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The following three themes have been fully integrated throughout the Pearson Edexcel International Advanced Level in Mathematics series, so they can be applied alongside your learning.

1. Mathematical argument, language and proof
   - Rigorous and consistent approach throughout
   - Notation boxes explain key mathematical language and symbols

2. Mathematical problem-solving
   - Hundreds of problem-solving questions, fully integrated into the main exercises
   - Problem-solving boxes provide tips and strategies
   - Challenge questions provide extra stretch

3. Transferable skills
   - Transferable skills are embedded throughout this book, in the exercises and in some examples
   - These skills are signposted to show students which skills they are using and developing

Finding your way around the book

Each chapter starts with a list of Learning objectives

The Prior knowledge check helps make sure you are ready to start the chapter

Glossary terms will be identified by bold blue text on their first appearance

Each chapter is mapped to the specification content for easy reference

The real world applications of the maths you are about to learn are highlighted at the start of the chapter
Exercise questions are carefully graded to increase in difficulty and gradually bring you up to exam standard.

Transferable skills are signposted where they naturally occur in the exercises and examples.

Exercises are packed with exam-style questions to ensure you are ready for the exams.

Exam-style questions are flagged with 📖.

Problem-solving questions are flagged with 🟢.

Step-by-step worked examples focus on the key types of questions you’ll need to tackle.

Each section begins with an explanation and key learning points.

Challenge boxes give you a chance to tackle some more difficult questions.

Problem-solving boxes provide hints, tips and strategies, and Watch out boxes highlight areas where students often lose marks in their exams.

Each chapter ends with a Chapter review and a Summary of key points.

After every few chapters, a Review exercise helps you consolidate your learning with lots of exam-style questions.

A full practice paper at the back of the book helps you prepare for the real thing.
Qualification and content overview

Pure Mathematics 4 (P4) is a compulsory unit in the following qualifications:
International Advanced Level in Mathematics
International Advanced Level in Pure Mathematics

Assessment overview

The following table gives an overview of the assessment for this unit.
We recommend that you study this information closely to help ensure that you are fully prepared for this course and know exactly what to expect in the assessment.

<table>
<thead>
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<th>Unit</th>
<th>Percentage</th>
<th>Mark</th>
<th>Time</th>
<th>Availability</th>
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<tr>
<td>P4: Pure Mathematics 4</td>
<td>16(\frac{2}{5}) % of IAL</td>
<td>75</td>
<td>1 hour 30 mins</td>
<td>January, June and October First assessment June 2020</td>
</tr>
<tr>
<td>Paper code WMA14/01</td>
<td>5% of IAL</td>
<td>75</td>
<td>1 hour 30 mins</td>
<td>January, June and October First assessment June 2020</td>
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IAL: International Advanced A Level.

Assessment objectives and weightings

<table>
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<tr>
<th>AO1</th>
<th>Recall, select and use their knowledge of mathematical facts, concepts and techniques in a variety of contexts.</th>
<th>Minimum weighting in IAS and IAL</th>
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<td>AO2</td>
<td>Construct rigorous mathematical arguments and proofs through use of precise statements, logical deduction and inference and by the manipulation of mathematical expressions, including the construction of extended arguments for handling substantial problems presented in unstructured form.</td>
<td>30%</td>
</tr>
<tr>
<td>AO3</td>
<td>Recall, select and use their knowledge of standard mathematical models to represent situations in the real world; recognise and understand given representations involving standard models; present and interpret results from such models in terms of the original situation, including discussion of the assumptions made and refinement of such models.</td>
<td>10%</td>
</tr>
<tr>
<td>AO4</td>
<td>Comprehend translations of common realistic contexts into mathematics; use the results of calculations to make predictions, or comment on the context; and, where appropriate, read critically and comprehend longer mathematical arguments or examples of applications.</td>
<td>5%</td>
</tr>
<tr>
<td>AO5</td>
<td>Use contemporary calculator technology and other permitted resources (such as formulae booklets or statistical tables) accurately and efficiently; understand when not to use such technology, and its limitations. Give answers to appropriate accuracy.</td>
<td>5%</td>
</tr>
<tr>
<td>P4</td>
<td>Assessment objective</td>
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<td></td>
<td>AO1</td>
<td>AO2</td>
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<tr>
<td>Marks out of 75</td>
<td>25–30</td>
<td>25–30</td>
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<td>%</td>
<td>33(\frac{1}{3})–40</td>
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**Calculators**

Students may use a calculator in assessments for these qualifications. