Lesson 1: The discriminant

Passage A

The discriminant

If you square any real number, the result is greater than or equal to 0.
This means that if y is negative, $\sqrt{y}$ cannot be a real number. Look at the quadratic formula:

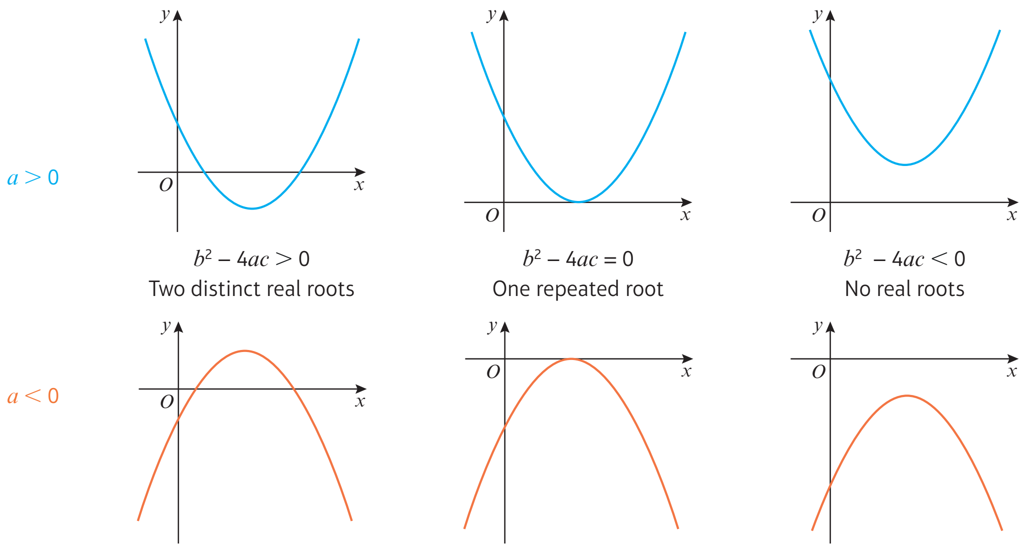
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If the value under the square root sign is negative, x cannot be a real number and there are no real solutions. If the value under the square root is equal to 0, both solutions will be the same.

- For the quadratic function $f(x) = ax^2 + bx + c$, the expression $b^2 - 4ac$ is called the discriminant. The value of the discriminant shows how many roots $f(x)$ has:
  - If $b^2 - 4ac > 0$ then $f(x)$ has two distinct real roots.
  - If $b^2 - 4ac = 0$ then $f(x)$ has one repeated root.
  - If $b^2 - 4ac < 0$ then $f(x)$ has no real roots.

Passage B

You can use the discriminant to check the shape of sketch graphs.
Below are some graphs of $y = f(x)$, where $f(x) = ax^2 + bx + c$.



Passage C

Example 1

Find the values of k for which $f(x) = x^2 + kx + 9 = 0$.

$x^2 + kx + 9 = 0$
Here $a = 1$, $b = k$ and $c = 9$
For equal roots, $b^2 - 4ac = 0$.
$k^2 - 4 \times 1 \times 9 = 0$
$k^2 - 36 = 0$
$k^2 = 36$
so $k = \pm 6$

Problem-solving
Use the condition given in the question to write a statement about the discriminant.

Substitute for a, b and c to get an equation with one unknown.

Solve to find the values of k.

Passage C

Example 2

Find the range of values of k for which $x^2 + 4x + k = 0$ has two distinct real solutions.

$x^2 + 4x + k = 0$
Here $a = 1$, $b = 4$ and $c = k$.
For two real solutions, $b^2 - 4ac > 0$.
$4^2 - 4 \times 1 \times k > 0$
$16 - 4k > 0$
$16 > 4k$
$4 > k$
So $k < 4$

This statement involves an **inequality**, so your answer will also be an inequality.

For any value of k less than 4, the equation will have two distinct real solutions.

Online Explore how the value of the discriminant changes with k using technology.

Passage D

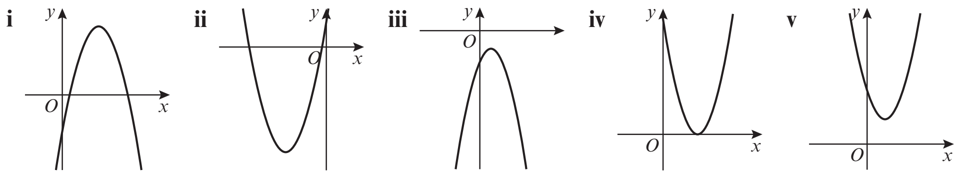
Exercise 1

1 **a** Calculate the value of the discriminant for each of these five functions:

i $f(x) = x^2 + 8x + 3$ **ii** $g(x) = 2x^2 - 3x + 4$ **iii** $h(x) = -x^2 + 7x - 3$

iv $j(x) = x^2 - 8x + 16$ **v** $k(x) = 2x - 3x^2 - 4$

b Using your answers to part **a**, match the same five functions to these sketch graphs.



