PEARSON EDEXCEL INTERNATIONAL A LEVEL

FURTHER PURE MATHEMATICS 1

Student Book

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Published by Pearson Education Limited, 80 Strand, London, WC2R 0RL.
www.pearsonglobalschools.com

Copies of official specifications for all Pearson qualifications may be found on the website: https://qualifications.pearson.com

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Typeset by Tech-Set Ltd, Gateshead, UK
Edited by Eric Pradel
Original illustrations © Pearson Education Limited 2018
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First published 2018

21 20 19 18
10 9 8 7 6 5 4 3 2 1

British Library Cataloguing in Publication Data
A catalogue record for this book is available from the British Library

ISBN 978 1 292244 64 8

Printed in Slovakia by Neografia

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ABOUT THIS BOOK

The following three themes have been fully integrated throughout the Pearson Edexcel International Advanced Level in Mathematics series, so they can be applied alongside your learning.

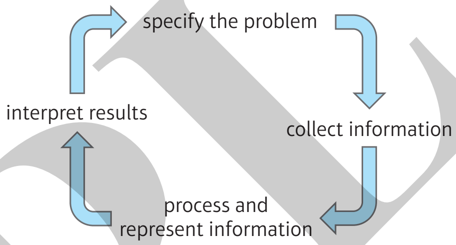
1. Mathematical argument, language and proof

- Rigorous and consistent approach throughout
- Notation boxes explain key mathematical language and symbols

2. Mathematical problem-solving

- Hundreds of problem-solving questions, fully integrated into the main exercises
- Problem-solving boxes provide tips and strategies
- Challenge questions provide extra stretch

The Mathematical Problem-Solving Cycle



3. Transferable skills

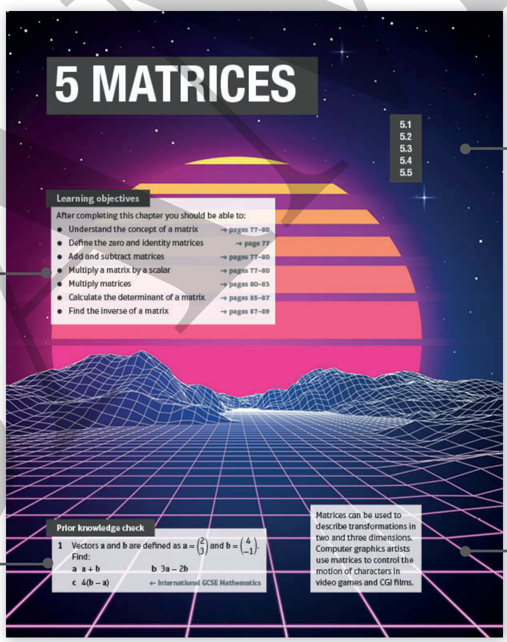
- Transferable skills are embedded throughout this book, in the exercises and in some examples
- These skills are signposted to show students which skills they are using and developing

Finding your way around the book

Each chapter starts with a list of *Learning objectives*

The *Prior knowledge check* helps make sure you are ready to start the chapter

Glossary terms will be identified by bold blue text on their first appearance



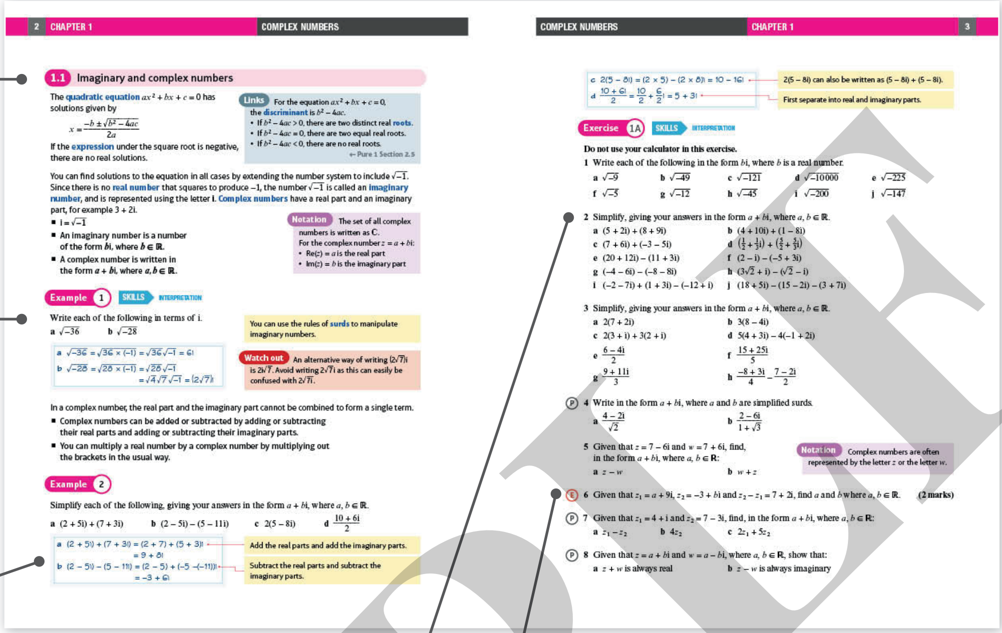
Each chapter is mapped to the specification content for easy reference

The real world applications of the mathematics you are about to learn are highlighted at the start of the chapter

Each section begins with explanation and key learning points

Transferable skills are signposted where they naturally occur in the exercises and examples

Step-by-step worked examples focus on the key types of questions you'll need to tackle



Exercise questions are carefully graded so they increase in difficulty and gradually bring you up to exam standard

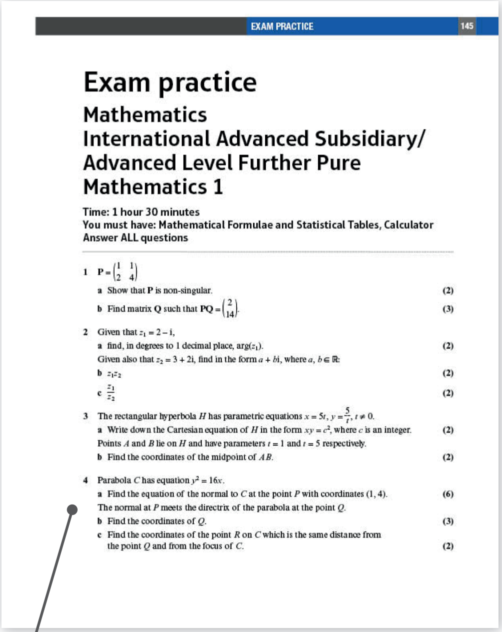
Exercises are packed with exam-style questions to ensure you are ready for the exams

Exam-style questions are flagged with Ⓔ

Problem-solving questions are flagged with Ⓟ

Each chapter ends with a *Chapter review* and a *Summary of key points*

After every few chapters, a *Review exercise* helps you consolidate your learning with lots of exam-style questions



A full practice paper at the back of the book helps you prepare for the real thing

QUALIFICATION AND ASSESSMENT OVERVIEW

Qualification and content overview

Further Pure Mathematics 1 (FP1) is a **compulsory** unit in the following qualifications:

International Advanced Subsidiary in Further Mathematics

International Advanced Level in Further Mathematics

Assessment overview

The following table gives an overview of the assessment for this unit.

We recommend that you study this information closely to help ensure that you are fully prepared for this course and know exactly what to expect in the assessment.

Unit	Percentage	Mark	Time	Availability
FP1: Further Pure Mathematics 1 Paper code WFM01/01	$33\frac{1}{3}$ % of IAS $16\frac{2}{3}$ % of IAL	75	1 hour 30 mins	January and June First assessment June 2019

IAS: International Advanced Subsidiary, IAL: International Advanced A Level.

Assessment objectives and weightings

		Minimum weighting in IAS and IAL
AO1	Recall, select and use their knowledge of mathematical facts, concepts and techniques in a variety of contexts.	30%
AO2	Construct rigorous mathematical arguments and proofs through use of precise statements, logical deduction and inference and by the manipulation of mathematical expressions, including the construction of extended arguments for handling substantial problems presented in unstructured form.	30%
AO3	Recall, select and use their knowledge of standard mathematical models to represent situations in the real world; recognise and understand given representations involving standard models; present and interpret results from such models in terms of the original situation, including discussion of the assumptions made and refinement of such models.	10%
AO4	Comprehend translations of common realistic contexts into mathematics; use the results of calculations to make predictions, or comment on the context; and, where appropriate, read critically and comprehend longer mathematical arguments or examples of applications.	5%
AO5	Use contemporary calculator technology and other permitted resources (such as formulae booklets or statistical tables) accurately and efficiently; understand when not to use such technology, and its limitations. Give answers to appropriate accuracy.	5%

Relationship of assessment objectives to units

FP1	Assessment objective				
	AO1	AO2	AO3	AO4	AO5
Marks out of 75	25–30	25–30	0–5	5–10	5–10
%	$33\frac{1}{3}$–40	$33\frac{1}{3}$–40	0–$6\frac{2}{3}$	$6\frac{2}{3}$–$13\frac{1}{3}$	$6\frac{2}{3}$–$13\frac{1}{3}$

Calculators

Students may use a calculator in assessments for these qualifications. Centres are responsible for making sure that calculators used by their students meet the requirements given in the table below.

Students are expected to have available a calculator with at least the following keys: +, −, ×, ÷, π, x^2, $\sqrt{x}$, $\frac{1}{x}$, x^y, ln x, e^x, $x!$, sine, cosine and tangent and their inverses in degrees and decimals of a degree, and in radians; memory.

Prohibitions

Calculators with any of the following facilities are prohibited in all examinations:

- databanks
- retrieval of text or formulae
- built-in symbolic algebra manipulations
- symbolic differentiation and/or integration
- language translators
- communication with other machines or the internet

Extra online content

Whenever you see an *Online* box, it means that there is extra online content available to support you.

SolutionBank

SolutionBank provides worked solutions for questions in the book.
Download all the solutions as a PDF or quickly find the solution you need online.

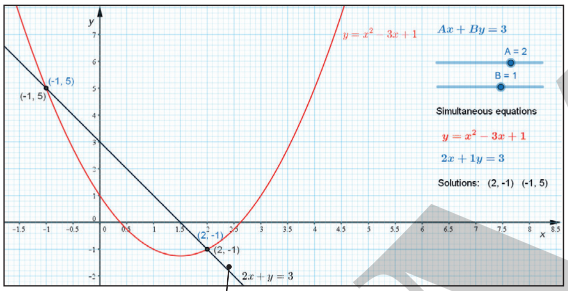
Use of technology

Explore topics in more detail, visualise problems and consolidate your understanding. Use pre-made GeoGebra activities or Casio resources for a graphic calculator.

Online Find the point of intersection graphically using technology.

