold into an open box

(a) Develop a cubic model for the volume of the open box that can be created using this method.

(b) Determine a reasonable domain for your model.

(c) Find the dimensions of the open box with the largest volume that can be created using this method.

(d) Calculate the maximum volume of the open box.

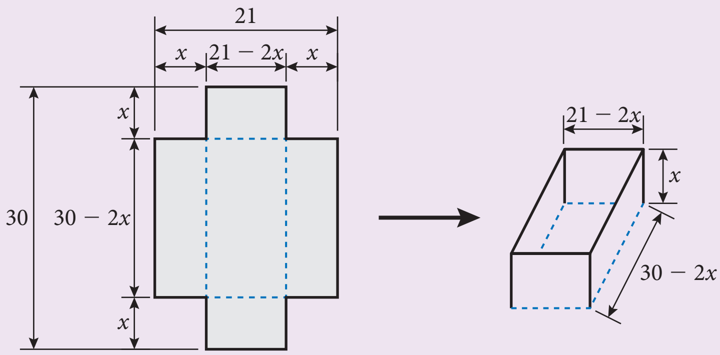
Solution

The first step in solving this problem is to find an appropriate model. Since we are interested in the volume of the box, it seems appropriate to start with a model for volume:

$$V = lwh$$

Then we need to think about what we know. The paper starts as 21×30 cm but those are not the dimensions of the open box. If we look at the second step in the first diagram, we can see that part of the width and length of the paper becomes the height for the box

(a)



The width and length of the paper get reduced by twice the length of the corners we cut out. Therefore, we know that

$$\text{length} = 30 - 2x, \text{ width} = 21 - 2x, \text{ height} = x$$

Now we can use those to develop a model for the volume of the box by substituting into the general model for volume:

$$V = lwh = (30 - 2x)(21 - 2x)(x)$$

(b) It doesn't make sense to remove a square with negative or zero length, so $x > 0$.

Is there an upper bound? What is the largest corner we can cut out? We are limited by the width of the paper. Since the paper is 21 cm wide, we need to cut less than $\frac{1}{2}(21) = 10.5$ cm from the edge in order to make an open box. Therefore, a reasonable domain is $0 < x < 10.5$

(c) Our goal is to find the maximum volume. To do this, we can use our GDC to graph the model and look for a maximum value.

Remember, when using a GDC to analyse a model, it's important to choose the viewing window carefully. Since the x axis represents the distance of each fold from the edge of the paper (which is also equal to the size of each square we cut out), we are only interested in positive x values. Also, because our paper is only 21 cm wide, x must be less than $\frac{21}{2} = 10.5$

Therefore, we set the window to $0 \leqslant x \leqslant 10.5$ and use the GDC's Zoom Fit to scale the y axis accordingly.

From the graph shown in Figure 6.16, we conclude that the value of x that produces the maximum volume is 4.06 cm. To find the dimensions, we need to go back to our expressions for the length, height, and width of the box. We can use these to obtain the missing dimensions:

$$\text{length} = 30 - 2x = 30 - 2(4.06) = 21.9 \text{ cm}$$

$$\text{width} = 21 - 2x = 21 - 2(4.06) = 12.9 \text{ cm}$$

$$\text{height} = x = 4.06 \text{ cm}$$

(d) The volume of the open box is given in the GDC output since the volume is the y coordinate in our GDC. Therefore, $V = 1140 \text{ cm}^3$.

If we expand the model for volume, we get

$$V = (30 - 2x)(21 - 2x)(x)$$
$$\Rightarrow V = 4x^3 - 102x^2 + 630x$$

This shows us that this is indeed a cubic model. However, since we are going to use our GDC to analyse this model, there is no need to expand the model.

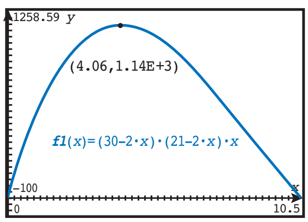


Figure 6.16

Remember that
$1.14\text{E}+3 = 1.14 \times 10^3 = 1140 \text{ cm}^3$

Exercise 6.3

1. Re-do Example 6.9 using a piece of paper with dimensions 8.5×11 inches (such as standard letter paper).

2. Re-do Example 6.9 using a piece of A3 paper, with approximate dimensions 30×42 cm.

t (seconds)	h (metres)
1	105
2	98
3	84
4	60
5	26

Table 6.4 Data for question 3

3. A rock falls off the top of a cliff. Let h be its height above ground in metres, after t seconds. Table 6.4 gives values of h and t.

Jane thinks that the function $f(t) = -0.25t^3 - 2.32t^2 + 1.93t + 106$ is a suitable model for the data. Use Jane's model to:

(a) write down the height of the cliff

(b) find the height of the rock after 4.5 seconds

(c) find after how many seconds the height of the rock is 30 m.

Kevin thinks that the function $g(t) = -5.2t^2 + 9.5t + 100$ is a better model for the data.

(d) Use Kevin's model to find the point at which the rock hits the ground.

(e) Create graphs of f, g, and the data given. By comparing the graphs of f and g with the plotted data, explain which function is a better model for the height of the falling rock.

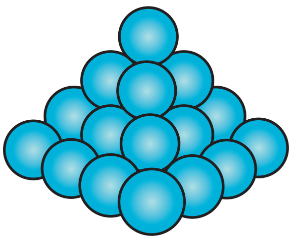


Figure 6.17 Diagram for question 4

4. An efficient way to stack cannonballs is as a pyramid with a square base, as shown in Figure 6.17.

The balls are stacked such that there is 1 ball on the first layer, 4 balls on the second layer, 9 balls on the third layer, and so on.

(a) Copy and complete the table.

Number of layers in stack, n	1	2	3	4
Total number of balls, B	1	5		

(b) The total number of balls, B, can be modelled by a cubic function $B = an^3 + bn^2 + cn$. Using your GDC, or otherwise, find the exact values of a, b, and c.

(c) Find the total number of balls in a stack with 10 layers.

(d) Find the number of layers required to stack 819 balls.

5. The cumulative number of HIV AIDS cases reported in the United States from 1983 to 1998 follows the cubic model

$$C = -222t^3 + 7260t^2 - 12\,700t + 13\,500$$

where C is the cumulative number of HIV AIDS cases, and t is the number of years since 1983.

(a) Find the cumulative number of reported cases in 1990.

(b) Find the year in which the number of cases exceeds 500 000.

(c) If this pattern continues, in what year will the maximum number of cases be reported?

(d) Give a reason why this model will probably not be correct past 1998.

6. A colony of bacteria is exposed to a drug that stimulates reproduction. The number of bacteria is given by the model $P = 1200 + 17t^2 - t^3$ where P is the number of bacteria present t minutes after the drug is introduced, $0 \leqslant t \leqslant 20$

   (a) Find the number of bacteria present when the drug is first introduced.

   (b) Find the number of bacteria present after 5 minutes.

   (c) At what time are there 1000 bacteria?

   (d) Find the maximum number of bacteria and the time at which this occurs.

   (e) At what time are there no bacteria remaining?

7. Researchers are monitoring how a particular drug causes patients' body temperatures to change. They measure the patients' body temperatures just before and 1 hour after the drug is given. After x milligrams per kg of body mass of a drug are given, body temperature increases according to the model $\Delta T(x) = 0.25x^2(1 - 0.1x)$, $0 \leqslant x \leqslant 10$, where ΔT is the change in °C.

   (a) How much will a patient's body temperature change when $4\,\text{mg}\,\text{kg}^{-1}$ are given?

   (b) Dosages between a and b $\text{mg}\,\text{kg}^{-1}$ will increase patients' body temperatures by at least 3°C. Find the values of a and b.

   (c) What is the maximum body temperature increase from this drug, and at what dosage does it occur?

6.4 Exponential models

Exponential models arise in situations where the rate of change is a constant **factor**, that is, when the next value is found by multiplying by a constant factor. This sort of change can produce surprising results.

Just how fast is exponential growth?

A classic fable about the game of chess goes something like this:

A great long time ago a wise man invented the game of chess. This king of this land was so pleased with this new game that he offered the wise man the riches of his kingdom as a gift. The wise man replied, 'I am a simple man, and my needs are modest. Instead of your riches, please place one grain of rice on the first square of the chessboard, two grains of rice on the second square, four grains of rice on the third square, and so on, doubling the number of grains of rice for each of the remaining squares.'

The king replied, 'What a silly man! I offer him the riches of my kingdom and all he asks for is a few grains of rice!'

How much rice did the wise man ask for? To investigate this, let's build a table. Since we are multiplying by 2 each time, we can write the multiplication using exponents as shown in Table 6.5.

Square	Grains of rice	Grains of rice
1	1	$1 = 2^0$
2	2	$1 \times 2 = 2^1$
3	4	$1 \times 2 \times 2 = 2^2$
4	8	$1 \times 2 \times 2 \times 2 = 2^3$

Table 6.5 Building the table

Now, we could keep multiplying by 2 each time until we get to the 64th square, but that seems like a lot of work. If we look carefully, we can see a pattern: the exponent of 2 is equal to one less than the number of the square. Now we can add a few more rows as shown inTable 6.6.

Square	Grains of rice	Grains of rice
1	1	1
2	2	2^1
3	4	2^2
4	8	2^3
…	…	…
n		2^{n-1}
64		2^{63}

Table 6.6 Adding rows to the table

So, we can conclude that the king must place 2^{63} grains of rice on the last square. How much rice is that? If we estimate that one kilogram of rice contains approximately 50 000 grains of rice, then we have

$\frac{2^{63}}{50\,000} = 184\,467\,440\,737\,095 = 1.84 \times 10^{14}$ kg of rice. Is that a lot?

According to the Food and Agriculture Organisation of the United Nations, the estimated worldwide production of rice in 2017 was 759.6 million tonnes, or 7.596×10^{11} kg of rice. Therefore, the amount of rice on the last square alone is $\frac{1.84 \times 10^{14}}{7.596 \times 10^{11}} \approx 240$ times more than the entire worldwide harvest of rice in 2017!

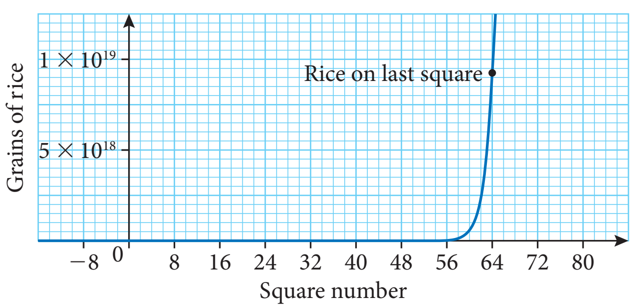


Figure 6.18 $y = 2^{x-1}$

So, in situations where exponential models apply, we can expect to see very fast increases or decreases. To see this more clearly, let's graph the model we developed, $y = 2^{x-1}$, shown in Figure 6.18.

Most calculators can't even graph numbers this large! At the scale of this graph, it looks like there are almost no grains of rice until somewhere around the 56th square. Only if we zoom in to the very first few squares can we see some of the initial growth, as shown in Figure 6.19.

So, both visually and numerically, we can see that exponential functions can describe very rapid change. We will need to keep this in mind when we consider the reasonableness of our models.

Of course, doubling the number of grains of rice on each square of a chessboard might seem like an extreme example. However, many real-life phenomena double in a fixed interval. For example, a single bacterium in ideal conditions can divide itself into two bacteria every 15 minutes! Furthermore, any amount that increases by a fixed fraction will also double in a fixed interval. For example, the mean university tuition increases in the USA is 4.2% per annum. This doesn't sound like much, but it means that the tuition rates have doubled about every 17 years. (Think: for a student at a US university, the cost of their children's education will be at least twice as much as their own, if they have children right away.)