E/P **2** Find the values of k for which $x^2 + 6x + k = 0$ has two real solutions. **(2 marks)**

E/P **3** Find the value of t for which $2x^2 - 3x + t = 0$ has exactly one solution. **(2 marks)**

E/P **4** Given that the function $f(x) = sx^2 + 8x + s$ has equal roots, find the value of the positive constant s. **(2 marks)**

E/P **5** Find the range of values of k for which $3x^2 - 4x + k = 0$ has no real solutions. **(2 marks)**

E/P **6** The function $g(x) = x^2 + 3px + (14p - 3)$, where p is an integer, has two equal roots.

a Find the value of p. **(2 marks)**

b For this value of p, solve the equation $x^2 + 3px + (14p - 3) = 0$. **(2 marks)**

E/P **7** $h(x) = 2x^2 + (k + 4)x + k$, where k is a real constant.

a Find the discriminant of $h(x)$ in terms of k. **(3 marks)**

b Hence or otherwise, prove that $h(x)$ has two distinct real roots for all values of k. **(3 marks)**

Problem-solving

If a question part says 'hence or otherwise' it is usually easier to use your answer to the previous question part.

Passage E

Simultaneous equations on graphs

You can represent the solutions of simultaneous equations graphically. As every point on a line or curve satisfies the equation of that line or curve, the points of **intersection** of two lines or curves satisfy both equations simultaneously.

- Solutions to a pair of simultaneous equations represent the points of intersection of their graphs.

Passage F

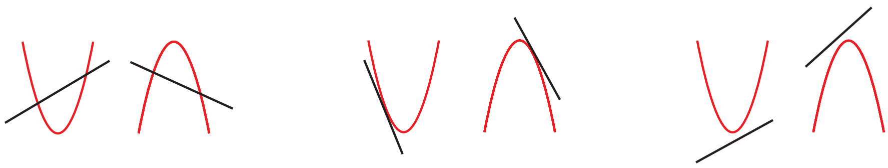
The graph of a linear equation and the graph of a quadratic equation can either:

- intersect twice
- intersect once
- not intersect

After substituting, you can use the discriminant of the resulting quadratic equation to determine the number of points of intersection.

Passage G

- For a pair of simultaneous equations that produce a quadratic equation of the form $ax^2 + bx + c = 0$:
  - $b^2 - 4ac > 0$ two real solutions
  - $b^2 - 4ac = 0$ one real solution
  - $b^2 - 4ac < 0$ no real solutions



Passage H

Example 3

The line with equation $y = 2x + 1$ meets the curve with equation $kx^2 + 2y + (k - 2) = 0$ at exactly one point. Given that k is a positive constant

a find the value of k

b for this value of k, find the coordinates of the point of intersection.

Online Explore how the value of k affects the line and the curve using technology.

a

$y = 2x + 1 \quad (1)$

$kx^2 + 2y + (k - 2) = 0 \quad (2)$

$kx^2 + 2(2x + 1) + (k - 2) = 0$

$kx^2 + 4x + 2 + k - 2 = 0$

$kx^2 + 4x + k = 0$

Substitute $y = 2x + 1$ into equation (2) and simplify the quadratic equation. The resulting quadratic equation is in the form $ax^2 + bx + c = 0$ with $a = k$, $b = 4$ and $c = k$.

$4^2 - 4 \times k \times k = 0$

Problem-solving You are told that the line meets the curve at exactly one point, so use the discriminant of the resulting quadratic. There will be exactly one solution, so $b^2 - 4ac = 0$.

$16 - 4k^2 = 0$

$k^2 - 4 = 0$

$(k - 2)(k + 2) = 0$

Factorise the quadratic to find the values of k.

$k = 2$ or $k = -2$

So $k = 2$

The solution is $k = +2$, as k is a positive constant.

b

$2x^2 + 4x + 2 = 0$

$x^2 + 2x + 1 = 0$

$(x + 1)(x + 1) = 0$

$x = -1$

Substitute $k = +2$ into the quadratic equation $kx^2 + 4x + k = 0$. Simplify and factorise to find the x-coordinate.

$y = 2(-1) + 1 = -1$

Substitute $x = -1$ into linear equation (1) to find the y-coordinate.

Point of intersection is $(-1, -1)$.

Check your answer by substituting into equation (2):

$2x^2 + 2y = 0$

$2(-1)^2 + 2(-1) = 2 - 2 = 0$ ✓

Passage I

Exercise 2

Given the simultaneous equations

$2x - y = 1$

$x^2 + 4ky + 5k = 0$

where k is a non-zero constant

a show that $x^2 + 8kx + k = 0$.

Given that $x^2 + 8kx + k = 0$ has equal roots

b find the value of k

c for this value of k, find the solution of the simultaneous equations.

Passage J

13 Find the values of k for which $kx^2 + 8x + 5 = 0$ has real roots.

14 The equation $2x^2 + 4kx - 5k = 0$, where k is a constant, has no real roots.
Prove that k satisfies the inequality $-\frac{5}{2} < k < 0$.