GeoGebra

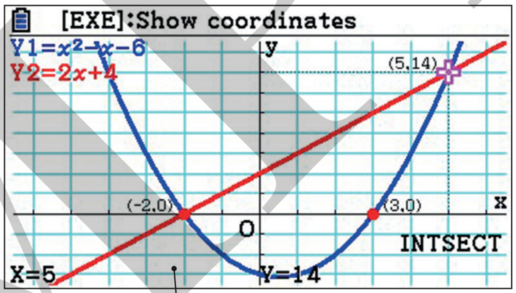
GeoGebra-powered interactives



Interact with the mathematics you are learning using GeoGebra's easy-to-use tools

CASIO

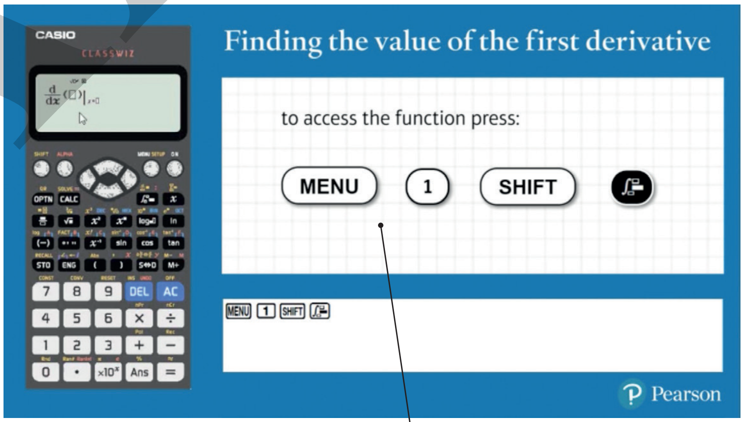
Graphic calculator interactives



Explore the mathematics you are learning and gain confidence in using a graphic calculator

Calculator tutorials

Our helpful video tutorials will guide you through how to use your calculator in the exams. They cover both Casio's scientific and colour graphic calculators.



Step-by-step guide with audio instructions on exactly which buttons to press and what should appear on your calculator's screen

Online Work out each coefficient quickly using the ${}^nC_r$ and power functions on your calculator.

1 COMPLEX NUMBERS

1.1
1.2
1.3
1.4
1.5
1.6

Learning objectives

After completing this chapter you should be able to:

- Understand and use the definitions of imaginary and complex numbers → page 2
- Add and subtract complex numbers → pages 2–3
- Find solutions to any quadratic equation with real coefficients → pages 4–5
- Multiply complex numbers → pages 5–6
- Understand the definition of a complex conjugate → pages 7–8
- Divide complex numbers → pages 7–8
- Show complex numbers on an Argand diagram → pages 9–10
- Find the modulus and argument of a complex number → pages 11–14
- Write a complex number in modulus-argument form → pages 15–16
- Solve quadratic equations that have complex roots → pages 16–18
- Solve cubic or quartic equations that have complex roots → pages 18–22

Prior knowledge check

1 Simplify each of the following:

a $\sqrt{50}$ **b** $\sqrt{108}$ **c** $\sqrt{180}$ ← Pure 1 Section 1.5

2 In each case, determine the number of distinct real roots of the equation $f(x) = 0$.

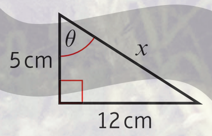
a $f(x) = 3x^2 + 8x + 10$

b $f(x) = 2x^2 - 9x + 7$

c $f(x) = 4x^2 + 12x + 9$ ← Pure 1 Section 2.3

3 For the triangle shown, find the values of:

a x **b** θ



← International GCSE Mathematics

4 Find the solutions of $x^2 - 8x + 6 = 0$, giving your answers in the form $a \pm \sqrt{b}$ where a and b are integers. ← Pure 1 Section 2.1

5 Write $\frac{7}{4-\sqrt{3}}$ in the form $p + q\sqrt{3}$ where p and q are rational numbers. ← Pure 1 Section 1.6

Complex numbers contain a real part and an imaginary part. Engineers and physicists often describe quantities with two components using a single complex number. This allows them to model complicated situations such as air flow over a cyclist.

1.1 Imaginary and complex numbers

The **quadratic equation** $ax^2 + bx + c = 0$ has solutions given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If the **expression** under the square root is negative, there are no real solutions.

Links For the equation $ax^2 + bx + c = 0$, the **discriminant** is $b^2 - 4ac$.
- If $b^2 - 4ac > 0$, there are two distinct real **roots**.
- If $b^2 - 4ac = 0$, there are two equal real roots.
- If $b^2 - 4ac < 0$, there are no real roots.

← Pure 1 Section 2.5

You can find solutions to the equation in all cases by extending the number system to include $\sqrt{-1}$. Since there is no **real number** that squares to produce −1, the number $\sqrt{-1}$ is called an **imaginary number**, and is represented using the letter **i**. **Complex numbers** have a real part and an imaginary part, for example 3 + 2i.

- $\mathrm{i} = \sqrt{-1}$
- **An imaginary number is a number of the form bi, where $b \in \mathbb{R}$.**
- **A complex number is written in the form $a + b$i, where $a, b \in \mathbb{R}$.**

Notation The set of all complex numbers is written as $\mathbb{C}$.
For the complex number $z = a + b\mathrm{i}$:
- $\mathrm{Re}(z) = a$ is the real part
- $\mathrm{Im}(z) = b$ is the imaginary part

Example 1 SKILLS INTERPRETATION

Write each of the following in terms of i.

a $\sqrt{-36}$ **b** $\sqrt{-28}$

a $\sqrt{-36} = \sqrt{36 \times (-1)} = \sqrt{36}\sqrt{-1} = 6\mathrm{i}$

b $\sqrt{-28} = \sqrt{28 \times (-1)} = \sqrt{28}\sqrt{-1}$
$= \sqrt{4}\sqrt{7}\sqrt{-1} = (2\sqrt{7})\mathrm{i}$

You can use the rules of **surds** to manipulate imaginary numbers.

Watch out An alternative way of writing $(2\sqrt{7})\mathrm{i}$ is $2\mathrm{i}\sqrt{7}$. Avoid writing $2\sqrt{7}\mathrm{i}$ as this can easily be confused with $2\sqrt{7\mathrm{i}}$.

In a complex number, the real part and the imaginary part cannot be combined to form a single term.

- **Complex numbers can be added or subtracted by adding or subtracting their real parts and adding or subtracting their imaginary parts.**
- **You can multiply a real number by a complex number by multiplying out the brackets in the usual way.**

Example 2

Simplify each of the following, giving your answers in the form $a + b$i, where $a, b \in \mathbb{R}$.

a $(2 + 5\mathrm{i}) + (7 + 3\mathrm{i})$ **b** $(2 - 5\mathrm{i}) - (5 - 11\mathrm{i})$ **c** $2(5 - 8\mathrm{i})$ **d** $\frac{10 + 6\mathrm{i}}{2}$

a $(2 + 5\mathrm{i}) + (7 + 3\mathrm{i}) = (2 + 7) + (5 + 3)\mathrm{i}$
$= 9 + 8\mathrm{i}$

Add the real parts and add the imaginary parts.

b $(2 - 5\mathrm{i}) - (5 - 11\mathrm{i}) = (2 - 5) + (-5 - (-11))\mathrm{i}$
$= -3 + 6\mathrm{i}$

Subtract the real parts and subtract the imaginary parts.

c $2(5-8i)=(2\times5)-(2\times8)i=10-16i$ — $2(5-8i)$ can also be written as $(5-8i)+(5-8i)$.

d $\frac{10+6i}{2}=\frac{10}{2}+\frac{6}{2}i=5+3i$ — First separate into real and imaginary parts.

Exercise 1A

SKILLS INTERPRETATION

Do not use your calculator in this exercise.

1 Write each of the following in the form bi, where b is a real number.

a $\sqrt{-9}$ **b** $\sqrt{-49}$ **c** $\sqrt{-121}$ **d** $\sqrt{-10000}$ **e** $\sqrt{-225}$

f $\sqrt{-5}$ **g** $\sqrt{-12}$ **h** $\sqrt{-45}$ **i** $\sqrt{-200}$ **j** $\sqrt{-147}$

2 Simplify, giving your answers in the form $a+bi$, where $a, b \in \mathbb{R}$.

a $(5+2i)+(8+9i)$ **b** $(4+10i)+(1-8i)$

c $(7+6i)+(-3-5i)$ **d** $\left(\frac{1}{2}+\frac{1}{3}i\right)+\left(\frac{5}{2}+\frac{5}{3}i\right)$

e $(20+12i)-(11+3i)$ **f** $(2-i)-(-5+3i)$

g $(-4-6i)-(-8-8i)$ **h** $(3\sqrt{2}+i)-(\sqrt{2}-i)$

i $(-2-7i)+(1+3i)-(-12+i)$ **j** $(18+5i)-(15-2i)-(3+7i)$

3 Simplify, giving your answers in the form $a+bi$, where $a, b \in \mathbb{R}$.

a $2(7+2i)$ **b** $3(8-4i)$

c $2(3+i)+3(2+i)$ **d** $5(4+3i)-4(-1+2i)$

e $\frac{6-4i}{2}$ **f** $\frac{15+25i}{5}$

g $\frac{9+11i}{3}$ **h** $\frac{-8+3i}{4}-\frac{7-2i}{2}$

(P) **4** Write in the form $a+bi$, where a and b are simplified surds.

a $\frac{4-2i}{\sqrt{2}}$ **b** $\frac{2-6i}{1+\sqrt{3}}$

5 Given that $z=7-6i$ and $w=7+6i$, find, in the form $a+bi$, where $a, b \in \mathbb{R}$:

a $z-w$ **b** $w+z$

Notation Complex numbers are often represented by the letter z or the letter w.

(E) **6** Given that $z_1=a+9i$, $z_2=-3+bi$ and $z_2-z_1=7+2i$, find a and b where $a, b \in \mathbb{R}$. **(2 marks)**

(P) **7** Given that $z_1=4+i$ and $z_2=7-3i$, find, in the form $a+bi$, where $a, b \in \mathbb{R}$:

a z_1-z_2 **b** $4z_2$ **c** $2z_1+5z_2$

(P) **8** Given that $z=a+bi$ and $w=a-bi$, where $a, b \in \mathbb{R}$, show that:

a $z+w$ is always real **b** $z-w$ is always imaginary

You can use complex numbers to find solutions to any quadratic equation with real **coefficients**.

- **If $b^2 - 4ac < 0$ then the quadratic equation $ax^2 + bx + c = 0$ has two distinct complex roots, neither of which are real.**

Example 3 SKILLS PROBLEM-SOLVING

Solve the equation $z^2 + 9 = 0$.

$z^2 = -9$

$z = \pm\sqrt{-9} = \pm\sqrt{9 \times (-1)} = \pm\sqrt{9}\sqrt{-1} = \pm 3i$

$z = +3i,\ z = -3i$

Note that just as $z^2 = 9$ has two roots +3 and −3, $z^2 = -9$ also has two roots +3i and −3i.

Example 4

Solve the equation $z^2 + 6z + 25 = 0$.