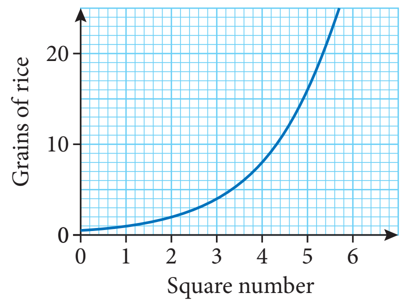


Figure 6.19 Zooming in on the graph

Developing exponential models

Exponential models can sometimes be developed from examining a table, as in the case of the grains-of-rice-on-a-chessboard example. Other times we know about or can hypothesise a percentage growth, doubling time, or growth factor. Any of these can be used to develop an exponential model. Example 6.10 looks at a situation where the growth rate is known.

Example 6.10

Suppose that a population of wombats is increasing at 7% per annum. The population at the beginning of 2018 is recorded to be 240 individuals. Assume that this rate of growth remains constant.

(a) Develop a model for the population of wombats over time.

(b) Use your model to predict the number of wombats at the beginning of 2025.

(c) How long will it take the population of wombats to double?

(d) An ecologist estimates that this region can sustain at most 1000 wombats. In what year will the population reach 1000 wombats?

Solution

(a) To develop a model, we start by making a table and see if we can see a pattern. We start by simply adding 7% for each year. However, to simplify our model, we start with year 0 as 2018.

Year	Years since 2018	Calculation of number of wombats	Number of wombats
2018	0	240	240
2019	1	$240 + 240 \times 0.07$	256.8
2020	2	$256.8 + 256.8 \times 0.07$	274.8
2021	3	$274.8 + 274.8 \times 0.07$	294.0

It is common in models where we are looking over time to think of the starting date as time 0. This could be the starting hour, day, month, or year, but doing so will simplify the calculations in our models. However, we have to be careful to convert back to actual times or dates when interpreting our results.

Although it doesn't make sense to have 256.**8** wombats, we keep the decimal value in order to make our subsequent calculations more accurate. On your GDC, you should store the previous results to maintain full precision in your calculations, as shown below.

24+240·0.07	256.8
256.8+256.8·0.07	274.776
274.776+274.776·0.07	294.01

There doesn't seem to be an obvious pattern so far. However, if we make a change to how we calculate the number of wombats we might have better luck. The key insight relies on a clever factorisation of our calculation:

$240 + 240(0.07)$	Original expression
$= 240(1) + 240(0.07)$	Multiplying by 1 does not change the value of the expression
$= 240(1 + 0.07)$	Distributive property
$= 240(1.07)$	Simplify

Now we rewrite our table:

Year	Years since 2018	Calculation of number of wombats	Number of wombats
2018	0	240	240
2019	1	240(1.07)	256.8
2020	2	256.8(1.07)	274.8
2021	3	274.8(1.07)	294.0

Finally, notice that we are multiplying 1.07 by the previous result each time. But, each previous result is also from multiplying by 1.07. So, we can then write the table like this:

Year	Years since 2018	Calculation of number of wombats	Number of wombats
2018	0	240	240
2019	1	240(1.07)	256.8
2020	2	240(1.07)(1.07)	274.8
2021	3	240(1.07)(1.07)(1.07)	294.0

Now we see that we are repeating the same multiplication, adding another factor of 1.07 in each row. This means that the exponent of the 1.07 factor is equal to the year since 2018. This allows us to write an exponential model in the final row of our table:

Year	Years since 2018	Calculation of number of wombats	Number of wombats
2018	0	240	240
2019	1	$240(1.07)^1$	256.8
2020	2	$240(1.07)^2$	274.8
2021	3	$240(1.07)^3$	294.0
$2018 + n$	n	$240(1.07)^n$	$240(1.07)^n$

Expressed as a function, our model is $P(n) = 240(1.07)^n$ where $P(n)$ is the population of wombats n years since 2018.

(b) Since 2025 is $2025 - 2018 = 7$ years since 2018, our model predicts that the population of wombats will be $P(7) = 240(1.07)^7 = 385$ wombats.

(c) To double, we need to solve the equation $P(n) = 480$, i.e., $480 = 240(1.07)^n$. We can use a GDC to solve this equation. There are two common methods.

GDC graphical method

Graph the left-hand side and the right-hand side of the equation $480 = 240(1.07)^n$ as two separate functions. We need to think carefully about a suitable viewing window. For the x axis, a domain of $0 \leqslant x \leqslant 20$ is a good start. Since the y axis is the number of wombats, and we know we are starting with 240 and looking for the point when the population is 480, we could choose $200 \leqslant y \leqslant 600$ as a place to start and use the Intersect feature to find the solution, as shown in Figure 6.20.

The population of wombats will reach 480 after 10.2 years. Therefore, the population will double every 10.2 years.

GDC numerical solver method

We could also use the numerical solver on a GDC (see Figure 6.21).

Again, we see that the population of wombats will double after 10.2 years.

(d) To find the year during which the population reaches 1200 wombats, solve $P(n) = 1200 \Rightarrow 1200 = 240(1.07)^n$ to obtain $n = 23.8$ (3 s.f.)

Therefore, the population of wombats will reach 1200 after 23.8 years. But what year is that? It is $2018 + 23.8 = 2041.8$, that is, during the year 2041.

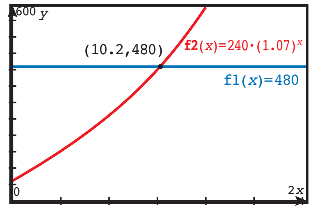


Figure 6.20 GDC graphical method

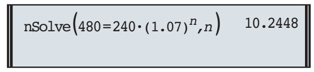


Figure 6.21 GDC numerical solver method

Using a numerical solver feature on a GDC is a quick way to find a solution to an equation if you know that there is only one solution. (You can tell numerical solvers to look near a starting value that you guess, but it can be hard to guess where to start!) So, be sure that you have thought about the equation you are trying to solve and are confident that there is, in fact, only one solution. Otherwise, a numerical solver can give you a solution that doesn't make sense in the context of the equation. In this case, it is often worth taking the time to use the GDC to make a good graph of the model by setting the viewing window appropriately.

We've now looked at two examples of exponential models relating to growth. The technique we used for developing the model is general. (In fact, it even works when there is a constant rate of decrease.) If we look back at the previous two examples, we can see that the general form of an exponential model is as follows.

General form of an exponential model

A quantity that is changing at a fixed fractional rate p per fixed period can be modelled by the exponential function

$$f(x) = k(1 + p)^x$$

where k is the initial value of the quantity.

In the general model, the fixed period we refer to may be days, hours, years, generations, chessboard squares, etc. Remember that p is the fraction of the previous value that is added each time. For example, if a quantity is doubling every period, we are adding 100% so $p = 1$. A quantity that increases by 10% every period has $p = 0.10$

A quantity that decreases by 15% every period has $p = -0.15$ (we call this **exponential decay**).

Sometimes, we add a constant c to the exponential model to obtain $f(x) = k(1 + p)^x + c$

The value of c represents the asymptotic value of the quantity. We will see this in the next example.

Interpreting exponential models

As with any model, we need to be able to understand what an exponential model can tell us and how it might be useful. In this example, we will look at how to interpret an exponential model.

Example 6.11

The value of a certain new car decreases according to the model $V(t) = 39\,000(0.72)^t + 1000$ where $V(t)$ represents the value in euros of the car t years after is it purchased.

(a) Find the original purchase price of the car.

(b) Interpret the meaning of the value 1000 in the model.

(c) Calculate when the value of the car will be half its original value.

Solution

(a) The original purchase price is
$V(0) = 39\,000(0.72)^0 + 1000 = 39\,000 + 1000 = 40\,000$ euros.

(b) The expression $39\,000(0.72)^t$ will tends toward zero as t increases. Therefore, the value 1000 in the model represents the eventual or residual value of the car. (This is probably the value of the car as scrap!)

(c) Since the original price of the car was 40 000, we are looking for $V(t) = 20\,000$ hence $20\,000 = 39\,000(0.72)^t + 1000$

Our GDC numerical solver is useful here (see Figure 6.22).

Therefore, the car will be worth half its original value after about 2.19 years.

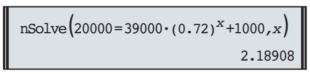


Figure 6.22 GDC numerical solver

Approaching a constant value

If a quantity is approaching a constant value c from an initial value $k + c$, and the difference $k - c$ is changing at a fixed fractional rate p per fixed period, the growth or decay in the quantity can be modelled by the exponential function

$f(x) = k(1 + p)^x + c$

Notice that when we use an exponential model of the form $f(x) = k(1 + p)^x + c$, the value k is no longer the initial value.

The generic statement of the model makes sense: if we consider Example 6.11, the original purchase price of €40 000 is not decreasing by a fixed percentage. Rather, it is the amount above the residual value of €1000 that is decreasing at a fixed percentage per period. Table 6.7 illustrates this idea.

Years since purchase	Value (€)	Percent change in total value of car	Value (€) above residual value of €1000	Percent change in value above residual value of car
0	40 000		39 000	
1	29 080	−27.3%	28 080	−28.0%
2	21 218	−27.0%	20 218	−28.0%
3	15 557	−26.7%	14 557	−28.0%
4	11 481	−26.2%	10 481	−28.0%

Table 6.7

It's common in science applications to use a different form of the exponential model. Instead of changing the base of the exponent by adding the fractional rate p to 1, we add a parameter r as a factor in the exponent and use the special base e: $f(x) = ke^{rx}$. The value of r is often determined through experimental evidence. It turns out that the two models are equivalent, as Example 6.12 will show.

Example 6.12

A student measures the temperature of a cup of coffee at regular intervals in a room where the temperature is 20°C. Table 6.8 shows her data.

(a) Develop an exponential model of the form $f(x) = ke^{rx} + c$, where k, r, and c are constants to be determined.

(b) Use your model to find the temperature of the coffee after 8 minutes.

(c) Use your model to predict when the temperature of the coffee will reach room temperature, to 3 significant figures.

(d) Find the rate of decrease in the difference between the coffee temperature and room temperature as a percentage per minute.

Time (minutes)	Temperature (°C)
0	95
1	79.00
2	66.41
3	56.51
4	48.72

Table 6.8 Data for Example 6.12

Solution

(a) We know the coffee is cooling, so our model must be a decreasing exponential function. We also know that a decreasing exponential function will tend towards zero unless a constant is added, so the expression ke^{rx} must represent the (decreasing) difference between the room temperature and the coffee, while c represents room temperature (20°C). Therefore, $c = 20$. Since $f(0) = 95$, we have

$$95 = ke^{r(0)} + 20 \quad \Rightarrow \quad k = 75$$

Thus, so far we have $f(x) = 75e^{rx} + 20$

Using the data value from the first minute, we know $f(1) = 79.0$ so

$79.0 = 75e^{r(1)} + 20$	Substitute known values
$0.787 = e^r$	Subtract 20 and divide by 75
$\ln 0.787 = r \ln e$	Apply natural logarithm and use $\ln a^n = n \ln a$
$-0.240 = r$	Evaluate with GDC

Therefore, the model is

$$f(x) = 75e^{-0.240x} + 20$$

(b) After 8 minutes, the coffee temperature is $f(8) = 75e^{-0.240(8)} + 20 = 31.0$°C

(c) Theoretically, the coffee cu p will never reach room temperature according to the model. However, it will be within three significant figures of room temperature when the temperature is less than 20.05°C. So, we are looking for the time when $f(x) = 20.05$

Algebraic approach

$20.05 = 75e^{-0.240x} + 20$	Substitute known values
$6.67 \times 10^{-4} = e^{-0.240x}$	Subtract 20 and divide by 75
$\ln(6.67 \times 10^{-4}) = -0.240x \ln e$	Apply natural logarithm and use $\ln a^n = n \ln a$
$30.5 = x$	Evaluate with GDC, divide by -0.240

Therefore the coffee temperature will be within 3 significant figures of room temperature after 30.5 minutes.

GDC approach

We can enter the equation directly into the numeric solver to obtain the same result as above.

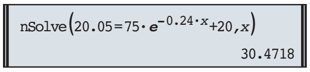


Figure 6.23 GDC numerical solver

(d) At first, it might seem that we could simply calculate the percentage decrease from the data table. If we try this, we get this table.

Time (minutes)	Temperature (°C)	Percentage change from previous minute
0	95	
1	79.00	−16.8%
2	66.41	−15.9%
3	56.51	−14.9%
4	48.72	−13.8%

This doesn't make sense, since the percentage change must be constant. We need to read the question more carefully: 'Find the rate of decrease in the difference between the coffee temperature and room temperature…' Let's try another table.

Time (minutes)	Temperature (°C)	Difference between coffee temperature and room temperature	Percentage change from previous minute
0	95	75	
1	79.00	59.00	−21.3%
2	66.41	46.41	−21.3%
3	56.51	36.51	−21.3%
4	48.72	28.72	−21.3%

Therefore, the percentage change per minute in the difference between the coffee temperature and room temperature is −21.3%

Notice that in part (d), what we are doing is removing the effect of the room temperature and considering the cooling rate of the coffee. Algebraically, that means we are considering the expression $75e^{-0.240x}$. If we are looking for a percentage rate, we could convert this expression into a percentage-decrease model by using a law of exponents:

$75e^{-0.240x}$	Original expression
$= 75(e^{-0.240})^x$	Using the law $(x^a)^b = x^{ab}$
$= 75(0.787)^x$	Evaluate $e^{-0.240}$ using GDC

Notice that $1 - 0.213 = 0.787$, confirming our result that the coffee temperature difference is decreasing by a fixed rate of 21.3%

Graphical interpretation

It's worth looking at the graphs of the last few models. Notice that in each case, the location of the horizontal asymptotes is given by the value of c in the model $f(x) = k(1 + p)^x + c$

For the wombats in Example 6.10, $c = 0$ so the asymptote is at $y = 0$. In that context, the horizontal asymptote is not meaningful because it occurs in negative 'years since 2018,' so there doesn't seem to be a useful interpretation of it. For the car in Example 6.11 (not shown), $c = 1000$ so the asymptote is at $y = 1000$. In that context, it represents the eventual (residual) value of the car.

For the coffee in Example 6.12 (Figure 6.25), $c = 20$ so the asymptote is at $y = 20$. In that context, it represents the room temperature, which is the eventual temperature of the coffee.

A common use of exponential functions is half-life calculations.

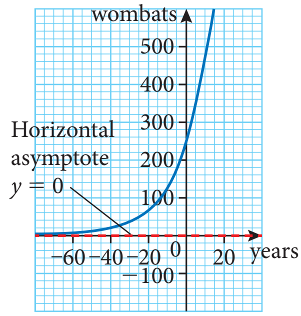


Figure 6.24

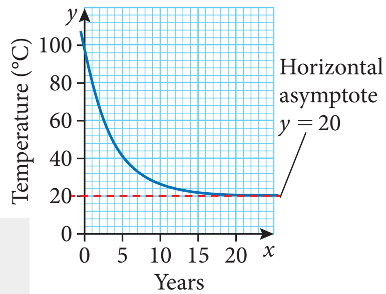


Figure 6.25

> Half-life refers to a process of radioactive decay. A substance's half-life is the average amount of time it takes for the activity of a substance to decrease to half its previous value. Activity refers to the number of nuclei decaying at any time, measured in decays per second or Becquerel (Bq).
>
> For example, the half-life of fermium-253 is 3 days. If the activity of a sample of fermium-253 is 100 Bq, then after 3 days the activity will have decreased to 50 Bq. After another 3 days, the activity will be 25 Bq.

Horizontal asymptotes

For an exponential model of the form

$f(x) = k(1 + p)^x + c$ or
$f(x) = ke^{rx} + c$

the graph of f has a horizontal asymptote at $y = c$

Example 6.13

All living things on Earth continuously take in carbon. Since a tiny fraction of the carbon on Earth is the naturally occurring radioactive isotope carbon-14, the fraction of carbon-14 in all living things is constant. However, once an organism dies, it stops taking in carbon and the carbon-14 in its body begins to decay. Because of this, scientists can use carbon-14 dating to determine how many years have passed since an organism died.

A model for carbon-14 dating is $A(t) = ke^{-0.000121t}$, where $A(t)$ is the activity of carbon-14 remaining from an initial activity k, after t years.

(a) Show that the half-life of carbon-14 is 5730 years, to 3 significant figures.

(b) A tissue sample from a body discovered on the border between Austria and Italy has an average activity of 6.92 Bq. A sample of live tissue of the same size has an average activity of 12 Bq. Determine the age of the sample.

Solution

(a) Using the model, $A(t) = ke^{-0.000121t}$, a half-life is the time taken for the activity of carbon-14 to halve.

Therefore, we can set $k = 2$ and $A(t) = 1$ to find
$1 = 2e^{-0.000121t} \quad \Rightarrow \quad 5728.5 = t$

Therefore, to 3 significant figures, the half-life of carbon-14 is 5730 years.

(b) Using the model, we obtain $6.92 = 12e^{-0.000121t} \quad \Rightarrow \quad 4550 = t$

Therefore, the age of the tissue sample is 4550 years.

Exercise 6.4

1. When a skydiver is falling towards the Earth, they will accelerate until the force of gravity is equal to air resistance. The velocity of the skydiver at this point is called terminal velocity. A skydiver records the difference between her velocity and terminal velocity every 5 seconds and obtains the data shown in the table.

Free fall time (s)	0	5	10	15	20
Difference between velocity and terminal velocity ($\mathrm{m\,s^{-1}}$)	56	23.4	9.79	4.10	1.71

(a) Develop an exponential model for this data.

(b) Find the difference between velocity and terminal velocity at 7 seconds.

(c) When is the skydiver first within $5\ \mathrm{m\,s^{-1}}$ of terminal velocity?

(d) Predict the time when the skydiver will be within $1\ \mathrm{m\,s^{-1}}$ of terminal velocity.

2. The number of bacteria in two colonies, A and B, starts increasing at the same time. The number of bacteria in colony A after t hours is modelled by the function $A(t) = 12e^{0.4t}$

(a) Find the number of bacteria in colony A after four hours.

(b) How long does it take for the number of bacteria in colony A to reach 400?

The number of bacteria in colony B after t hours is modelled by the function $B(t) = 24e^{kt}$ After four hours, there are 60 bacteria in colony B.

(c) Find the value of k.

(d) The number of bacteria in colony A first exceeds the number of bacteria in colony B after n hours, where $n \in \mathbb{Z}$. Find the value of n.

3. The concentration of medication in a patient's bloodstream is given by $C(t) = 9(0.5)^{0.021t}$, where C is in milligrams per litre t minutes after taking the medicine.

(a) Write down $C(0)$.

(b) Find the concentration of medication left in the patients' bloodstream after 40 minutes.

(c) A patient takes the medicine at 14:00. The patient should take the medicine again when the concentration of medication reaches 0.350 milligrams per litre. What time should the patient take the medicine again?

4. A large lizard known as a Gila Monster is about 16 cm long at birth. For the first 8 years of its life, the Gila Monster's length increases by about 8% each year.

(a) Write a function to model the length L of a Gila Monster t years after birth.

(b) Estimate the length of a 3-year-old Gila Monster.

(c) Estimate the age of a 25-cm long Gila Monster.

5. DDT is a toxic insecticide that was widely used in the past. The function $A = A_0 e^{kt}$ can be used to describe the amount A of DDT left in an area t years after an initial application of A_0 units.

(a) Given that the half-life of DDT is about 15 years, find the value of k.

(b) Find the amount of DDT left after 2 years when 50 units were initially applied to an area.

(c) A sample of soil is tested and 35 units of DDT are found. It is known that the last application of DDT was 20 years ago. Find the initial amount of DDT applied to this soil.

(d) In one area, 120 units of DDT are applied. Given that the safe level of DDT is 40 units, for how long will the area be unsafe?

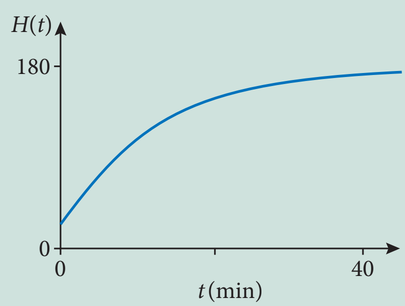
6. In certain soil conditions, the half-life of the pesticide glyphosate is about 45 days. A scientist studying how this chemical decays wrote in his notes that the data in an experiment could be modelled with the function $A(t) = 500(0.5)^t$

(a) Find $A(0)$ and interpret in context.

(b) Explain what the variable t represents in this context.

(c) Find $A(1)$ and interpret in context.

7. Sameera is baking a birthday cake. She places the cake mix in a preheated oven. The temperature in the centre of the cake mix in °C is modelled by the function $H(t) = 180 - a(1.08)^{-t}$ where time t is in minutes after the mix is placed in the oven. The graph of $H(t)$ is given.



(a) Write down what the value of 180 represents in the context of the question.

(b) The temperature in the centre of the cake mix was 22°C when placed in the oven. Find the value of a.

(c) The cake is removed from the oven 29 minutes after the temperature in the centre of the cake has reached 150°C.

Find the total time that the baking tin is in the oven.

8. The number of fish, N, in a pond is decreasing according to the model $N(t) = ab^{-t} + 40, t \geqslant 0$ where a and b are positive constants, and t is the time in months since the number of fish in the pond was first counted. The fish are first counted in January. Some data collected is shown in the table.

Month	January	May	September
Fish	850	100	?

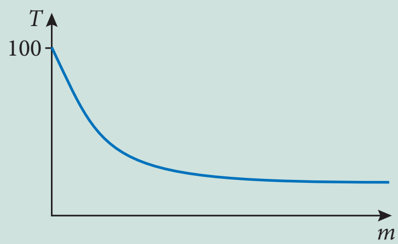
(a) Find the value of a and of b.

(b) Use your model to estimate the number of fish in September.

(c) Use your model to find the first month when there were less than 50 fish.

(d) The number of fish in the pond will not decrease below p. Write down the value of p.

9. A cup of boiling water is placed in a room to cool. The temperature of the room is 20°C. This situation can be modelled by the exponential function $T = a + b(k^{-m})$ where T is the temperature of the water, in °C, and m is the number of minutes for which the cup has been placed in the room. A sketch of the situation is given.



(a) Explain why $a = 20$

Table 6.9 shows some data about the temperature of the water.

Time (minutes)	Temperature (°C)
0	100
2	85
5	?
?	35

Table 6.9 Data for question 9

(b) Find the value of b and of k.

(c) Find the temperature of the water 5 minutes after it has been placed in the room.

(d) Find the total time needed for the water to reach a temperature of 35°C. Give your answer in minutes and seconds, correct to the nearest second.

10. A potato is placed in an oven heated to a temperature of 200°C. The temperature of the potato, in °C, is modelled by the function $p(t) = 200 - 190(0.97)^t$, where t is the time, in minutes, that the potato has been in the oven.

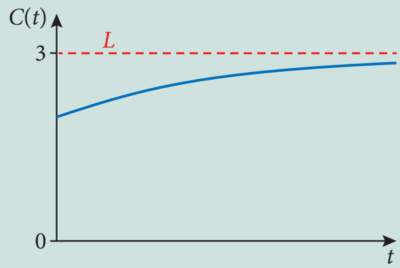
(a) Write down the temperature of the potato at the moment it is placed in the oven.

(b) Find the temperature of the potato half an hour after it has been placed in the oven.

(c) After the potato has been in the oven for k minutes, its temperature is 40°C.

Find the value of k.

11. The amount of electrical charge in a rechargeable battery is modelled by the function $C(t) = 3 - 2.4^{-t}$ where $C(t)$ is the amount of charge after t hours of charging.



(a) Write down the amount of electrical charge in the battery at $t = 0$

(b) The line L is the horizontal asymptote to the graph. Write down the equation of L.

(c) The charging of the battery should stop when it is within 1% of the maximum charge possible. After how many hours should the charging stop?

12. The number of cells, C, in a culture is given by the equation $C = p \times 2^{0.5t} + q$, where t is the time in hours measured from 12:00 on Monday and p and q are constants. The number of cells in the culture at 12:00 on Monday is 47. The number of cells in the culture at 16:00 on Monday is 53.

(a) Write down two equations in p and q .

(b) Calculate the value of p and of q.

(c) Find the number of cells in the culture at 22:00 on Monday.

6.5 Direct and inverse variation

Direct variation and inverse variation are two types of models that occur so often in real life that they deserve a special look.

In the following two sections, we will examine direct and inverse variation models more carefully.

Direct variation

Suppose a ball is dropped from the roof of a building. The distance the ball has fallen varies directly with the square of the time since it was dropped. Symbolically, we write

$$d = at^2$$

where d is the distance travelled, t is the time elapsed, and a is a **constant of variation** that we will determine.

At the instant the ball is dropped, it has travelled 0 metres. One second later, it has travelled 4 metres.

In general, **direct variation** occurs when one quantity increases along with another quantity. **Inverse variation** occurs when one quantity decreases along with another quantity.

If you have studied the falling-object model or physics, you might protest that the constant of variation must be equal to $\frac{1}{2}g = 4.91$ m s^{-1}! However, remember that the falling-object model only perfectly describes objects falling in a vacuum or with negligible air resistance. In Earth's atmosphere, the air resistance will decrease the acceleration of the object, so that its acceleration will be less than 4.91 m s^{-1}. The ball in this example could be a lightweight perforated plastic ball.

By two seconds, it has travelled 16 metres. By three seconds, it has travelled 36 metres. We can express this data as ordered pairs:

$$(0, 0),\ (1, 4),\ (2, 16),\ (3, 36)$$

How can we find the constant of variation? We simply need to substitute one of the ordered pairs into the model:

$$4 = a(1)^2 \ \Rightarrow\ a = 4$$

Therefore the direct variation model is

$$d = 4t^2$$

To b✓e sure, we can check the other points for agreement:

$$(2, 16) \ \Rightarrow\ d = 4(2)^2 = 16 \ ✓$$

$$(3, 36) \ \Rightarrow\ d = 4(3)^2 = 36 \ ✓$$

Notice that we cannot use the point (0, 0) to find the constant of variation. Why not? The point (0, 0) produces the equation $0 = 0$

Direct variation models are simplified polynomial models of the form

$$y = ax^n \text{ where } n \in \mathbb{Z},\ n > 0$$

The constant a is sometimes called **the constant of variation**.

In common usage, direct variation is often used only to refer to linear direct variation models, where $y = ax$

Here we use a broader definition, as shown in the key fact box.

Example 6.14

The cost C of a phone call varies directly with the length of the call m in minutes. A recent 5-minute phone call cost \$0.60.

(a) Write a direct variation model for this situation.

(b) Use your model to predict the cost of a 7-minute phone call.

(c) Find the length of a call that cost \$1.56.

(d) Write down the cost per minute for phone calls.

Solution

(a) Since a 5-minute phone call cost \$0.60, we have

$$C = am \ \Rightarrow\ 0.60 = a(5) \ \Rightarrow\ a = 0.12 \text{ therefore the model is } C = 0.12m$$

(b) A 7-minute phone call would cost $C = 0.12(7) = \$0.84$

(c) A call that cost \$1.56 must have $1.56 = 0.12m \ \Rightarrow\ m = 13$ minutes.

(d) From the model, we see that $a = 0.12$. Therefore, calls cost \$0.12 per minute.

Example 6.15

The volume of a sphere varies directly with the cube of the radius of the sphere. You are given that a sphere with a radius of 14 cm has a volume of 11 500 cm^3 (3 s.f.).

(a) Find a direct variation model.

(b) Use your model to find the volume of a sphere with a radius of 7 cm, to 3 significant figures.

(c) Using 6 significant figures, find the percentage error between your model and the volume given by the formula $V = \frac{4}{3}\pi r^3$ for a sphere with radius 7 cm, to three significant figures.

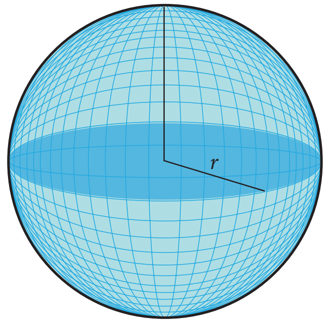


Figure 6.26 Diagram for Example 6.15

Solution

(a) Since the volume varies directly with the cube of the radius, we have

$$V = ar^3$$

Given that a sphere with radius 14 cm has a volume of 11 500 cm^3, we can solve for the constant of variation:

$$11\,500 = a(14)^3 \;\Rightarrow\; a = 4.19 \text{ (3 s.f.)}$$

Therefore, our model is $V = 4.19r^3$

(b) For a sphere of radius 7 cm, we have

$$V = 4.19(7)^3 = 1437.17 \approx 1440 \text{ cm}^3 \text{ (3 s.f.)}$$

(c) The theoretical volume of a 7 cm sphere is $V = \frac{4}{3}\pi(7)^3 = 1436.76 \text{ cm}^3$

There percentage error is therefore $\frac{|1436.76 - 1437.17|}{1436.76} \times 100 = 0.0285\%$

Inverse variation

Suppose you want to hire a DJ for a party. The cost of the DJ is \$500 for the night. To cover the cost, you plan to sell tickets. What price should you set for the tickets, based on how many you think you will sell? This is a case of inverse variation, because the more tickets you sell, the lower the price per ticket can be in order to cover the cost of hiring the DJ. Consider a few data points: if you only sell one ticket, the ticket price will need to be \$500 to cover costs (but it would be a lonely party!). If you can sell two tickets, then each one can be \$250 (party for two!). If you can sell 10 tickets, then each one is $\frac{500}{10} = \$50$

For 20 tickets, each one is $\frac{500}{20} = \$25$. Ah-ha! We have already developed the model without even thinking (too hard) about it: to cover costs, the ticket price is the total cost divided by the number of tickets you can sell. Symbolically:

$$P = \frac{500}{n}$$

where P is the ticket price and n is the number of tickets you must sell.

In Figure 6.27, we have drawn the graph as a series of dots but have not connected them. Why not? It's not possible to sell 1.5 tickets – we can only sell whole tickets. So, we plot the graph as a series of points rather than a smooth function. However, for convenience, we often draw the graph as a smooth function even though that may not strictly represent reality.

This produces an interesting graph, shown in Figure 6.27. Price per ticket varies inversely with the number of tickets sold.

Notice that in Figure 6.27 the graph approaches – but does not intersect – the horizontal axis. This makes sense, because as we sell more and more tickets, we could reduce the price per ticket. But, if we give away the tickets for free, we can't possibly cover the cost of the DJ.

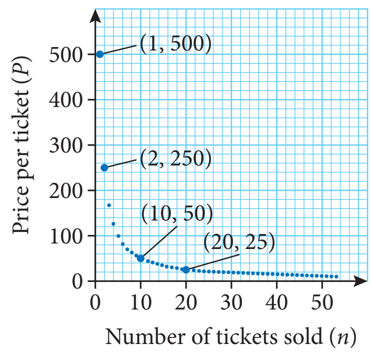


Figure 6.27 Price per ticket varies inversely with the number of tickets sold

In this example, we can rewrite our model using a negative exponent:

$$P = \frac{500}{n} \Rightarrow P = 500n^{-1}$$

Inverse variation models are of the form $y = ax^n$ where $n \in \mathbb{Z}, n \leqslant 0$

The constant a is sometimes called the **constant of variation**.

This leads us to the definition of inverse variation models shown in the key fact box.

Notice that the only difference between direct and inverse variation models is the sign of the exponent.

While it is quite common to state a weight in kg, it is technically incorrect. Kilogram (kg) is a unit of mass, not of weight. Weight is a force, which can be measured in Newtons (N) or kilogram-force (kgf); 1 kgf is equal to 9.81 N. However, kgf is a non-standard unit.

Example 6.16

The weight of an object varies inversely with the square of the distance from the centre of the Earth. At sea level (6370 km from the centre of the Earth), an astronaut weighs 600 N.

(a) Find a model for the weight of the astronaut based on the distance from the centre of the Earth.

(b) The International Space Station orbits at an altitude of 408 km. Find the weight of the astronaut while on board the International Space Station.

(a) Find the required distance from the centre of the Earth for the astronaut to weigh half what she weighs at sea level.

Solution

The answer to (b) might seem counter-intuitive: if the astronaut still weighs 530 N on the International Space Station, why do we always see videos of astronauts floating? The answer is that the space station and the astronauts inside it are actually falling towards the Earth at the same rate, so the astronauts seem to be 'floating' inside the space station..

(a) Since the weight varies inversely with the square of the distance, we can write $w = \frac{a}{d^2}$ where w is the weight in kg, d is the distance from the centre of the Earth, and a is the constant of variation that we will determine. We find a by substituting known values:

$$w = \frac{a}{d^2} \Rightarrow 600 = \frac{a}{6370^2} \Rightarrow a = 2.43 \times 10^{10}$$

Therefore our model is $w = \frac{2.43 \times 10^{10}}{d^2}$

(b) At an altitude of 408 km, the distance from the centre of the Earth is $6370 + 408 = 6778$ km

Therefore the weight of the astronaut would be $w = \frac{2.43 \times 10^{10}}{6778^2} = 530$ N

(c) We need to solve

$$300 = \frac{2.43 \times 10^{10}}{d^2} \Rightarrow 300d^2 = 2.43 \times 10^{10}$$

$$\Rightarrow d = \sqrt{\frac{2.43 \times 10^{10}}{300}} = 9010\,\text{km (3 s.f.)}$$

Exercise 6.5

1. Choose always/sometimes/never:

(a) Direct variation models always/sometimes/never pass through the origin (0, 0).

(b) Direct variation models are always/sometimes/never a type of polynomial model.

(c) Direct variation models are always/sometimes/never a type of linear model.

(d) Direct variation models are always/sometimes/never a type of exponential model.

2. Choose always/sometimes/never:

(a) Inverse variation models always/sometimes/never pass through the origin (0, 0).

(b) Inverse variation models are always/sometimes/never a type of polynomial model.

(c) Inverse variation models are always/sometimes/never a type of linear model.

(d) Inverse variation models are always/sometimes/never a type of exponential model.

3. Given that y varies directly with x, and $y = 462$ when $x = 11$, find:

(a) y when $x = 5$ **(b)** x when $y = 672$

4. Given that y varies directly with the square of x, and $y = 10$ when $x = 5$, find:

(a) y when $x = 20$ **(b)** x when $y = 40$

5. Given that y varies directly with the cube of x, and $y = 250$ when $x = 5$, find:

(a) y when $x = 8$ **(b)** x when $y = 128$

6. Given that y varies inversely as x, and $y = 10$ when $x = 5$, find:

(a) y when $x = 20$ **(b)** x when $y = 0.5$

7. Given that y varies inversely as the square of x, and $y = 10$ when $x = 5$, find:

(a) y when $x = 20$ **(b)** x when $y = 2.5$

8. Given that y varies inversely as the cube of x, and $y = 54$ when $x = 5$, find:

(a) y when $x = 15$ **(b)** x when $y = 250$

9. The height of an image produced by a projector varies directly with the distance from the screen. The image is 1.5 metres tall when the projector is 2 metres from the screen.

(a) Find a direct variation model for the size of the image S given the distance d from the screen.

(b) Hence predict the size of the image for a projector 7 metres from the screen.

(c) Use your model to find the distance required to project a 10-metre image.

10. The velocity of a falling object varies directly with the amount of time it has been falling.

(a) Given that an object that has been falling for two seconds has a velocity of 19.6 m s^{-1}, calculate the constant of variation.

(b) Calculate the velocity of an object that has been falling for 4 seconds.

(c) Terminal velocity for a skydiver is approximately 200 km h^{-1}. Calculate the approximate number of seconds it takes a skydiver to reach terminal velocity according to this model.

11. The position of a falling object varies directly with the square of the number of seconds it has been falling. We assume the initial velocity is zero.

(a) Given that an object that has been falling for 10 seconds towards the surface of the Moon travels 162 m, find the constant of variation.