Method 1 (Completing the square)

$z^2 + 6z = (z + 3)^2 - 9$

Because $(z + 3)^2 = (z + 3)(z + 3) = z^2 + 6z + 9$

$z^2 + 6z + 25 = (z + 3)^2 - 9 + 25 = (z + 3)^2 + 16$

$(z + 3)^2 + 16 = 0$

$(z + 3)^2 = -16$

$z + 3 = \pm\sqrt{-16} = \pm 4i$

$\sqrt{-16} = \sqrt{16 \times (-1)} = \sqrt{16}\sqrt{-1} = \pm 4i$

$z = -3 \pm 4i$

$z = -3 + 4i,\ z = -3 - 4i$

You can use your calculator to find the complex roots of a quadratic equation like this one.

Method 2 (Quadratic formula)

$$z = \frac{-6 \pm \sqrt{6^2 - 4 \times 1 \times 25}}{2}$$

Using $z = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$= \frac{-6 \pm \sqrt{-64}}{2}$$

$$z = \frac{-6 \pm 8i}{2} = -3 \pm 4i$$

$\sqrt{-64} = \sqrt{64 \times (-1)} = \sqrt{64}\sqrt{-1} = \pm 8i$

$z = -3 + 4i,\ z = -3 - 4i$

Exercise 1B SKILLS PROBLEM-SOLVING

Do not use your calculator in this exercise.

1 Solve each of the following equations. Write your answers in the form $\pm bi$.

a $z^2 + 121 = 0$ **b** $z^2 + 40 = 0$ **c** $2z^2 + 120 = 0$

d $3z^2 + 150 = 38 - z^2$ **e** $z^2 + 30 = -3z^2 - 66$ **f** $6z^2 + 1 = 2z^2$

2 Solve each of the following equations.
Write your answers in the form $a \pm b$i.

a $(z - 3)^2 - 9 = -16$

b $2(z - 7)^2 + 30 = 6$

c $16(z + 1)^2 + 11 = 2$

Hint The left-hand side of each equation is in completed square form already. Use **inverse operations** to find the values of z.

3 Solve each of the following equations. Write your answers in the form $a \pm b$i.

a $z^2 + 2z + 5 = 0$ **b** $z^2 - 2z + 10 = 0$ **c** $z^2 + 4z + 29 = 0$

d $z^2 + 10z + 26 = 0$ **e** $z^2 + 5z + 25 = 0$ **f** $z^2 + 3z + 5 = 0$

4 Solve each of the following equations. Write your answers in the form $a \pm b$i.

a $2z^2 + 5z + 4 = 0$ **b** $7z^2 - 3z + 3 = 0$ **c** $5z^2 - z + 3 = 0$

5 The solutions to the quadratic equation $z^2 - 8z + 21 = 0$ are z_1 and z_2.
Find z_1 and z_2, giving each in the form $a \pm \mathrm{i}\sqrt{b}$.

E/P **6** The equation $z^2 + bz + 11 = 0$, where $b \in \mathbb{R}$, has distinct non-real complex roots.
Find the range of possible values of b. **(3 marks)**

1.2 Multiplying complex numbers

You can multiply complex numbers using the same technique that you use for multiplying brackets in algebra. You can use the fact that $\mathrm{i} = \sqrt{-1}$ to simplify powers of i.

- $\mathrm{i}^2 = -1$

Example 5

Express each of the following in the form $a + b$i, where a and b are real numbers.

a $(2 + 3\mathrm{i})(4 + 5\mathrm{i})$ **b** $(7 - 4\mathrm{i})^2$

a $(2 + 3\mathrm{i})(4 + 5\mathrm{i}) = 2(4 + 5\mathrm{i}) + 3\mathrm{i}(4 + 5\mathrm{i})$ — Multiply out the two brackets the same as you would with real numbers.
$= 8 + 10\mathrm{i} + 12\mathrm{i} + 15\mathrm{i}^2$
$= 8 + 10\mathrm{i} + 12\mathrm{i} - 15$ — Use the fact that $\mathrm{i}^2 = -1$.
$= (8 - 15) + (10\mathrm{i} + 12\mathrm{i})$ — Add real parts and add imaginary parts.
$= -7 + 22\mathrm{i}$

b $(7 - 4\mathrm{i})^2 = (7 - 4\mathrm{i})(7 - 4\mathrm{i})$
$= 7(7 - 4\mathrm{i}) - 4\mathrm{i}(7 - 4\mathrm{i})$ — Multiply out the two brackets the same as you would with real numbers.
$= 49 - 28\mathrm{i} - 28\mathrm{i} + 16\mathrm{i}^2$
$= 49 - 28\mathrm{i} - 28\mathrm{i} - 16$ — Use the fact that $\mathrm{i}^2 = -1$.
$= (49 - 16) + (-28\mathrm{i} - 28\mathrm{i})$ — Add real parts and add imaginary parts.
$= 33 - 56\mathrm{i}$

Example 6 SKILLS ANALYSIS

Simplify: **a** i^3 **b** i^4 **c** $(2i)^5$

a $i^3 = i \times i \times i = i^2 \times i = -i$

b $i^4 = i \times i \times i \times i = i^2 \times i^2 = (-1) \times (-1) = 1$

c $(2i)^5 = 2i \times 2i \times 2i \times 2i \times 2i$

$= 32(i \times i \times i \times i \times i) = 32(i^2 \times i^2 \times i)$

$= 32 \times (-1) \times (-1) \times i = 32i$

$i^2 = -1$

$(2i)^5 = 2^5 \times i^5$
First work out $2^5 = 32$.

Exercise 1C SKILLS EXECUTIVE FUNCTION

Do not use your calculator in this exercise.

1 Simplify each of the following, giving your answers in the form $a + bi$.

a $(5 + i)(3 + 4i)$ **b** $(6 + 3i)(7 + 2i)$ **c** $(5 - 2i)(1 + 5i)$
d $(13 - 3i)(2 - 8i)$ **e** $(-3 - i)(4 + 7i)$ **f** $(8 + 5i)^2$
g $(2 - 9i)^2$ **h** $(1 + i)(2 + i)(3 + i)$
i $(3 - 2i)(5 + i)(4 - 2i)$ **j** $(2 + 3i)^3$

Hint For part **h**, begin by multiplying the first pair of brackets.

(P) **2 a** Simplify $(4 + 5i)(4 - 5i)$, giving your answer in the form $a + bi$.
b Simplify $(7 - 2i)(7 + 2i)$, giving your answer in the form $a + bi$.
c Comment on your answers to parts **a** and **b**.
d Show that $(a + bi)(a - bi)$ is a real number for any real numbers a and b.

(P) **3** Given that $(a + 3i)(1 + bi) = 25 - 39i$, find two possible pairs of values for a and b.

4 Write each of the following in its simplest form.
a i^6 **b** $(3i)^4$ **c** $i^5 + i$ **d** $(4i)^3 - 4i^3$

(P) **5** Express $(1 + i)^6$ in the form $a - bi$, where a and b are **integers** to be found.

(P) **6** Find the value of the real part of $(3 - 2i)^4$.

(P) **7** $f(z) = 2z^2 - z + 8$
Find: **a** $f(2i)$ **b** $f(3 - 6i)$

Problem-solving
You can use the **binomial** theorem to expand $(a + b)^n$. ← Pure 2 Section 4.3

(E/P) **8** $f(z) = z^2 - 2z + 17$
Show that $z = 1 - 4i$ is a solution to $f(z) = 0$. **(2 marks)**

9 a Given that $i^1 = i$ and $i^2 = -1$, write i^3 and i^4 in their simplest forms.
b Write i^5, i^6, i^7 and i^8 in their simplest forms.
c Write down the value of:
i i^{100} **ii** i^{253} **iii** i^{301}

Challenge
a Expand $(a + bi)^2$.
b Hence, or otherwise, find $\sqrt{40 - 42i}$, giving your answer in the form $a - bi$, where a and b are positive integers.

Notation The **principal square root** of a complex number, $\sqrt{z}$, has a positive real part.

1.3 Complex conjugation

- For any complex number $z = a + bi$, the **complex conjugate** of the number is defined as $z^* = a - bi$.

Notation Together, z and z^* are called a **complex conjugate pair**.

Example 7 SKILLS INTERPRETATION

Given that $z = 2 - 7i$,

a write down z^* **b** find the value of $z + z^*$ **c** find the value of zz^*.

a $z^* = 2 + 7i$

Change the sign of the imaginary part from – to +.

b $z + z^* = (2 - 7i) + (2 + 7i)$
$= (2 + 2) + (-7 + 7)i = 4$

Notation Notice that $z + z^*$ is real.

c $zz^* = (2 - 7i)(2 + 7i)$
$= 2(2 + 7i) - 7i(2 + 7i)$
$= 4 + 14i - 14i - 49i^2$
$= 4 + 49 = 53$

Remember $i^2 = -1$.

Notation Notice that zz^* is real.

For any complex number z, the **product** of z and z^* is a real number. You can use this property (i.e. characteristic) to **divide two complex numbers**. To do this, you multiply both the numerator and the denominator by the complex conjugate of the denominator and then simplify the result.

Links The method used to divide complex numbers is similar to the method used to rationalise a denominator when simplifying surds. ← Pure 1 Section 1.5

Example 8

Write $\frac{5 + 4i}{2 - 3i}$ in the form $a + bi$.

$$\frac{5 + 4i}{2 - 3i} = \frac{5 + 4i}{2 - 3i} \times \frac{2 + 3i}{2 + 3i}$$

The complex conjugate of the denominator is $2 + 3i$. Multiply both the numerator and the denominator by the complex conjugate.

$$= \frac{(5 + 4i)(2 + 3i)}{(2 - 3i)(2 + 3i)}$$

zz^* is real, so $(2 - 3i)(2 + 3i)$ will be a real number.

$$= \frac{10 + 8i + 15i + 12i^2}{4 + 6i - 6i - 9i^2}$$

You can enter complex numbers directly into your calculator to multiply or divide them quickly.

$$= \frac{10 - 12 + 23i}{4 + 9}$$

$$= \frac{-2 + 23i}{13} = -\frac{2}{13} + \frac{23}{13}i$$

Divide each term in the numerator by 13.

Exercise 1D SKILLS INTERPRETATION

Do not use your calculator in this exercise.