(b) Calculate the distance an object falls in 5 seconds.

(c) Calculate the time required for an object to fall 200 m.

12. The radius of a satellite's orbit around the Earth varies inversely with the square of the velocity of the satellite.

(a) Given that a satellite that travels at a velocity of 7700 m s^{-1} has an orbital radius of 6.75×10^6 m, calculate the constant of variation.

(b) Find the orbital velocity of a satellite with an orbital radius of 7.0×10^6 m.

(c) Find the orbital radius of a satellite with a velocity of 8000 m s^{-1}.

A regular dodecahedron has 12 pentagonal faces.

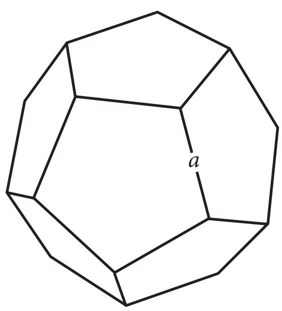


Figure 6.28 Dodecahedron

13. The volume of a regular dodecahedron varies directly with the cube of the length of an edge a.

(a) Give that the volume of a dodecahedron is 958 cm^3 with the edge length $a = 5$, find the constant of variation.

(b) Find the volume of a dodecahedron when the edge length is 8 cm.

(c) Find the edge length of a dodecahedron with volume 100 m^3.

14. The power a wind turbine can generate varies directly with the cube of the wind speed. A certain turbine can generate 314 W when the wind speed is 8 m s^{-1}.
   (a) Find the constant of variation.
   (b) Find the power generated when the wind speed is 12 m s^{-1}.
   (c) Find the wind speed necessary to generate 2000 W.

6.6 Trigonometric models

Trigonometric models are well-suited to describing phenomena that repeat themselves: tides, seasonal temperatures, motion of wheels, etc. In this section, we will examine the models $y = a\sin(bx) + d$ and $y = a\cos(bx) + d$

Exploration

To be able to use the sine and cosine functions to model effectively, you need to understand how the parameters a, b, and d affect the graphs of the functions and the difference between the graphs of sine and cosine. This is best done by using a graphing application such as Geogebra and experimenting with different values of these parameters. Your goal is to observe carefully the effect each parameter has on the shape of the graph. Follow the suggestions below to help you to understand what a, b, and d do. Make sure you work in **degree** mode.

1. What does *a* do?

Use your graphing program to generate a graph of $y = a\sin x$. Experiment with different values of a and notice the effect on the graph. Make sure to set a to at least the following four values: $a = 1, \frac{1}{2}, 2, -2$

You should notice that $|a|$ is equal to the **amplitude** of the graph, which is half of the **wave height**, as shown in Figure 6.29. Also, if a is negative, then the graph is reflected across the x axis.

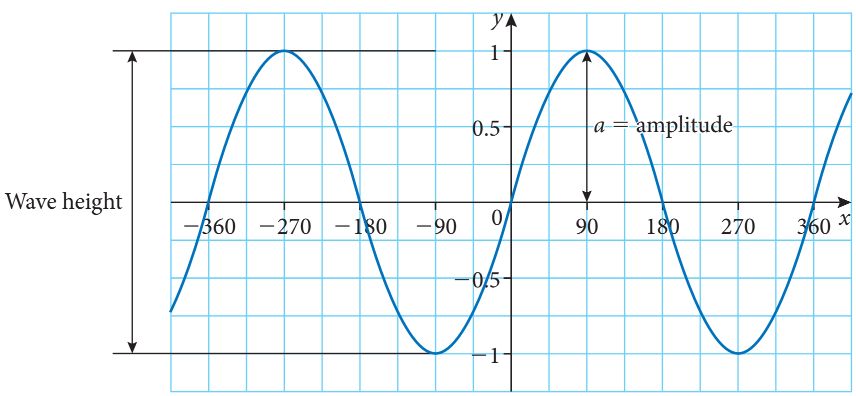


Figure 6.29 Amplitude is half of wave height

2. What does b do?

Use your graphing program to generate a graph of $y = \sin(bx)$. Experiment with different values of b and notice the effect on the graph. Make sure to set b to at least the following four values: $b = 1, \frac{1}{2}, 2, -2$

Value of b	Length of period (p)
1	360°
2	180°
$\frac{1}{2}$	720°
−2	180°

Table 6.10 Values of b and length of period

You should notice that b controls the **wavelength** of the graph. In mathematics and science, we refer to this as the **period** of the function. The period is the distance it takes for the graph to repeat itself – the distance between two adjacent local maxima or local minima, for the sine and cosine functions.

But, how exactly does b affect the period? If we write down the values of b and the length of the period, we get Table 6.10. What can we observe? First we notice that the sign of b does not seem to make a difference to the period since the period is the same for both 2 and −2. Next, we see that as the absolute value of b increases, the period gets shorter. This suggests that the length of the period and the value of b vary inversely. We can also see from the table that the product of $|b|$ and the period is a constant 360°. Therefore, we conclude that $|b|p = 360° \Rightarrow p = \frac{360°}{|b|}$. This is shown in Figure 6.30.

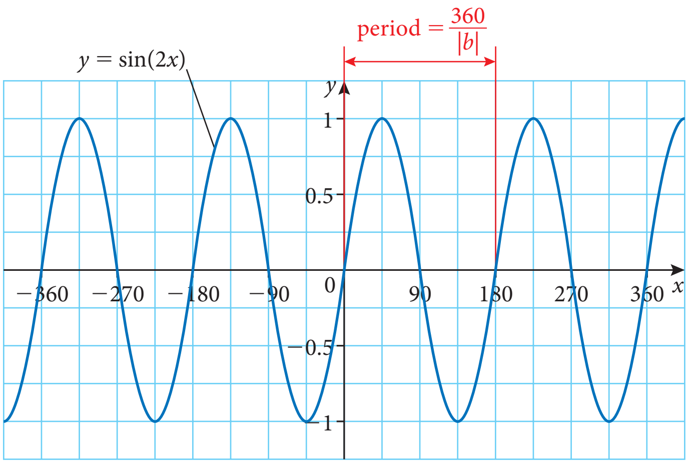


Figure 6.30 Relationship between period and b

Also, notice that if b is negative, then the graph is reflected across the y axis.

3. What does d do?

Use your graphing program to generate a graph of $y = \sin(x) + d$. Experiment with different values of d and notice the effect on the graph. Make sure to set d to at least the following four values: $d = 0, \frac{1}{2}, 2, -2$

You should notice that d controls the vertical position of the graph. Specifically, the value of d gives us the position of the **principal axis** of the function. The principal axis is a horizontal line with equation $y = d$ (Figure 6.31).

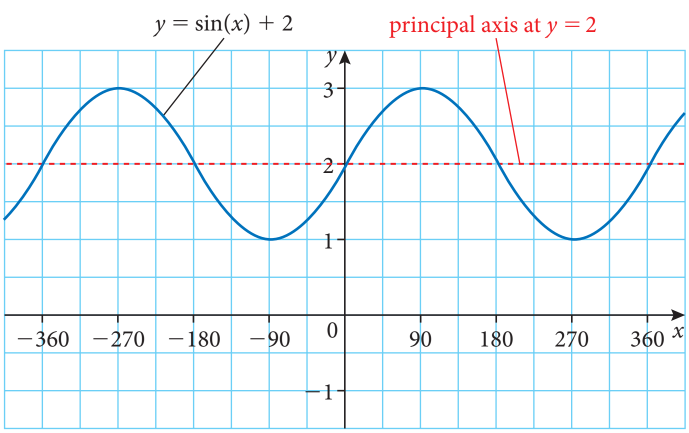


Figure 6.31 The principal axis of a periodic function

What is the difference between the graphs of sine and cosine?

Notice that in the questions above, we have only used sine. This is because the graphs of sine and cosine are very similar – the properties of a, b, and d are the same for both. However, there is an important difference between the graphs of sine and cosine. Graph the functions $y = \cos x$ and $y = \sin x$ and look carefully at the graphs.

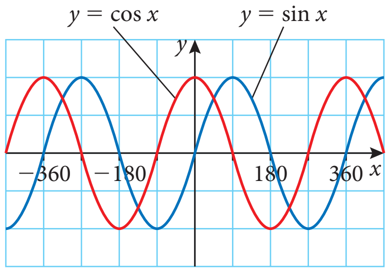


Figure 6.32 Comparison of sine and cosine graphs

Look at the behaviour of both functions when $x = 0$, as shown in Figure 6.32. Cosine has a maximum at $x = 0$ and then decreases. For sine, when $x = 0$ the graph increases from where the principal axis meets $x = 0$. Is that always true? It's not – remember how the parameters a, b, and d affect the graphs. For the cosine function, because the value of a can cause the graph to be reflected across the principal axis, when $a < 0$ cosine will have a minimum at $x = 0$ and then increase. For the sine function, when $a < 0$ the curve will decrease from where the principal axis meets $x = 0$. These graphs are shown in Figure 6.33.

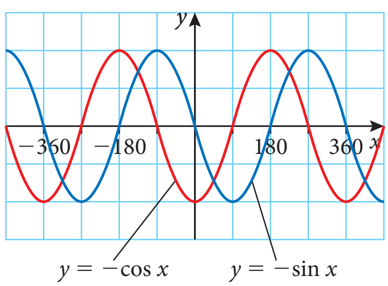


Figure 6.33 When $a < 0$, the graphs of sine and cosine are reflected across the x axis. This causes a change in behaviour around $x = 0$ that is important for modelling

The key findings of the exploration are summarised here.

For the trigonometric models $y = a\sin(bx) + d$ and $y = a\cos(bx) + d$

- Amplitude $= |a|$
  The amplitude is equal to half the wave height.
- For $a < 0$, the graph is reflected across the x axis.
- Period $= \dfrac{360°}{|b|}$
- For $b < 0$, the graph is reflected across the y axis.
- The principal axis is at $y = d$
- Use a sine model when you want the initial value of the model to be equal to the location of the principal axis (half-way between the maximum and minimum values). If the model should initially increase, let $a > 0$. If the model should initially decrease, let $a < 0$.
- Use a cosine model when you want the initial value of the model to be a maximum (let $a > 0$) or minimum (let $a < 0$).

Developing trigonometric models

Now that we have an understanding of the parameters of trigonometric models, we can model some real-life periodic phenomena.

Example 6.17

A reflector is attached to a bicycle wheel at a point about 17 cm from the centre of the wheel. The radius of the wheel is approximately 30 cm. The wheel rotates clockwise and takes 4 seconds to complete one revolution.

(a) Assuming that the reflector is at the top-most position at time $t = 0$ develop a model for the height h of the reflector above the ground at any time t in seconds.

(b) Revise your model so that the reflector is at the bottom-most position at time $t = 0$

(c) Revise your model so that the reflector is at the right-most position at time $t = 0$

(d) Revise your model for a point on the outer edge of the wheel. Assume that the point is at the left-most position at time $t = 0$

Solution

(a) Since the radius of the wheel is 30 cm, and the reflector is 17 cm from the centre of the wheel, we know that the minimum height of the reflector is $30 - 17 = 13$ cm. At the top-most position, the reflector will be $30 + 17 = 47$ cm from the ground. Since the amplitude is half of the wave height, it must be $\frac{47 - 13}{2} = 17$ cm, so $a = 17$ or $a = -17$.

That makes sense since the reflector is located 17 cm from the centre of the wheel! Also, the principal axis must be halfway between the minimum and maximum values, so it must be at $\frac{47 + 13}{2} = 30$ cm, hence $d = 30$. That makes sense because the centre of the wheel is 30 cm above the ground. Since it takes 4 seconds for the wheel to complete one revolution, the period of our model must equal 4. We can then find $|b|$:

$$\text{Period} = \frac{360}{|b|} \Rightarrow 4 = \frac{360}{|b|} \Rightarrow |b| = \frac{360}{4} = 90$$

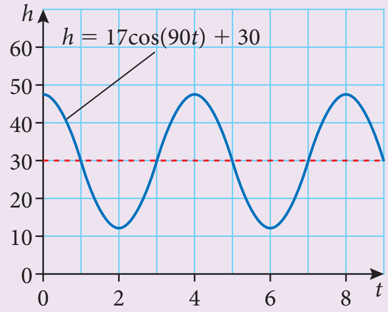
So $b = 90$ or $b = -90$

Finally, we need to decide if a sine or cosine model is better suited for this situation. Since we are told that the reflector begins at the top-most point, our model must begin at a maximum. Therefore, we choose the cosine function, and we keep both a and b positive since no reflection across the x or y axis is needed. Thus, we have $a = 17$, $b = 90$, and $d = 30$, and our model is

$$h = 17\cos(90t) + 30$$

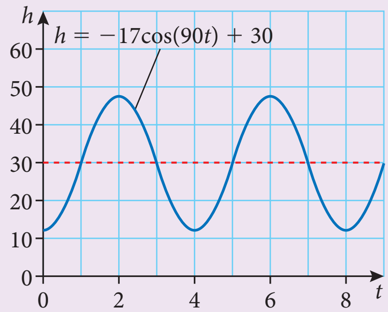
We can check by graphing two complete periods ($0 \leqslant t \leqslant 8$)

The graph appears to fit the behaviour we expect.



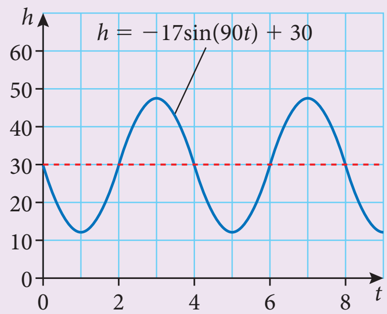
(b) For the reflector to start at the bottom-most position, the model must have a minimum as an initial value. In this case, we simply let a be negative, and revise our model accordingly:

$h = -17\cos(90t) + 30$
as shown in the graph.



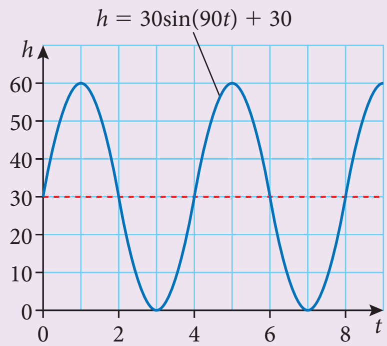
(c) For the reflector to start at the right-most position, the model must have an initial value equal to the location of the principal axis. Therefore, we must choose a sine model. Also, we are told that the wheel rotates clockwise, which will cause the height of the reflector to initially decrease. Therefore, we must let $a < 0$ and so our model is:

$h = -17\sin(90t) + 30$
as shown in the graph.



(d) A point on the outer edge of the wheel will have a similar model, but the amplitude of the model is now equal to the radius of the wheel. Also, since the point is in the left-most position at time $t = 0$, the model must increase initially. Therefore, we use a sine model with $a = +30$:

$h = 30\sin(90t) + 30$
as shown in the graph.



Example 6.18

The height of a person on Vienna's famous Riesenrad ferris wheel can be modelled by the function

$$h(t) = a\cos(36(t - 5)) + d$$

where $h(t)$ is the height in feet at time t minutes.

(a) Given that the diameter of the Riesenrad is 200 feet, deduce the value of a.

(b) The boarding platform at the lowest point of the Riesenrad is 12 feet above the ground. Find the value of d.

(c) Calculate the number of minutes until a person riding the Riesenrad reaches the top of the wheel.

(d) Calculate the time required for one complete rotation.

(e) Determine a reasonable domain and range for this model if each ride is exactly one complete revolution.

(f) In one 10-minute ride, during what interval of time will a person on the Riesenrad be at least 100 feet above the ground?

(g) Comment on the reasonableness of this model.

Solution

(a) Since the diameter of the Riesenrad is 200 feet, the wave height of the model must be 200, so the amplitude $a = 100$

(b) Since the minimum value of the function is 12 feet, the principal axis must be located at $12 + 100 = 112$. Therefore $d = 112$

(c) By graphing the model, we obtain the GDC screen shown in Figure 6.34.

We can see that a local maximum is at $t = 5$ minutes.

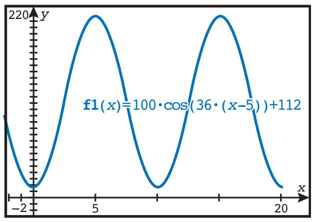


Figure 6.34 GDC screen for solution to Example 6.18 (c)

(d) Since the first maximum is reached 5 minutes after the first minimum (when the person boards the Riesenrad), we can conclude that one complete rotation takes 10 minutes. This agrees with the graph.

(e) Since one complete revolution takes 10 minutes, the domain for this model should be $0 \leqslant t \leqslant 10$. In that time, the person will travel from 12 feet to 212 feet, so the range is $12 \leqslant h(t) \leqslant 212$

(f) Again, we can use our GDC to solve this, by adding a second function with the constant value of 100, as shown in Figure 6.35.

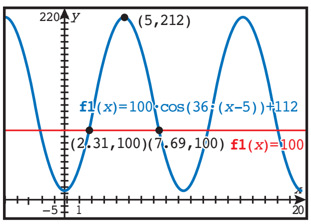


Figure 6.35 GDC screen for solution to Example 6.18 (f)

Then, by using the Intersection command, we can find the times when a person goes above and below 100 feet.

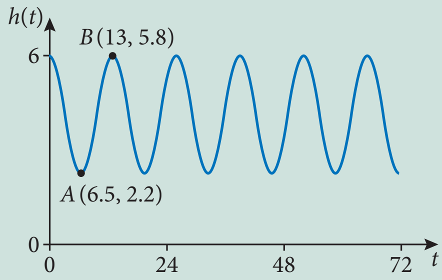
We see that a person would be at least 100 feet above the ground for $2.31 \leqslant t \leqslant 7.69$ minutes.

We do not consider the next period, since the question asked only for the first ten minutes.

(g) Although this model is reasonable in principal, it assumes that the rotation speed of the wheel is constant. In reality, the wheel would need to stop at regular intervals to allow passengers to get on and off the ride.

Exercise 6.6

1. At Cabot Cove the height of the water in metres is read from a depth gauge in the water. The depth can be modelled by the function $h(t) = p\cos(q \times t) + r$, where t is the number of hours after high tide at 15:00 on 10 March 2018. The diagram shows the graph of h, for $0 \leqslant h \leqslant 72$



The point $A(6.5, 2.2)$ represents the first low tide and $B(13, 5.8)$ represents the next high tide.

(a) How much time is there between the first low tide and the next high tide?

(b) Find the difference in height between low tide and high tide.

(c) Find the value of p, q, and r.

(d) There are two high tides on 11 March 2018. At what time does the second high tide occur?

(e) Calculate the number of minutes where the height of the water is at least 5.5 metres on 11 March 2018.

2. The depth of water at a dock can be modelled by the function $d(t) = a\cos(bt) + 6.2$, for $0 \leqslant t \leqslant 18$, where $d(t)$ is the depth in metres t hours after high tide. At high tide, the depth is 9.0 m. Low tide is 6.25 hours later, at which time the depth is 3.4 m.

(a) Find the value of a and the value of b.

(b) Use the model to find the depth of the water 9 hours after high tide.

(c) A certain sailboat needs water at least 4.5 m deep. How many hours after high tide can it stay at the dock before it must leave?

3. The height of a seat on a Ferris wheel can be modelled by the function $h(t) = -22\cos(40t) + 23$, for $t \geq 0$, where $h(t)$ is the height in metres after t minutes.

(a) Find the height of the seat when $t = 0$

(b) The seat first reaches a height of 35 m after k minutes. Find k

(c) Calculate the time needed for the seat to complete a full rotation.

4. The London Eye, a large Ferris wheel, has a diameter of 120 metres. One revolution of the wheel takes 30 minutes. Let A be a point at the bottom of the wheel, at ground level.

(a) Write down the height of A above ground level after:

(i) 15 minutes **(ii)** 20 minutes.

(b) The function $h(t) = a\cos(bt) + c$ gives the height in metres of point A after t minutes. Find a , b, and c.

(c) Find the number of minutes that A is more than 100 metres above the ground during one rotation.

5. A water-wheel rotates clockwise. Point P is at the right-most position on the wheel.

The diameter of the wheel is 16 metres. The centre of the wheel, C, is 5 metres above the water level. The height of point P above the water level is modelled by $h(t) = a\sin(bt) + c$

(a) Find the value of a

(b) It takes 90 seconds for the wheel to rotate once. Find the value of b

(c) Calculate the number of seconds point P is underwater during one rotation.

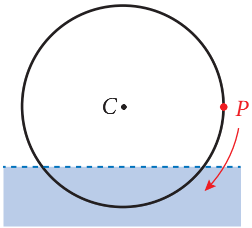


Figure 6.36 Diagram for question 5

6. The number of hours of daylight in San Diego, California, can be modelled by the function $d(t) = a\sin(bt) + c$, where $d(t)$ is the number of hours of daylight t days after 21 March. The longest day has 14.4 hours of daylight and the shortest day has 9.6 hours of daylight. On 21 March there are 12 hours of daylight. Assume 365 days in one year.

(a) Find the value of a and of c

(b) Assuming that the cycle of day lengths repeats exactly every year, find the value of b in the form $\frac{p}{q}$, where $p, q \in \mathbb{Z}$

(c) Find how many days in a year have more than 14 hours of daylight.

7. The temperature in degrees Celsius during a 24-hour period is shown on the graph in Figure 6.37 and is given by the function $f(t) = a\cos(bt) + c$, where a, b, and c are constants, t is the time in hours, and (bt) is measured in degrees.

(a) Write down the value of a.

(b) Find the value of b.

(c) Write down the value of c.

(d) Write down the interval of time during which the temperature is increasing from -4°C to -2°C.

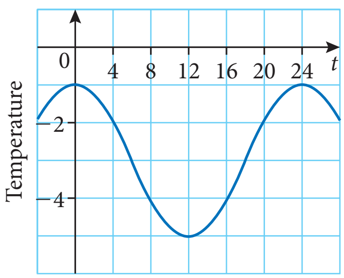
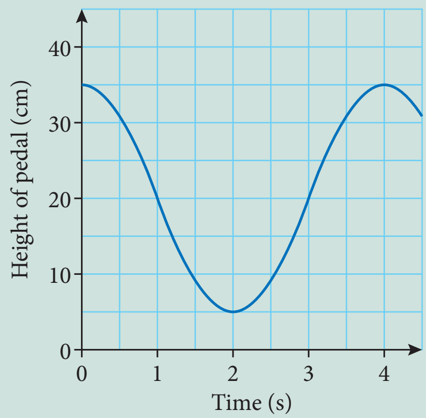


Figure 6.37 Diagram for question 7

8. The height, $h(t)$, in centimetres, of a bicycle pedal above the ground at time t seconds is a cosine function of the form $h(t) = A\cos(bt) + C$, where (bt) is measured in degrees. The graph of this function for $0 \leqslant t \leqslant 4.3$ is shown here.



(a) Write down the maximum height of the pedal above the ground.

(b) Write down the minimum height of the pedal above the ground.

(c) Find the amplitude of the function.

(d) Hence, or otherwise, find the value of A and of C.

(e) Write down the period of the function $h(t)$, including units.

(f) Hence find the value of b.

(g) Calculate the first value of t for which the height of the pedal above the ground is 30 cm.

(h) Calculate the number of times the pedal rotates in one minute.

9. The depth, in metres, of water in a harbour is given by the function $d = 4\sin(0.5t) + 7$, where t is in minutes, $0 \leqslant t \leqslant 1440$

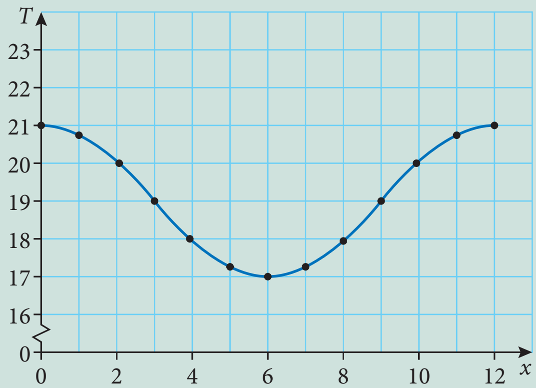
(a) Write down the amplitude of d.