1 Write down the complex conjugate z^* for:

a $z = 8 + 2\text{i}$ **b** $z = 6 - 5\text{i}$ **c** $z = \frac{2}{3} - \frac{1}{2}\text{i}$ **d** $z = \sqrt{5} + \text{i}\sqrt{10}$

2 Find $z + z^*$ and zz^* for:

a $z = 6 - 3\text{i}$ **b** $z = 10 + 5\text{i}$ **c** $z = \frac{3}{4} + \frac{1}{4}\text{i}$ **d** $z = \sqrt{5} - 3\text{i}\sqrt{5}$

3 Write each of the following in the form $a + b\text{i}$.

a $\dfrac{3 - 5\text{i}}{1 + 3\text{i}}$ **b** $\dfrac{3 + 5\text{i}}{6 - 8\text{i}}$ **c** $\dfrac{28 - 3\text{i}}{1 - \text{i}}$ **d** $\dfrac{2 + \text{i}}{1 + 4\text{i}}$

4 Write $\dfrac{(3 - 4\text{i})^2}{1 + \text{i}}$ in the form $x + \text{i}y$, where $x, y \in \mathbb{R}$.

5 Given that $z_1 = 1 + \text{i}$, $z_2 = 2 + \text{i}$ and $z_3 = 3 + \text{i}$, write each of the following in the form $a + b\text{i}$.

a $\dfrac{z_1 z_2}{z_3}$ **b** $\dfrac{(z_2)^2}{z_1}$ **c** $\dfrac{2z_1 + 5z_3}{z_2}$

E **6** Given that $\dfrac{5 + 2\text{i}}{z} = 2 - \text{i}$, find z in the form $a + b\text{i}$. **(2 marks)**

7 Simplify $\dfrac{6 + 8\text{i}}{1 + \text{i}} + \dfrac{6 + 8\text{i}}{1 - \text{i}}$, giving your answer in the form $a + b\text{i}$.

8 $w = \dfrac{4}{8 - \text{i}\sqrt{2}}$

Express w in the form $a + b\text{i}\sqrt{2}$, where a and b are **rational numbers**.

9 $w = 1 - 9\text{i}$

Express $\dfrac{1}{w}$ in the form $a + b\text{i}$, where a and b are rational numbers.

10 $z = 4 - \text{i}\sqrt{2}$

Use algebra to express $\dfrac{z + 4}{z - 3}$ in the form $p + q\text{i}\sqrt{2}$, where p and q are rational numbers.

E/P **11** The complex number z satisfies the equation $(4 + 2\text{i})(z - 2\text{i}) = 6 - 4\text{i}$.
Find z, giving your answer in the form $a + b\text{i}$ where a and b are rational numbers. **(4 marks)**

E/P **12** The complex numbers z_1 and z_2 are given by $z_1 = p - 7\text{i}$ and $z_2 = 2 + 5\text{i}$, where p is an integer.
Find $\dfrac{z_1}{z_2}$ in the form $a + b\text{i}$, where a and b are rational, and are given in terms of p. **(4 marks)**

E **13** $z = \sqrt{5} + 4\text{i}$. z^* is the complex conjugate of z.
Show that $\dfrac{z}{z^*} = a + b\text{i}\sqrt{5}$, where a and b are rational numbers to be found. **(4 marks)**

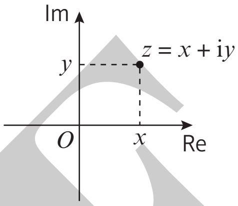
E/P **14** The complex number z is defined by $z = \dfrac{p + 5\text{i}}{p - 2\text{i}}$, $p \in \mathbb{R}$, $p > 0$.
Given that the real part of z is $\frac{1}{2}$,

a find the value of p **(4 marks)**

b write z in the form $a + b\text{i}$, where a and b are real. **(1 mark)**

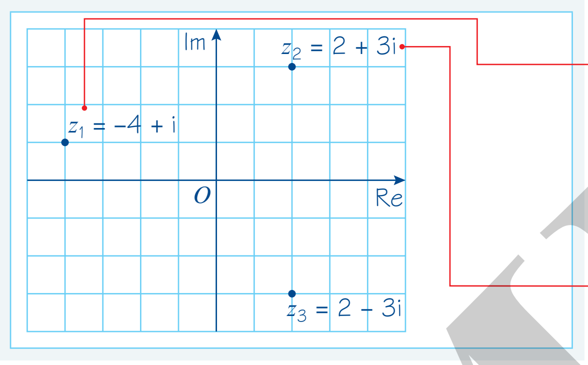
1.4 Argand diagrams

- **You can represent complex numbers on an Argand diagram. The x-axis on an Argand diagram is called the real axis and the y-axis is called the imaginary axis. The complex number $z = x + iy$ is represented on the diagram by the point $P(x, y)$, where x and y are Cartesian coordinates.**



Example 9 SKILLS INTERPRETATION

Show the complex numbers $z_1 = -4 + i$, $z_2 = 2 + 3i$ and $z_3 = 2 - 3i$ on an Argand diagram.



The real part of each number describes its horizontal position, and the imaginary part describes its vertical position. For example, $z_1 = -4 + i$ has real part -4 and imaginary part 1.

Note that z_2 and z_3 are complex conjugates. On an Argand diagram, complex conjugate pairs are symmetrical about the real axis.

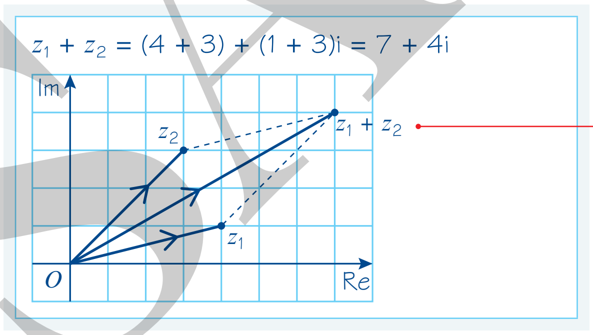
Complex numbers can also be represented as vectors on an Argand diagram.

- **The complex number $z = x + iy$ can be represented as the vector $\begin{pmatrix} x \\ y \end{pmatrix}$ on an Argand diagram.**

You can add or subtract complex numbers on an Argand diagram by adding or subtracting their **corresponding** (i.e. equivalent) vectors.

Example 10

$z_1 = 4 + i$ and $z_2 = 3 + 3i$. Show z_1, z_2 and $z_1 + z_2$ on an Argand diagram.

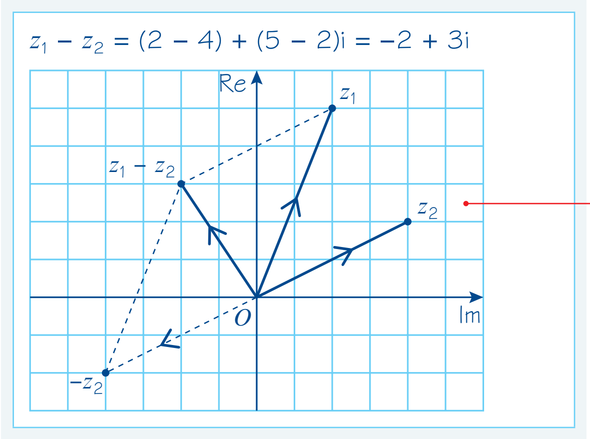


The **vector** representing $z_1 + z_2$ is the diagonal of the parallelogram with vertices at O, z_1 and z_2. You can use vector addition to find $z_1 + z_2$:

$$\begin{pmatrix} 4 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$$

Example 11

$z_1 = 2 + 5i$ and $z_2 = 4 + 2i$. Show z_1, z_2 and $z_1 - z_2$ on an Argand diagram.



The vector corresponding to z_2 is $\begin{pmatrix}4\\2\end{pmatrix}$, so the vector corresponding to $-z_2$ is $\begin{pmatrix}-4\\-2\end{pmatrix}$.

The vector representing $z_1 - z_2$ is the diagonal of the parallelogram with vertices at O, z_1 and $-z_2$.

Online Explore adding and subtracting complex numbers on an Argand diagram using GeoGebra.

Exercise 1E

SKILLS INTERPRETATION

1 Show these numbers on an Argand diagram.

a $7 + 2i$ **b** $5 - 4i$ **c** $-6 - i$ **d** $-2 + 5i$

e $3i$ **f** $\sqrt{2} + 2i$ **g** $-\frac{1}{2} + \frac{5}{2}i$ **h** -4

2 $z_1 = 11 + 2i$ and $z_2 = 2 + 4i$. Show z_1, z_2 and $z_1 + z_2$ on an Argand diagram.

3 $z_1 = -3 + 6i$ and $z_2 = 8 - i$. Show z_1, z_2 and $z_1 + z_2$ on an Argand diagram.

4 $z_1 = 8 + 4i$ and $z_2 = 6 + 7i$. Show z_1, z_2 and $z_1 - z_2$ on an Argand diagram.

5 $z_1 = -6 - 5i$ and $z_2 = -4 + 4i$. Show z_1, z_2 and $z_1 - z_2$ on an Argand diagram.

(P) **6** $z_1 = 7 - 5i$, $z_2 = a + bi$ and $z_3 = -3 + 2i$, where $a, b \in \mathbb{Z}$. Given that $z_3 = z_1 + z_2$,

a find the values of a and b **b** show z_1, z_2 and z_3 on an Argand diagram.

(P) **7** $z_1 = p + qi$, $z_2 = 9 - 5i$ and $z_3 = -8 + 5i$, where $p, q \in \mathbb{Z}$. Given that $z_3 = z_1 + z_2$,

a find the values of p and q **b** show z_1, z_2 and z_3 on an Argand diagram.

(E) **8** The solutions to the quadratic equation $z^2 - 6z + 10 = 0$ are z_1 and z_2.

a Find z_1 and z_2, giving your answers in the form $p \pm qi$, where p and q are integers. **(3 marks)**

b Show, on an Argand diagram, the points representing the complex numbers z_1 and z_2. **(2 marks)**

(E/P) **9** $f(z) = 2z^3 - 19z^2 + 64z - 60$

a Show that $f\left(\frac{3}{2}\right) = 0$. **(1 mark)**

b Use algebra to solve $f(z) = 0$ completely. **(4 marks)**

c Show all three solutions on an Argand diagram. **(2 marks)**

Challenge

SKILLS CREATIVITY

a Find all the solutions to the equation $z^6 = 1$.

b Show each solution on an Argand diagram.

c Show that each solution lies on a circle with centre (0, 0) and radius 1.

Hint There will be six distinct roots in total. Write $z^6 = 1$ as $(z^3 - 1)(z^3 + 1) = 0$, then find three distinct roots of $z^3 - 1 = 0$ and three distinct roots of $z^3 + 1 = 0$.

1.5 Modulus and argument

The **modulus** or absolute value of a complex number is the magnitude (i.e. size) of its corresponding vector.

- The modulus of a complex number, $|z|$, is the distance from the **origin** to that number on an Argand diagram. For a complex number $z = x + iy$, the modulus is given by $|z| = \sqrt{x^2 + y^2}$.

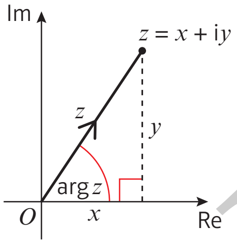
Notation The modulus of the complex number z is written as r, $|z|$ or $|x + iy|$.

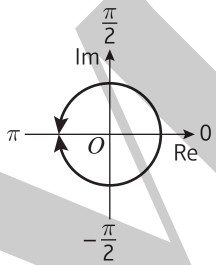
The **argument** of a complex number is the angle its corresponding vector makes with the positive real axis.

- The argument of a complex number, arg z, is the angle between the positive real axis and the line joining that number to the origin on an Argand diagram, measured in an anticlockwise direction (i.e. moving in the opposite direction to the hands of a clock). For a complex number $z = x + iy$, the argument, θ, satisfies $\tan\theta = \frac{y}{x}$.