(b) Find the maximum value of d.

(c) Find the period of d. Give your answer in hours.

(d) On Tuesday, the minimum value of d occurs at 14:00. Find when the next maximum value of d occurs.

10. The graph represents the temperature ($T°$ Celsius) in Washington measured at midday during a period of thirteen consecutive days starting at Day 0. These points also lie on the graph of the function $T(x) = a + b\cos(cx)$, $0 \leqslant x \leqslant 12$, where $a, b, c \in \mathbb{R}$

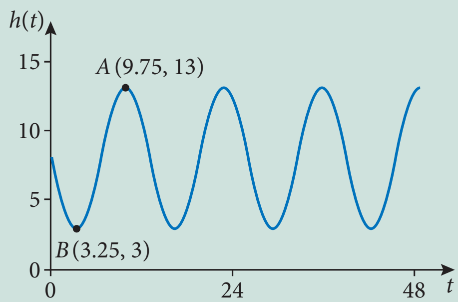


(a) Find the value of a and of b.

(b) Find the value of c.

(c) Using the graph, or otherwise, write down the part of the domain for which the midday temperature is less than 18.5°C.

11. At Kabulonga Beach the height of the water in metres is modelled by the function $h(t) = p\sin(q \times t) + r$, where t is the number of hours after 8:00 hours on 19 September 2018. The diagram shows the graph of h, for $0 \leqslant h \leqslant 48$



The point $A(9.75, 13)$ represents the first high tide and $B(3.25, 3)$ represents the first low tide.

(a) How much time is there between the first low tide and the first high tide?

(b) Find the difference in height between low tide and high tide.

(c) Give a reason why p must be negative.

(d) Find the values of p, q, and r.

(e) There are two high tides on 20 September 2018. At what time does the second high tide occur?

6.7 Modelling skills

Days since birth	Mass (kg)
0	3.36
4	3.4
17	3.69
45	5.29
73	6.15

Table 6.11 Baby mass recordings

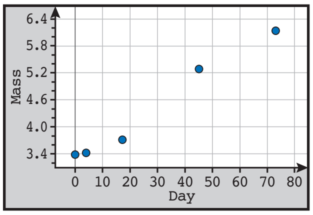


Figure 6.38 Mass versus days since birth for a young child

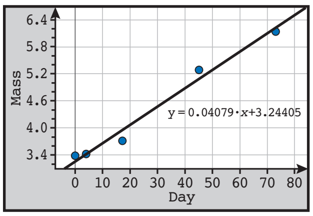


Figure 6.39 Linear model for mass in terms of days after birth

In the real world, it's critical to be able to identify, develop, evaluate, and use models responsibly. In the preceding sections of this chapter we have examined different models and touched on many of the uses and limitations of models. However, it's also important to be able to choose an appropriate model.

One of the authors recorded the mass of his young baby at a several points after she was born. The data is given in Table 6.11.

A graph of this data is shown in Figure 6.38.

The data appears approximately linear, so the author used technology to generate a linear model for this data:

$m = 0.0408d + 3.24$, where m is mass in kg, d days after birth.

The model suggests that she is gaining 0.0408 kg per day, with an initial mass of 3.24 kg. Graphically, the model appears to fit the data reasonably well, as shown in Figure 6.39.

Of course, the author was very interested in the eventual mass of his young baby girl. How much would she weigh she was one year old? Five years old? Ten years old? He used the model to make some predictions, shown in Table 6.12 (he even added leap days to be sure).

Days since birth	365 (1 year)	1826 (5 years)	3652 (10 years)
Mass (kg)	18.1	77.7	152

Table 6.12 Mass projections

Are these results reasonable? A mass of 77.7 kg for a five-year-old girl is alarming, and a mass of 152 kg for a ten-year-old girl is definitely a cause for concern!

What went wrong? The mathematical mechanics are perfectly correct.

There are two flaws in our reasoning. One, our **model choice** (linear model) assumes that the rate of change is constant (in this case, 40.8 g per day), which is not correct; we know that the growth rate will decrease as the child gets older. Two, we **extrapolated** beyond known data. We will discuss both of these issues in this section.

Choosing a model

One of the most challenging parts of modelling in the real world is choosing an appropriate model. Often, we try to use our understanding of a situation when we are choosing and developing a model. Other times we may look at the shape of the data (as seen on a graph) to try to help us choose a model.

Here are some guidelines to help, with examples from the introduction to this chapter:

- Consider the rate of change.
  - Is it constant? Try a linear model.

    Example: On a long flight, the airspeed of a plane is constant, so the distance remaining to the destination can be described by a **linear** model.
  - Is the quantity increasing/decreasing by a fixed percentage or ratio? Try an exponential model.

    Examples: Algae in a polluted lake doubles every 3 days; a bank account earns 5% quarterly interest, compounded annually; the value of a car is decreasing by 25% per year; what will the activity of a radioisotope be after a given time.
  - Is the quantity increasing/decreasing at a linearly increasing/decreasing rate? Try a quadratic model.

    Examples: The price to manufacture x units of some product decreases linearly, the revenue from selling x units can be described by a quadratic model; the velocity of a falling object changes linearly, the position follows a quadratic model.
- Consider the nature of the phenomenon.
  - Does it relate volume to a linear quantity? Try a cubic model.

    Examples: The volume of a balloon relative to its diameter can be described by a cubic model; electricity generated by a wind turbine based on wind speed.
  - Is it cyclical/periodic/repeating? Try a trigonometric model.

    Examples: tide height; person on Ferris wheel; seasonal average temperatures.
  - Does one variable appear to be in constant ratio to a positive or negative power of the other variable? Consider a direct or inverse variation model.

    Examples: A DJ charges a fixed amount to provide music for a party, the cost per person can be described by an inverse variation model; the distance a dropped object has travelled varies with the square of elapsed time, use a direct variation model.
- Consider the shape of the data.

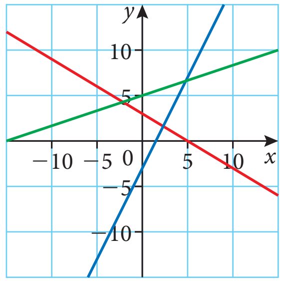


Figure 6.40 Linear models

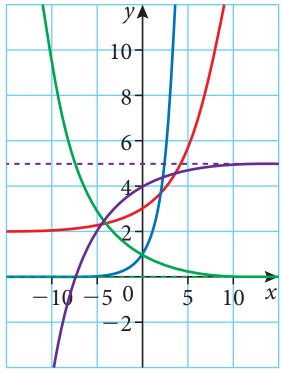


Figure 6.41 Exponential models

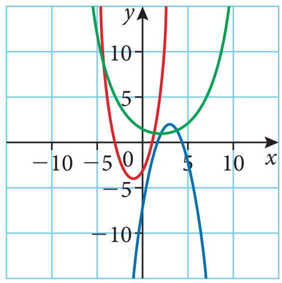


Figure 6.42 Quadratic models

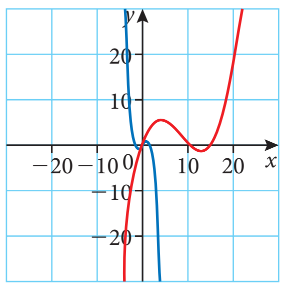


Figure 6.43 Cubic models

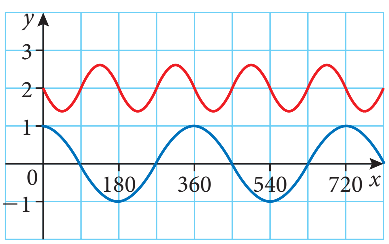


Figure 6.44 Trigonometric models

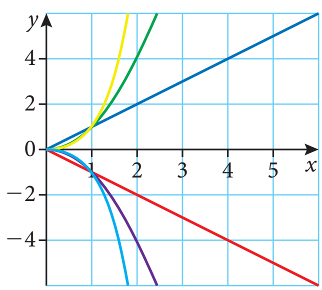


Figure 6.45 Direct variation models

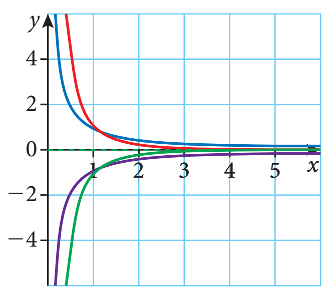


Figure 6.46 Inverse variation models

Example 6.19

For each of the following examples, choose an appropriate model and state a reason why you chose it.

(a) Modelling a person's mass as a function of their height.

(b) The number of hours of daylight in Tokyo each day of the year.

(c) The population of bacteria in a Petri dish over time.

(d) The resale value of a computer over time.

(e) The distance from Earth for a Voyager space probe travelling away at a constant rate.

(f) The height of the golf ball Alan Shepard hit on the moon in 1971.

(g) The number of payments required to repay a loan as a function of the amount of the monthly payment.

Solution

(a) Since weight is based on volume, and volume varies with the cube of height, we should use a cubic direct variation model of the form $W = kh^3$

(b) Number of hours of daylight per day is a periodic phenomenon, so a trigonometric model is appropriate.

(c) The population of bacteria in a Petri dish will grow exponentially (until the maximum population for the dish is reached), so an exponential model is appropriate.

(d) A computer's value will depreciate exponentially, so an exponential model is appropriate.

(e) Since the Voyager is travelling away from Earth at a constant rate, a linear model is appropriate.

(f) The golf ball will follow a quadratic falling-object model as it is still subject to gravity (just less than on Earth).

(g) If we increase the amount of the monthly payment, the number of payments we have to make decreases, and vice-versa. Since the total repayment amount (the value of the loan) is the product of the number of payments and the payment amount, this will be an inverse variation model of the form $V = an \Rightarrow n = \frac{V}{a}$ where V is the value of the loan (a constant), a is the amount of each payment, and n is the number of payments.

Testing a model

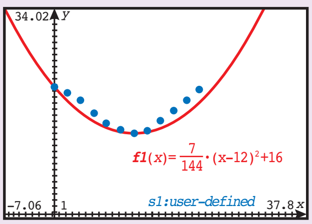
Once a model is developed (using the techniques in this chapter), it is important to test it. In fact, in real life, it is often the case that after testing a model you decide to develop an alternative model or revise your existing model. How do we test a model? We can test a model by looking at the fit of the model for the known data. If the model does not seem to fit well, it can be a sign that we do not fully understand the phenomenon we are trying to model. In that case, we may decide to try a different model, or perhaps more research is necessary!

Developing useful models is the work of many people: economists, doctors, aid workers, actuaries, and many other researchers. This chapter is a basic introduction to the art and science of modelling.

It is important to understand that, in the real world, for many topics, there is often no 'right' model. Instead, for real-life applications we often judge a model by its usefulness. What does it tell us about the data we have? Does it tell us something about the nature of the phenomena we are modelling? Does it allow us to make meaningful predictions? Are the predictions surprising but plausible? What assumptions are we making? Asking these sorts of questions is a crucial skill for the practicing mathematical modeller. In the context of this course, we can only simulate the sort of scenarios you might encounter in the real world in the hope that you might be critical and thoughtful in your future encounters with mathematical models.

Example 6.20

Anna is attempting to model the temperature of a pond at her school. She has collected 12 points of data, measuring the temperature of the pond every 2 hours. Based on her data, she believes a quadratic model may be useful so she has used the vertex and y-intercept to fit a model. The graph shows her data points and model.



(a) Give two reasons why Anna's choice of model may not be appropriate.

(b) Suggest a better model and give a reason why it would be more appropriate.

Solution

(a) Anna's model doesn't seem to fit the data very well. Also, the temperature of the pond probably varies during the day and night in a periodic way, which would not suggest a quadratic model.