Notation The argument of the complex number z is written as arg z. It is usually given in radians, where
- 2π radians = 360°
- π radians = 180°

← Pure 1 Section 7.1





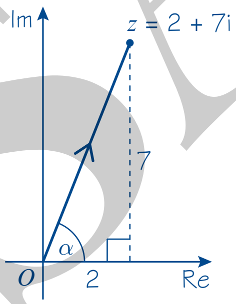
The argument θ of any complex number is usually given in the range $-\pi < \theta \leqslant \pi$. This is sometimes referred to as the **principal argument**.

Example 12 SKILLS PROBLEM-SOLVING

Given the complex number $z = 2 + 7i$, find:

a the modulus of z **b** the argument of z, giving your answer in **radians** to 2 d.p.

a



Sketch the Argand diagram, showing the position of the number.

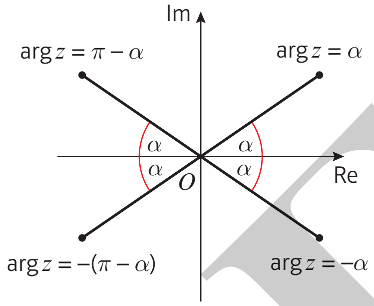
Modulus: $|z| = |2 + 7i| = \sqrt{2^2 + 7^2}$
$= \sqrt{53}$

b Argument: $\tan\alpha = \frac{7}{2}$ $\alpha = 1.2924...$ radians

arg z = 1.29 radians (2 d.p.)

If z does not lie in the first **quadrant**, you can use an Argand diagram to help you find its argument.

- Let α be the positive **acute** angle made with the real axis by the line joining the origin and z.
  - If z lies in the first quadrant, then $\arg z = \alpha$.
  - If z lies in the second quadrant, then $\arg z = \pi - \alpha$.
  - If z lies in the third quadrant, then $\arg z = -(\pi - \alpha)$.
  - If z lies in the fourth quadrant, then $\arg z = -\alpha$.

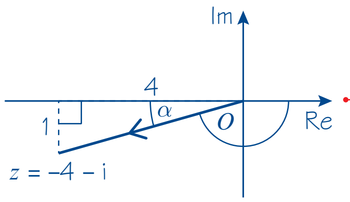


Example 13

Given the complex number $z = -4 - \mathrm{i}$, find:

a the modulus of z **b** the argument of z, giving your answer in radians to 2 d.p.

a



Sketch the Argand diagram, showing the position of the number.

Modulus: $|z| = |-4 - \mathrm{i}| = \sqrt{(-4)^2 + (-1)^2}$
$= \sqrt{17}$

b Argument: $\tan\alpha = \frac{1}{4}$ $\alpha = 0.2449...$ radians
$\arg z = -(\pi - 0.2449)$
$= -2.90$ radians (2 d.p.)

Here z is in the third quadrant, so the required argument is $-(\pi - \alpha)$.

You can use the following rule to multiply the moduli of complex numbers quickly.

For any two complex numbers z_1 and z_2,

$$|z_1||z_2| = |z_1 z_2|.$$

The proof of this result is beyond the scope of this book.

Example 14

$z_1 = 3 + 4\mathrm{i}$ and $z_2 = 5 - 12\mathrm{i}$

Find:

a the modulus of z_1 and the modulus of z_2

b $z_1 z_2$

c hence, find $|z_1 z_2|$ and verify that $|z_1||z_2| = |z_1 z_2|$

a $|z_1| = \sqrt{3^2 + 4^2} = 5$
$|z_2| = \sqrt{5^2 + (-12)^2} = 13$

The modulus of a complex number $|a + b\mathrm{i}| = \sqrt{a^2 + b^2}$

b $z_1 z_2 = (3 + 4\mathrm{i})(5 - 12\mathrm{i})$
$= 15 - 36\mathrm{i} + 20\mathrm{i} - 48\mathrm{i}^2$
$= 63 - 16\mathrm{i}$

Don't forget that $\mathrm{i}^2 = -1$.

c $|z_1 z_2| = \sqrt{63^2 + (-16)^2} = 65$
$|z_1||z_2| = 5 \times 13 = 65$

This is not a proof. However the result is verified and works in every case.

Exercise 1F SKILLS PROBLEM-SOLVING

1 For each of the following complex numbers,

i find the modulus, writing your answer in surd form if necessary

ii find the argument, writing your answer in radians to 2 decimal places.

a $z = 12 + 5\text{i}$ **b** $z = \sqrt{3} + \text{i}$ **c** $z = -3 + 6\text{i}$

d $z = 2 - 2\text{i}$ **e** $z = -8 - 7\text{i}$ **f** $z = -4 + 11\text{i}$

g $z = 2\sqrt{3} - \text{i}\sqrt{3}$ **h** $z = -8 - 15\text{i}$

Hint In part **c**, the complex number is in the second quadrant, so the argument will be $\pi - \alpha$. In part **d**, the complex number is in the fourth quadrant, so the argument will be $-\alpha$.

2 For each of the following complex numbers,

i find the modulus, writing your answer in surd form

ii find the argument, writing your answer in terms of π.

a $2 + 2\text{i}$ **b** $5 + 5\text{i}$ **c** $-6 + 6\text{i}$ **d** $-a - a\text{i}, a \in \mathbb{R}$

3 The complex number z_1 is such that $z_1 = 3 + 5\text{i}$.

a Find $|z_1|$. **b** Find $(z_1)^2$. **c** Hence verify that $|(z_1)^2| = |z_1|^2$.

4 The complex number z_1 is such that $z_1 = \dfrac{26}{3 + 2\text{i}}$.

a Write z_1 in the form $a + b\text{i}$, where a and b are integers.

b Find $|z_1|$.

c Given that $|z_1 z_2| = 26\sqrt{13}$, find $|z_2|$.

d Given also that $z_2 = 5 + p\text{i}$, find the possible values of p.

(E) **5** $z = -40 - 9\text{i}$

a Show z on an Argand diagram. **(1 mark)**

b Calculate arg z, giving your answer in radians to 2 decimal places. **(2 marks)**

(E) **6** $z = 3 + 4\text{i}$

a Show that $z^2 = -7 + 24\text{i}$. **(2 marks)**

Find, showing your working:

b $|z^2|$ **(2 marks)**

c arg (z^2), giving your answer in radians to 2 decimal places. **(2 marks)**

d Show z and z^2 on an Argand diagram. **(1 mark)**

(E) **7** The complex numbers z_1 and z_2 are given by $z_1 = 4 + 6\text{i}$ and $z_2 = 1 + \text{i}$.

Find, showing your working:

a $\dfrac{z_1}{z_2}$ in the form $a + b\text{i}$, where a and b are real **(3 marks)**

b $\left|\dfrac{z_1}{z_2}\right|$ **(2 marks)**

c $\arg \dfrac{z_1}{z_2}$, giving your answer in radians to 2 decimal places. **(2 marks)**

E/P **8** The complex numbers z_1 and z_2 are such that $z_1 = 3 + 2p\text{i}$ and $\frac{z_1}{z_2} = 1 - \text{i}$, where p is a real **constant**.

a Find z_2 in the form $a + b\text{i}$, giving the real numbers a and b in terms of p. **(3 marks)**

Given that $\arg z_2 = \tan^{-1} 5$,

b find the value of p **(2 marks)**

c find the value of $|z_2|$ **(2 marks)**

d show z_1, z_2 and $\frac{z_1}{z_2}$ on a single Argand diagram. **(2 marks)**

E **9** Given the complex number $z = \frac{26}{2 - 3\text{i}}$, find:

a z in the form $a + \text{i}b$, where $a, b \in \mathbb{R}$ **(2 marks)**

b z^2 in the form $a + \text{i}b$, where $a, b \in \mathbb{R}$ **(2 marks)**

c $|z|$ **(2 marks)**

d $\arg(z^2)$, giving your answer in radians to 2 decimal places. **(2 marks)**

E/P **10** Given that $z_1 = 4 + 2\text{i}$, $z_2 = 2 + 4\text{i}$, $z_3 = a + b\text{i}$, where $a, b \in \mathbb{R}$,

a find the exact value of $|z_1 + z_2|$. **(2 marks)**

Given that $w = \frac{z_1 z_3}{z_2}$,

b find w in terms of a and b, giving your answer in the form $x + \text{i}y$, where $x, y \in \mathbb{R}$. **(4 marks)**

Given also that $w = \frac{21}{5} - \frac{22}{5}\text{i}$, find:

c the values of a and b **(3 marks)**

d $\arg w$, giving your answer in radians to 2 decimal places. **(2 marks)**

E/P **11** The complex number w is given by $w = 6 + 3\text{i}$. Find:

a $|w|$ **(1 mark)**

b $\arg w$, giving your answer in radians to 2 decimal places. **(2 marks)**

Given that $\arg(\lambda + 5\text{i} + w) = \frac{\pi}{4}$ where λ is a real constant,

c find the value of λ. **(2 marks)**

E **12** Given the complex number $z = -1 - \text{i}\sqrt{3}$, find:

a $|z|$ **(1 mark)**

b $\left|\frac{z}{z^*}\right|$ **(4 marks)**

c $\arg z$, $\arg(z^*)$ and $\arg\frac{z}{z^*}$, giving your answers in terms of π. **(3 marks)**

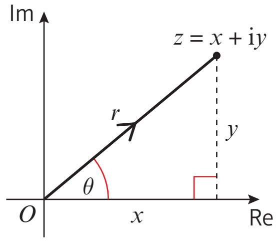
E/P **13** The complex numbers w and z are given by $w = k + \text{i}$ and $z = -4 + 5k\text{i}$, where k is a real constant. Given that $\arg(w + z) = \frac{2\pi}{3}$, find the exact value of k. **(6 marks)**

E/P **14** The complex numbers w and z are defined such that $\arg w = \frac{\pi}{10}$, $|w| = 5$ and $\arg z = \frac{2\pi}{5}$.
Given that $\arg(w + z) = \frac{\pi}{5}$, find the value of $|z|$. **(4 marks)**

1.6 Modulus–argument form of complex numbers

You can write any complex number in terms of its modulus and argument.

- **For a complex number z with $|z| = r$ and $\arg z = \theta$, the modulus–argument form of z is $z = r(\cos\theta + \text{i}\sin\theta)$.**



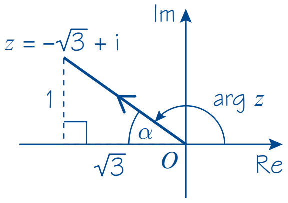
From the right-angled triangle, $x = r\cos\theta$ and $y = r\sin\theta$.

$z = x + \text{i}y = r\cos\theta + \text{i}r\sin\theta = r(\cos\theta + \text{i}\sin\theta)$

This formula works for a complex number in any quadrant of the Argand diagram. The argument, θ, is usually given in the range $-\pi < \theta \leqslant \pi$, although the formula works for any value of θ measured anticlockwise from the positive real axis.

Example 15 SKILLS INTERPRETATION

Express $z = -\sqrt{3} + \text{i}$ in the form $r(\cos\theta + \text{i}\sin\theta)$, where $-\pi < \theta \leqslant \pi$.



Sketch the Argand diagram, showing the position of the number.
Here z is in the second quadrant, so the required argument is $\pi - \alpha$.

$r = \sqrt{(-\sqrt{3})^2 + 1^2} = \sqrt{4} = 2$

$\theta = \arg z = \pi - \arctan\left(\frac{1}{\sqrt{3}}\right) = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

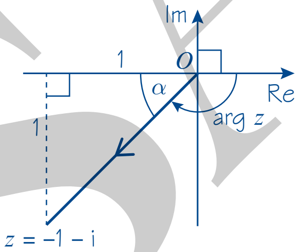
Find r and θ.

Therefore, $z = 2\left(\cos\frac{5\pi}{6} + \text{i}\sin\frac{5\pi}{6}\right)$

Apply $z = r(\cos\theta + \text{i}\sin\theta)$.

Example 16

Express $z = -1 - \text{i}$ in the form $r(\cos\theta + \text{i}\sin\theta)$, where $-\pi < \theta \leqslant \pi$.



Sketch the Argand diagram, showing the position of the number.
Here z is in the third quadrant, so the required argument is $-(\pi - \alpha)$.

$r = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$

$\theta = \arg z = -\pi + \arctan\left(\frac{1}{1}\right) = -\pi + \frac{\pi}{4} = -\frac{3\pi}{4}$

Find r and θ.

Therefore, $z = \sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right) + \text{i}\sin\left(-\frac{3\pi}{4}\right)\right)$

Apply $z = r(\cos\theta + \text{i}\sin\theta)$.

Exercise 1G SKILLS INTERPRETATION

1 Express the following in the form $r(\cos\theta + \mathrm{i}\sin\theta)$, where $-\pi < \theta \leqslant \pi$.
Give the exact values of r and θ where possible, or values to 2 d.p. otherwise.

a $2 + 2\mathrm{i}$ **b** $3\mathrm{i}$ **c** $-3 + 4\mathrm{i}$ **d** $1 - \mathrm{i}\sqrt{3}$

e $-2 - 5\mathrm{i}$ **f** -20 **g** $7 - 24\mathrm{i}$ **h** $-5 + 5\mathrm{i}$

2 Express these in the form $r(\cos\theta + \mathrm{i}\sin\theta)$, giving exact values of r and θ where possible, or values to 2 d.p. otherwise.

a $\dfrac{3}{1 + \mathrm{i}\sqrt{3}}$ **b** $\dfrac{1}{2 - \mathrm{i}}$ **c** $\dfrac{1 + \mathrm{i}}{1 - \mathrm{i}}$

3 Express the following in the form $x + \mathrm{i}y$, where $x, y \in \mathbb{R}$.

a $5\left(\cos\dfrac{\pi}{2} + \mathrm{i}\sin\dfrac{\pi}{2}\right)$ **b** $\dfrac{1}{2}\left(\cos\dfrac{\pi}{6} + \mathrm{i}\sin\dfrac{\pi}{6}\right)$ **c** $6\left(\cos\dfrac{5\pi}{6} + \mathrm{i}\sin\dfrac{5\pi}{6}\right)$

d $3\left(\cos\left(-\dfrac{2\pi}{3}\right) + \mathrm{i}\sin\left(-\dfrac{2\pi}{3}\right)\right)$ **e** $2\sqrt{2}\left(\cos\left(-\dfrac{\pi}{4}\right) + \mathrm{i}\sin\left(-\dfrac{\pi}{4}\right)\right)$ **f** $-4\left(\cos\dfrac{7\pi}{6} + \mathrm{i}\sin\dfrac{7\pi}{6}\right)$

(E) **4** **a** Express the complex number $z = 4\left(\cos\left(\dfrac{2\pi}{3}\right) + \mathrm{i}\sin\left(\dfrac{2\pi}{3}\right)\right)$ in the form $x + \mathrm{i}y$, where $x, y \in \mathbb{R}$. **(2 marks)**

b Show the complex number z on an Argand diagram. **(1 mark)**

(E) **5** The complex number z is such that $|z| = 7$ and $\arg z = \dfrac{11\pi}{6}$. Find z in the form $p + q\mathrm{i}$, where p and q are exact real numbers to be found. **(3 marks)**

(E) **6** The complex number z is such that $|z| = 5$ and $\arg z = -\dfrac{4\pi}{3}$. Find z in the form $a + b\mathrm{i}$, where a and b are exact real numbers to be found. **(3 marks)**

1.7 Roots of quadratic equations

- **For real numbers a, b and c, if the roots of the quadratic equation $az^2 + bz + c = 0$ are non-real complex numbers, then they occur as a conjugate pair.**

Another way of stating this is that for a real-valued quadratic **function** f(z), if z_1 is a root of f(z) = 0 then $z_1{}^*$ is also a root. You can use this fact to find one root if you know the other, or to find the original equation.

- **If the roots of a quadratic equation are α and β, then you can write the equation as $(z - \alpha)(z - \beta) = 0$**

  or $z^2 - (\alpha + \beta)z + \alpha\beta = 0$

Notation Roots of complex-valued **polynomials** are often written using Greek letters such as α (alpha), β (beta) and γ (gamma).

Example 17 SKILLS EXECUTIVE FUNCTION

Given that $\alpha = 7 + 2i$ is one of the roots of a quadratic equation with real coefficients,

a state the value of the other root, β

b find the quadratic equation

c find the values of $\alpha + \beta$ and $\alpha\beta$ and interpret the results.

a $\beta = 7 - 2i$

α and β will always be a complex conjugate pair.

b
$$(z - \alpha)(z - \beta) = 0$$
$$(z - (7 + 2i))(z - (7 - 2i)) = 0$$
$$z^2 - z(7 - 2i) - z(7 + 2i) + (7 + 2i)(7 - 2i) = 0$$
$$z^2 - 7z + 2iz - 7z - 2iz + 49 - 14i + 14i - 4i^2 = 0$$
$$z^2 - 14z + 49 + 4 = 0$$
$$z^2 - 14z + 53 = 0$$

The quadratic equation with roots α and β is $(z - \alpha)(z - \beta) = 0$.

Collect like terms. Use the fact that $i^2 = -1$.

c $\alpha + \beta = (7 + 2i) + (7 - 2i)$
$= (7 + 7) + (2 + (-2))i = 14$

The coefficient of z in $z^2 - 14z + 53$ is $-(\alpha + \beta)$.

$\alpha\beta = (7 + 2i)(7 - 2i) = 49 - 14i + 14i - 4i^2$
$= 49 + 4 = 53$

The constant term in $z^2 - 14z + 53$ is $\alpha\beta$.

Hint For $z = a + bi$, you should learn the results:

$z + z^* = 2a$

$zz^* = a^2 + b^2$

You can use these to find the quadratic equation quickly.

Exercise 1H SKILLS EXECUTIVE FUNCTION

1 The roots of the quadratic equation $z^2 + 2z + 26 = 0$ are α and β.

Find: **a** α and β **b** $\alpha + \beta$ **c** $\alpha\beta$

2 The roots of the quadratic equation $z^2 - 8z + 25 = 0$ are α and β.

Find: **a** α and β **b** $\alpha + \beta$ **c** $\alpha\beta$

(E) **3** Given that $2 + 3i$ is one of the roots of a quadratic equation with real coefficients,

a write down the other root of the equation **(1 mark)**

b find the quadratic equation, giving your answer in the form $z^2 + bz + c = 0$ where b and c are real constants. **(3 marks)**

(E) **4** Given that $5 - i$ is a root of the equation $z^2 + pz + q = 0$, where p and q are real constants,

a write down the other root of the equation **(1 mark)**

b find the value of p and the value of q. **(3 marks)**

(E/P) **5** Given that $z_1 = -5 + 4i$ is one of the roots of the quadratic equation $z^2 + bz + c = 0$, where b and c are real constants, find the values of b and c. **(4 marks)**

(E/P) **6** Given that $1 + 2i$ is one of the roots of a quadratic equation with real coefficients, find the equation, giving your answer in the form $z^2 + bz + c = 0$, where b and c are integers to be found. **(4 marks)**

E/P **7** Given that $3 - 5\text{i}$ is one of the roots of a quadratic equation with real coefficients, find the equation, giving your answer in the form $z^2 + bz + c = 0$, where b and c are real constants. **(4 marks)**

E/P **8** $z = \dfrac{5}{3 - \text{i}}$

a Find z in the form $a + b\text{i}$, where a and b are real constants. **(1 mark)**

Given that z is a complex root of the quadratic equation $z^2 + pz + q = 0$, where p and q are rational numbers,

b find the value of p and the value of q. **(4 marks)**

E/P **9** Given that $z = 5 + q\text{i}$ is a root of the equation $z^2 - 4pz + 34 = 0$, where p and q are positive real constants, find the value of p and the value of q. **(4 marks)**

1.8 Solving cubic and quartic equations

You can generalise the rule for the roots of quadratic equations to any polynomial with real coefficients.

- **If $\mathrm{f}(z)$ is a polynomial with real coefficients, and z_1 is a root of $\mathrm{f}(z) = 0$, then $z_1{}^*$ is also a root of $\mathrm{f}(z) = 0$.**

Notation If z_1 is real, then $z_1{}^* = z_1$.

You can use this property (i.e. characteristic) to find roots of **cubic** and **quartic** equations with real coefficients.

- **An equation of the form $az^3 + bz^2 + cz + d = 0$ is called a cubic equation, and has three roots.**
- **For a cubic equation with real coefficients, either:**
  - **all three roots are real, or**
  - **one root is real and the other two roots form a complex conjugate pair.**

Watch out A real-valued cubic equation might have two or three repeated real roots.

Example 18 SKILLS EXECUTIVE FUNCTION

Given that -1 is a root of the equation $z^3 - z^2 + 3z + k = 0$,

a find the value of k **b** find the other two roots of the equation.

a If -1 is a root,

$(-1)^3 - (-1)^2 + 3(-1) + k = 0$

$-1 - 1 - 3 + k = 0$

$k = 5$

b -1 is a root of the equation, so $z + 1$ is a factor of $z^3 - z^2 + 3z + 5$.

$$
\begin{array}{r}
z^2 - 2z + 5 \\
z + 1 \,\big)\, z^3 - z^2 + 3z + 5 \\
z^3 + z^2 \phantom{{}+3z+5} \\
-2z^2 + 3z \phantom{{}+5} \\
-2z^2 - 2z \phantom{{}+5} \\
5z + 5 \\
5z + 5 \\
0
\end{array}
$$

Problem-solving Use the factor theorem to help: if $\mathrm{f}(\alpha) = 0$, then α is a root of the polynomial and $z - \alpha$ is a factor of the polynomial.

Use long division (or inspection) to find the quadratic factor.