(b) Since the temperature of the pond probably varies periodically, a trigonometric model would be more appropriate.

Interpolation versus extrapolation

As in the example at the start of the section, we can quickly get into trouble when we extrapolate. But what exactly is extrapolation? How can we tell when we are using a model appropriately and when we are extrapolating? One way to check is to look at the range of known data, as in Example 6.21.

Example 6.21

A used-car dealer has recently sold five of the same kind of car, called the Canyonero. All five were in similar condition, but the ages of the cars varied. The data are shown in the table.

Age of car (years)	Sale price ($)
2	20 000
3	14 500
4	12 000
6	6100
10	2000

The dealer decides to use the model $P = 20\,000(0.75)^{t-2}$ to describe the price of the Canyonero (P represents the price and t represents the age in years).

(a) Use the model to predict the price of a new Canyonero. Comment on the usefulness of your prediction.

(b) Use the model to predict the price of an 8-year-old Canyonero. Comment on the usefulness of your prediction.

(c) Use the model to predict the price of a 15-year-old Canyonero. Comment on the usefulness of your prediction.

Solution

(a) According to the model, the price of a new Canyonero would be $P = 20\,000\,(0.75)^{0-2}$ = \$35 600 (3 s.f.). Although this may seem like a reasonable prediction, our data only includes cars from 2 to 10 years old, so we cannot be confident in this prediction. Therefore, it is not a useful prediction.

(b) According to the model, the price of an 8-year-old Canyonero would be $P = 20\,000\,(0.75)^{8-2}$ = \$3 560 (3 s.f.). Since 8 years is within the range of the age of cars in our data, we can be reasonably confident in this prediction. Therefore, this is a useful prediction.

(c) According to the model, the price of a 15-year-old Canyonero would be $P = 20\,000\,(0.75)^{15-2}$ = \$475 (3 s.f.). Not only does this value seem like a small amount, but it is not within the range of the age of cars in our data. Therefore, it is not a useful prediction.

As you can see in Example 6.21, it is important to note that we decide whether a prediction is interpolation or extrapolation based on the range of our independent variable. This is easier to see when we look at a graph.

As shown in Figure 6.47, any predictions based on values within the range of the independent variable are called **interpolations**. Therefore, any predictions for cars between 2 and 10 years old would be interpolations and therefore relatively safe and useful.

On the other hand, any predictions based on values outside the range of the independent variable are called **extrapolations**. Therefore, any predictions for cars newer than 2 years old or older than 10 years old are considered extrapolations. Extrapolated predictions are less reliable and therefore less useful, since we simply can't be sure what the value of the car might be outside the range that we have observed.

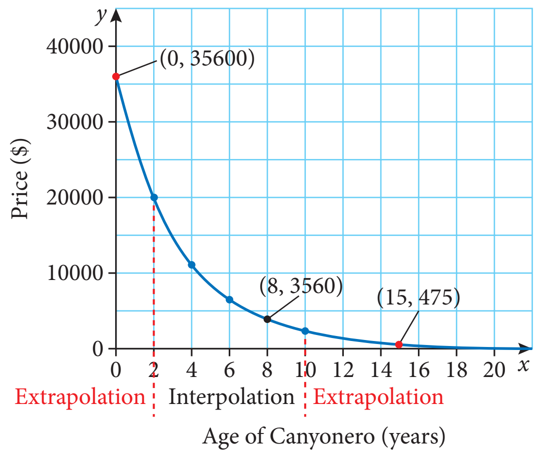


Figure 6.47 Interpolation is based on the range of the independent variable

Although it is tempting to look at the value of a prediction to decide whether a prediction is useful or not, it is critical to consider first whether the value of the independent variable is within the range of known values of the independent variable. That is, we can decide whether a value is interpolation or extrapolation before we make a prediction. Prediction based on extrapolation should always be used with caution!

Interpolations are predictions based on values within the range of known values of the independent variable. **Extrapolations** are predictions based on values outside the range of known values of the independent variable.

Given a model $f(x)$ based on data with minimum x value a and maximum x value b, $a \leqslant x \leqslant b$

- If $a \leqslant c \leqslant b$, then $f(c)$ is an **interpolated value**.
- If $c < a$ or $c > b$, then $f(c)$ is an **extrapolated value**.

Exercise 6.7

1. Identify appropriate models for the following situations.

(a) The acceleration produced by exerting a constant force on various masses.

(b) The total cost of fuel purchased at a fixed price per litre.

(c) The area of a rectangle where the length is twice as large as the width.

(d) Average monthly temperature over a year in Tokyo.

(e) The volume of a cube as a function of its side length.

(f) The value of a savings account growing at a fixed percentage rate.

(g) The height of tide at a beach.

(h) The value of a new car over time.

(i) The cost for hiring a bus, per person.

(j) The cost of a wedding as a function of the number of guests.

2. For the following data sets, identify the valid domain for interpolation. Assume we are treating the first row in the table as the independent variable.

(a)

Time in dive (s)	2	4	6
Velocity (m s^{-1})	10	53	90

(b)

x	41	33	100	76	11
y	33.4	31.8	52.3	47.3	16.6

(c)

n	8.2	3.9	5.5	1.3	8.5
$P(n)$	15.5	11.5	12.3	6.0	15.2

3. In general, what are the key feature(s) that we might see to suggest that we should use each of the following models:

(a) Linear **(b)** Quadratic **(c)** Trigonometric
(d) Exponential **(e)** Inverse

Chapter 6 practice questions

1. A city is concerned about pollution, and decides to look at the number of people using taxis. At the end of the year 2000, there were 280 taxis in the city. After n years the number of taxis, T, in the city is given by $T = 280 \times 1.12^n$

(a) Find the number of taxis in the city at the end of 2005.

(b) Find the year in which the number of taxis is double the number of taxis there were at the end of 2000.

At the end of 2000 there were 25 600 people in the city who used taxis. After n years the number of people, P, in the city who used taxis is given by $P = \dfrac{2\,560\,000}{10 + 90e^{-0.1n}}$

(c) Find the value of P at the end of 2005, giving your answer to the nearest whole number.

(d) After seven complete years, will the value of P be double its value at the end of 2000? Justify your answer.

Let R be the ratio of the number of people using taxis in the city to the number of taxis. The city will reduce the number of taxis if $R < 70$

(e) Find the value of R at the end of 2000.

(f) After how many complete years will the city first reduce the number of taxis?

2. The profit (P) in Swiss Francs made by three students selling homemade lemonade is modelled by the function $P = -\frac{1}{20}x^2 + 5x - 30$ where x is the number of glasses of lemonade sold.

(a) Copy and complete the table.

x	0	10	20	30	40	50	60	70	80	90
P		15			90			75	50	

(b) On graph paper draw axes for x and P, placing x on the horizontal axis and P on the vertical axis. Use suitable scales. Draw the graph of P against x by plotting the points. Label your graph.

(c) Use your graph to find:

(i) the maximum possible profit

(ii) the number of glasses that need to be sold to make the maximum profit

(iii) the number of glasses that need to be sold to make a profit of 80 Swiss Francs

(iv) the amount of money initially invested by the three students.

(d) The three students Baljeet, Jane and Fiona share the profits in the ratio of 1:2:3 respectively. If they sold 40 glasses of lemonade, calculate Fiona's share of the profits.

3. The function $Q(t) = 0.003t^2 - 0.625t + 25$ represents the amount of energy in a battery after t minutes of use.

(a) Write down the amount of energy held by the battery immediately before it was used.

(b) Calculate the amount of energy available after 20 minutes.

(c) Given that $Q(10) = 19.05$, find the average amount of energy used per minute for the interval $10 \leqslant t \leqslant 20$

(d) Calculate the number of minutes it takes for the energy to reach zero.

4. Jashanti is saving money to buy a car. The price of the car, in US Dollars (USD), can be modelled by the equation $P = 8500(0.95)^t$

Jashanti's savings, in USD, can be modelled by the equation $S = 400t + 2000$

In both equations t is the time in months since Jashanti started saving for the car.

(a) Write down the amount of money Jashanti saves per month.

(b) Use your graphic display calculator to find how long it will take for Jashanti to have saved enough money to buy the car.

(c) Jashanti does not want to wait too long and wants to buy the car two months after she started saving. She decides to ask her parents for the extra money that she needs. Calculate how much extra money Jashanti needs.

5. A building company has many rectangular construction sites, of varying widths, along a road. The area, A, of each site is given by the function $A(x) = x(200 - x)$ where x is the width of the site in metres and $20 \leqslant x \leqslant 180$

(a) Site S has a width of 20 m. Write down the area of S.

(b) Site T has the same area as site S, but a different width. Find the width of T.

(c) When the width of the construction site is b metres, the site has a maximum area.

(i) Write down the value of b

(ii) Write down the maximum area.

(d) The range of $A(x)$ is $m \leqslant A(x) \leqslant n$
Write down the value of m and of n.

6. Water has a lower boiling point at higher altitudes. The relationship between the boiling point of water (T) and the height above sea level (h) can be described by the model $T = -0.0034h + 100$ where T is measured in degrees Celsius (°C) and h is measured in metres above sea level.

(a) Write down the boiling point of water at sea level.

(b) Use the model to calculate the boiling point of water at a height of 1.37 km above sea level.

(c) Water boils at the top of Mt. Everest at 70°C. Use the model to calculate the height above sea level of Mt. Everest.

7. The temperature in °C of a pot of water removed from a cooker is given by $T(m) = 20 + 70 \times 2.72^{-0.4m}$, where m is the number of minutes after the pot is removed from the cooker.

(a) Show that the temperature of the water when it is removed from the cooker is 90°C.

(b) Calculate the temperature of the water after 10 minutes.

(c) Calculate how long it takes for the temperature of the water to reach 56°C.

(d) Write down the temperature approached by the water after a long time. Justify your answer.

Consider the function $S(m) = 20m - 40$ for $2 \leqslant m \leqslant 6$

The function $S(m)$ represents the temperature of soup in a pot placed on the cooker two minutes after the water has been removed. The soup is then heated.

(e) Comment on the meaning of the constant 20 in the formula for $S(m)$ in relation to the temperature of the soup.

(f) Solve the equation $S(m) = T(m)$. Interpret the solution in context.

(g) Hence describe by using inequalities the set of values of m for which $S(m) > T(m)$

8. Shiyun bought a car in 1999. The value of the car, V, in USD, is depreciating according to the exponential model $V = 25\,000 \times 1.5^{-0.2t}$, $t \geqslant 0$, where t is the time, in years, that Shiyun has owned the car.

(a) Write down the value of the car when Shiyun bought it.

(b) Calculate the value of the car three years after Shiyun bought it. Give your answer correct to two decimal places.

(c) Calculate the time for the car to depreciate to half of its value when Shiyun bought it.

9. In an experiment it is found that a culture of bacteria triples in number every four hours. There are 200 bacteria at the start of the experiment.

(a) Write down the number of bacteria after 8 hours.

(b) Calculate how many bacteria there will be after one day.

(c) Find how long it will take for there to be two million bacteria.

10. The cost per person, in euros, when x people hire an airplane can be determined by the function $C(x) = x + \frac{200}{x}$

(a) Calculate the cost per person when 40 people hire the airplane.

(b) When the number of people who hire the airplane is a or b, the cost per person is 33 euros. Find the value of a and of b.

(c) When n people hire the airplane, the cost per person is the minimum possible value. Find n.

(d) Find the minimum cost per person, to the nearest 0.01 euro.

11. In Fairbanks, Alaska, there are 3.7 hours of daylight on 21 December. This is the shortest day of the year. The number of hours of daylight can be modelled by the function $L(d) = p\cos(q \times d) + r$, where $L(d)$ is the number of hours of daylight d days after 21 December.

(a) Write down the number of hours of daylight for the longest day of the year.

(b) Give a reason why p must be negative.

(c) Hence find the values of p and r.

(d) Assuming that the cycle of day length repeats exactly every 365 days, find the value of q in the form $\frac{a}{b}$, where $a, b \in \mathbb{Z}$

(e) Find the number of days during a year that have less than 6 hours of daylight.

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