$z^3 - z^2 + 3z + 5 = (z + 1)(z^2 - 2z + 5) = 0$

Solving $z^2 - 2z + 5 = 0$,

$z^2 - 2z = (z - 1)^2 - 1$

$z^2 - 2z + 5 = (z - 1)^2 - 1 + 5 = (z - 1)^2 + 4$

$(z - 1)^2 + 4 = 0$

$(z - 1)^2 = -4$

$z - 1 = \pm\sqrt{-4} = \pm 2i$

$z = 1 \pm 2i$

$z = 1 + 2i,\ z = 1 - 2i$

So the other two roots of the equation are $1 + 2i$ and $1 - 2i$.

The other two roots are found by solving the quadratic equation.

Solve by completing the square. Alternatively, you could use the quadratic formula.

The quadratic equation has complex roots which must be a conjugate pair.

You could write the equation as $(z + 1)[z - (1 + 2i)][z - (1 - 2i)] = 0$

- An equation of the form $az^4 + bz^3 + cz^2 + dz + e = 0$ is called a quartic equation, and has four roots.
- For a quartic equation with real coefficients, either:
  - all four roots are real, or
  - two roots are real and the other two roots form a complex conjugate pair, or
  - two roots form a complex conjugate pair and the other two roots also form a complex conjugate pair.

Watch out A real-valued quartic equation might have repeated real roots or repeated complex roots.

Example 19

Given that $3 + i$ is a root of the quartic equation $2z^4 - 3z^3 - 39z^2 + 120z - 50 = 0$, solve the equation completely.

Another root is $3 - i$.

So $(z - (3 + i))(z - (3 - i))$ is a factor of $2z^4 - 3z^3 - 39z^2 + 120z - 50$

$(z - (3 + i))(z - (3 - i)) = z^2 - z(3 - i) - z(3 + i) + (3 + i)(3 - i)$
$= z^2 - 6z + 10$

So $z^2 - 6z + 10$ is a factor of $2z^4 - 3z^3 - 39z^2 + 120z - 50$.

$(z^2 - 6z + 10)(az^2 + bz + c) = 2z^4 - 3z^3 - 39z^2 + 120z - 50$

Consider $2z^4$:
The only z^4 term in the expansion is $z^2 \times az^2$, so $a = 2$.

$(z^2 - 6z + 10)(2z^2 + bz + c) = 2z^4 - 3z^3 - 39z^2 + 120z - 50$

Consider $-3z^3$:
The z^3 terms in the expansion are $z^2 \times bz$ and $-6z \times 2z^2$,
so $bz^3 - 12z^3 = -3z^3$

$b - 12 = -3$

$b = 9$

so $(z^2 - 6z + 10)(2z^2 + 9z + c) = 2z^4 - 3z^3 - 39z^2 + 120z - 50$

Complex roots occur in conjugate pairs.

If α and β are roots of $f(z) = 0$, then $(z - \alpha)(z - \beta)$ is a factor of $f(z)$.

You can work this out quickly by noting that $[z - (a + bi)][z - (a - bi)] = z^2 - 2az + a^2 + b^2$

Problem-solving

It is possible to **factorise** a polynomial without using a formal **algebraic** method. Here, the polynomial is factorised by 'inspection' (i.e. looking carefully). By considering each term of the quartic separately, it is possible to work out the missing coefficients.

Consider −50:
The only constant term in the expansion is $10 \times c$, so $c = -5$.
$2z^4 - 3z^3 - 39z^2 + 120z - 50 = (z^2 - 6z + 10)(2z^2 + 9z - 5)$

Solving $2z^2 + 9z - 5 = 0$:
$(2z - 1)(z + 5) = 0$
$z = \frac{1}{2}, z = -5$

So the roots of $2z^4 - 3z^3 - 39z^2 + 120z - 50 = 0$ are:
$\frac{1}{2}$, −5, 3 + i and 3 − i

You can check this by considering the z and z^2 terms in the expansion.

Example 20 SKILLS EXECUTIVE FUNCTION

Show that $z^2 + 4$ is a factor of $z^4 - 2z^3 + 21z^2 - 8z + 68$.
Hence solve the equation $z^4 - 2z^3 + 21z^2 - 8z + 68 = 0$.

Using long division:

$$\begin{array}{r} z^2 - 2z + 17 \\ z^2 + 4 \overline{) z^4 - 2z^3 + 21z^2 - 8z + 68} \\ \underline{z^4 \qquad + 4z^2} \qquad\qquad \\ -2z^3 + 17z^2 - 8z \qquad \\ \underline{-2z^3 \qquad\quad - 8z} \qquad \\ 17z^2 \qquad + 68 \\ \underline{17z^2 \qquad + 68} \\ 0 \end{array}$$

There is no remainder and hence $z^2 + 4$ is a factor of $z^4 - 2z^3 + 21z^2 - 8z + 68$.

Alternatively, the quartic can be factorised by inspection:
$z^4 - 2z^3 + 21z^2 - 8z + 68$
$= (z^2 + 4)(az^2 + bz + c)$
$a = 1$, as the leading coefficient is 1.
The only z^3 term is formed by $z^2 \times bz$ so $b = -2$.
The constant term is formed by $4 \times c$, so $4c = 68$, and $c = 17$.

So $z^4 - 2z^3 + 21z^2 - 8z + 68 = (z^2 + 4)(z^2 - 2z + 17) = 0$
Either $z^2 + 4 = 0$ or $z^2 - 2z + 17 = 0$
Solving $z^2 + 4 = 0$:
$z^2 = -4$
$z = \pm 2i$
Solving $z^2 - 2z + 17 = 0$:
$(z - 1)^2 + 16 = 0$
$(z - 1)^2 = -16$
$z - 1 = \pm 4i$
$z = 1 \pm 4i$
So the roots of $z^4 - 2z^3 + 21z^2 - 8z + 68 = 0$ are:
2i, −2i, 1 + 4i and 1 − 4i

Solve by completing the square. Alternatively, you could use the quadratic formula.

Watch out You could use your calculator to solve $z^2 - 2z + 17 = 0$. However, you should still write down the equation you are solving, and both roots.

Exercise 1I

SKILLS EXECUTIVE FUNCTION

E **1** $f(z) = z^3 - 6z^2 + 21z - 26$

a Show that $f(2) = 0$. **(1 mark)**

b Hence solve $f(z) = 0$ completely. **(3 marks)**

E **2** $f(z) = 2z^3 + 5z^2 + 9z - 6$

a Show that $f\left(\frac{1}{2}\right) = 0$. **(1 mark)**

b Hence write $f(z)$ in the form $(2z - 1)(z^2 + bz + c)$, where b and c are real constants to be found. **(2 marks)**

c Use algebra to solve $f(z) = 0$ completely. **(2 marks)**

E/P **3** $g(z) = 2z^3 - 4z^2 - 5z - 3$

Given that $z = 3$ is a root of the equation $g(z) = 0$, solve $g(z) = 0$ completely. **(4 marks)**

E **4** $p(z) = z^3 + 4z^2 - 15z - 68$

Given that $z = -4 + \mathrm{i}$ is a solution to the equation $p(z) = 0$,

a show that $z^2 + 8z + 17$ is a factor of $p(z)$. **(2 marks)**

b Hence solve $p(z) = 0$ completely. **(2 marks)**

E **5** $f(z) = z^3 + 9z^2 + 33z + 25$

Given that $f(z) = (z + 1)(z^2 + az + b)$, where a and b are real constants,

a find the value of a and the value of b **(2 marks)**

b find the three roots of $f(z) = 0$ **(4 marks)**

c find the **sum** of the three roots of $f(z) = 0$. **(1 mark)**

E/P **6** $g(z) = z^3 - 12z^2 + cz + d = 0$, where $c, d \in \mathbb{R}$.

Given that 6 and $3 + \mathrm{i}$ are roots of the equation $g(z) = 0$,

a write down the other complex root of the equation **(1 mark)**

b find the value of c and the value of d. **(4 marks)**

E/P **7** $h(z) = 2z^3 + 3z^2 + 3z + 1$

Given that $2z + 1$ is a factor of $h(z)$, find the three roots of $h(z) = 0$. **(4 marks)**

E/P **8** $f(z) = z^3 - 6z^2 + 28z + k$

Given that $f(2) = 0$,

a find the value of k **(1 mark)**

b find the other two roots of the equation. **(4 marks)**

9 Find the four roots of the equation $z^4 - 16 = 0$.

E **10** $f(z) = z^4 - 12z^3 + 31z^2 + 108z - 360$

a Write $f(z)$ in the form $(z^2 - 9)(z^2 + bz + c)$, where b and c are real constants to be found. **(2 marks)**

b Hence find all the solutions to $f(z) = 0$. **(3 marks)**

P **11** $g(z) = z^4 + 2z^3 - z^2 + 38z + 130$

Given that $g(2 + 3i) = 0$, find all the roots of $g(z) = 0$.

E/P **12** $f(z) = z^4 - 10z^3 + 71z^2 + Qz + 442$, where Q is a real constant.

Given that $z = 2 - 3i$ is a root of the equation $f(z) = 0$,

a show that $z^2 - 6z + 34$ is a factor of $f(z)$ **(4 marks)**

b find the value of Q **(1 mark)**

c solve completely the equation $f(z) = 0$. **(2 marks)**

Challenge

Three of the roots of the equation $z^5 + bz^4 + cz^3 + dz^2 + ez + f = 0$, where $b, c, d, e, f \in \mathbb{R}$, are -2, $2i$ and $1 + i$. Find the values of b, c, d, e and f.

Chapter review 1

1 Given that $z_1 = 8 - 3i$ and $z_2 = -2 + 4i$, find, in the form $a + bi$, where $a, b \in \mathbb{R}$:

a $z_1 + z_2$

b $3z_2$

c $6z_1 - z_2$

E/P **2** The equation $z^2 + bz + 14 = 0$, where $b \in \mathbb{R}$, has no real roots.

Find the range of possible values of b. **(3 marks)**

3 The solutions to the quadratic equation $z^2 - 6z + 12 = 0$ are z_1 and z_2.

Find z_1 and z_2, giving each answer in the form $a \pm i\sqrt{b}$.

E/P **4** By using the binomial expansion, or otherwise, show that $(1 + 2i)^5 = 41 - 38i$. **(3 marks)**

E **5** $f(z) = z^2 - 6z + 10$

Show that $z = 3 + i$ is a solution to $f(z) = 0$. **(2 marks)**

6 You are given the complex numbers $z_1 = 4 + 2i$ and $z_2 = -3 + i$.

Express, in the form $a + bi$, where $a, b \in \mathbb{R}$:

a z_1^* **b** $z_1 z_2$ **c** $\frac{z_1}{z_2}$

7 Write $\frac{(7 - 2i)^2}{1 + i\sqrt{3}}$ in the form $x + iy$, where $x, y \in \mathbb{R}$.

E/P **8** Given that $\frac{4 - 7i}{z} = 3 + i$, find z in the form $a + bi$, where $a, b \in \mathbb{R}$. **(2 marks)**

9 You are given the complex number $z = \frac{1}{2 + \mathrm{i}}$.

Express in the form $a + b\mathrm{i}$, where $a, b \in \mathbb{R}$:

a z^2 **b** $z - \frac{1}{z}$

E/P **10** Given that $z = a + b\mathrm{i}$, show that $\frac{z}{z^*} = \left(\frac{a^2 - b^2}{a^2 + b^2}\right) + \left(\frac{2ab}{a^2 + b^2}\right)\mathrm{i}$ **(4 marks)**

E/P **11** The complex number z is defined by $z = \frac{3 + q\mathrm{i}}{q - 5\mathrm{i}}$, where $q \in \mathbb{R}$.

Given that the real part of z is $\frac{1}{13}$,

a find the possible values of q **(4 marks)**

b write the possible values of z in the form $a + b\mathrm{i}$, where a and b are real constants. **(1 mark)**

E/P **12** Given that $z = x + \mathrm{i}y$, find the value of x and the value of y such that $z + 4\mathrm{i}z^* = -3 + 18\mathrm{i}$, where z^* is the complex conjugate of z. **(5 marks)**

13 $z = 9 + 6\mathrm{i}$, $w = 2 - 3\mathrm{i}$

Express $\frac{z}{w}$ in the form $a + b\mathrm{i}$, where a and b are real constants.

E/P **14** The complex number z is given by $z = \frac{q + 3\mathrm{i}}{4 + q\mathrm{i}}$ where q is an integer.

Express z in the form $a + b\mathrm{i}$ where a and b are rational and are given in terms of q. **(4 marks)**

E **15** $\mathrm{f}(z) = z^2 + 5z + 10$

a Find the roots of the equation $\mathrm{f}(z) = 0$, giving your answers in the form $a \pm \mathrm{i}b$, where a and b are real numbers. **(3 marks)**

b Show these roots on an Argand diagram. **(1 mark)**

E **16** Given that $6 - 2\mathrm{i}$ is one of the roots of a quadratic equation with real coefficients,

a write down the other root of the equation **(1 mark)**

b find the quadratic equation, giving your answer in the form $z^2 + bz + c = 0$ where b and c are real constants. **(2 marks)**

E/P **17** Given that $z = 4 - k\mathrm{i}$ is a root of the equation $z^2 - 2mz + 52 = 0$, where k and m are positive real constants, find the value of k and the value of m. **(4 marks)**

E/P **18** $\mathrm{h}(z) = z^3 - 11z + 20$

Given that $2 + \mathrm{i}$ is a root of the equation $\mathrm{h}(z) = 0$, solve $\mathrm{h}(z) = 0$ completely. **(4 marks)**

E/P **19** $\mathrm{f}(z) = z^3 + 6z + 20$

Given that $\mathrm{f}(1 + 3\mathrm{i}) = 0$, solve $\mathrm{f}(z) = 0$ completely. **(4 marks)**

E/P **20** $\mathrm{f}(z) = z^3 + 3z^2 + kz + 48$, $k \in \mathbb{R}$

Given that $\mathrm{f}(4\mathrm{i}) = 0$,

a find the value of k **(2 marks)**

b find the other two roots of the equation. **(3 marks)**

E/P **21** $f(z) = z^3 + z^2 + 3z - 5$

Given that $f(-1 + 2i) = 0$,

a find all the solutions to the equation $f(z) = 0$ **(4 marks)**

b show all the roots of $f(z) = 0$ on a single Argand diagram **(2 marks)**

c prove that these three points are the vertices of a right-angled triangle. **(2 marks)**

E **22** $f(z) = z^4 - z^3 - 16z^2 - 74z - 60$

a Write $f(z)$ in the form $(z^2 - 5z - 6)(z^2 + bz + c)$, where b and c are real constants to be found. **(2 marks)**

b Hence find all the solutions to $f(z) = 0$. **(3 marks)**

E/P **23** $g(z) = z^4 - 6z^3 + 19z^2 - 36z + 78$

Given that $g(3 - 2i) = 0$, find all the roots of $g(z) = 0$. **(4 marks)**

E/P **24** $f(z) = z^4 - 2z^3 - 5z^2 + pz + 24$

Given that $f(4) = 0$,

a find the value of p **(1 mark)**

b solve completely the equation $f(z) = 0$. **(5 marks)**

E/P **25** $f(z) = z^4 - z^3 + 13z^2 - 47z + 34$

Given that $z = -1 + 4i$ is a solution to the equation,

a find all the solutions to the equation $f(z) = 0$ **(4 marks)**

b show all the roots on a single Argand diagram. **(2 marks)**

E **26** The real and imaginary parts of the complex number $z = x + iy$ satisfy the equation $(4 - 3i)x - (1 + 6i)y - 3 = 0$.

a Find the value of x and the value of y. **(3 marks)**

b Show z on an Argand diagram. **(1 mark)**

Find the values of:

c $|z|$ **(2 marks)**

d $\arg z$ **(2 marks)**

E **27** A complex number z is given by $z = a + 4i$ where a is a non-zero real number.

a Find $z^2 + 2z$ in the form $x + iy$, where x and y are real expressions in terms of a. **(4 marks)**

Given that $z^2 + 2z$ is real,

b find the value of a. **(1 mark)**

Using this value for a,

c find the values of the modulus and argument of z, giving the argument in radians and giving your answers correct to 3 significant figures. **(3 marks)**

d Show the complex numbers z, z^2 and $z^2 + 2z$ on a single Argand diagram. **(3 marks)**

E **28** The complex number z is defined by $z = \frac{3 + 5i}{2 - i}$.

Find:

a $|z|$ **(4 marks)**

b $\arg z$ **(2 marks)**

E **29** You are given the complex number $z = 1 + 2i$.

a Show that $|z^2 - z| = 2\sqrt{5}$. **(4 marks)**

b Find $\arg(z^2 - z)$, giving your answer in radians to 2 decimal places. **(2 marks)**

c Show z and $z^2 - z$ on a single Argand diagram. **(2 marks)**

E **30** You are given the complex number $z = \frac{1}{2 + i}$.

a Express in the form $a + bi$, where $a, b \in \mathbb{R}$:

i z^2 **ii** $z - \frac{1}{z}$ **(4 marks)**

b Find $|z^2|$. **(2 marks)**

c Find $\arg\left(z - \frac{1}{z}\right)$, giving your answer in radians to 2 decimal places. **(2 marks)**

E/P **31** Given is the complex number $z = \frac{a + 3i}{2 + ai}$, where $a \in \mathbb{R}$.

a Given that $a = 4$, find $|z|$. **(3 marks)**

b Show that there is only one value of a for which $\arg z = \frac{\pi}{4}$, and find this value. **(3 marks)**

E **32** Express $4 - 4i$ in the form $r(\cos\theta + i\sin\theta)$, where $r > 0$, $-\pi < \theta \leqslant \pi$, giving r and θ as exact values. **(3 marks)**

Challenge

SKILLS INNOVATION

a Explain why a cubic equation with real coefficients cannot have a repeated non-real root.

b By means of an example, show that a quartic equation with real coefficients can have a repeated non-real root.

Summary of key points

1 $i = \sqrt{-1}$ and $i^2 = -1$

2 An **imaginary number** is a number of the form bi, where $b \in \mathbb{R}$.

3 A **complex number** is written in the form $a + bi$, where $a, b \in \mathbb{R}$.

4 Complex numbers can be added or subtracted by adding or subtracting their real parts and adding or subtracting their imaginary parts.

5 You can multiply a real number by a complex number by multiplying out the brackets in the usual way.

6 If $b^2 - 4ac < 0$ then the quadratic equation $ax^2 + bx + c = 0$ has two distinct complex roots, neither of which is real.

7 For any complex number $z = a + b\mathrm{i}$, the **complex conjugate** of the number is defined as $z^* = a - b\mathrm{i}$.

8 For real numbers a, b and c, if the roots of the quadratic equation $az^2 + bz + c = 0$ are non-real complex numbers, then they occur as a conjugate pair.

9 If the roots of a quadratic equation are α and β, then you can write the equation as

$(z - \alpha)(z - \beta) = 0$ or $z^2 - (\alpha + \beta)z + \alpha\beta = 0$.

10 If $\mathrm{f}(z)$ is a polynomial with real coefficients, and z_1 is a root of $\mathrm{f}(z) = 0$, then $z_1{}^*$ is also a root of $\mathrm{f}(z) = 0$.

11 An equation of the form $az^3 + bz^2 + cz + d = 0$ is called a cubic equation, and has three roots.

For a cubic equation with real coefficients, either:

- all three roots are real, or
- one root is real and the other two roots form a complex conjugate pair.

12 An equation of the form $az^4 + bz^3 + cz^2 + dz + e = 0$ is called a quartic equation, and has four roots.

For a quartic equation with real coefficients, either:

- all four roots are real, or
- two roots are real and the other two roots form a complex conjugate pair, or
- two roots form a complex conjugate pair and the other two roots also form a complex conjugate pair.

13 You can represent complex numbers on an **Argand diagram**. The x-axis on an Argand diagram is called the **real axis** and the y-axis is called the **imaginary axis**. The complex number $z = x + \mathrm{i}y$ is represented on the diagram by the point $P(x, y)$, where x and y are Cartesian coordinates.

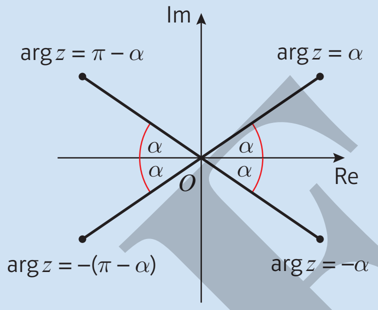
14 The complex number $z = x + \mathrm{i}y$ can be represented as the vector $\begin{pmatrix} x \\ y \end{pmatrix}$ on an Argand diagram.

15 The **modulus** of a complex number, $|z|$, is the distance from the origin to that number on an Argand diagram. For a complex number $z = x + \mathrm{i}y$, the modulus is given by $|z| = \sqrt{x^2 + y^2}$.

16 The **argument** of a complex number, $\arg z$, is the angle between the positive real axis and the line joining that number to the origin on an Argand diagram. For a complex number $z = x + \mathrm{i}y$, the argument, θ, satisfies $\tan\theta = \frac{y}{x}$.

17 Let α be the positive acute angle made with the real axis by the line joining the origin and z.

- If z lies in the first quadrant, then $\arg z = \alpha$.
- If z lies in the second quadrant, then $\arg z = \pi - \alpha$.
- If z lies in the third quadrant, then $\arg z = -(\pi - \alpha)$.
- If z lies in the fourth quadrant, then $\arg z = -\alpha$.



18 For a complex number z with $|z| = r$ and $\arg z = \theta$, the modulus–argument form of z is $z = r(\cos\theta + \mathrm{i}\sin\theta)$.

19 For any two complex numbers z_1 and z_2, $|z_1 z_2| = |z_1||z_2|$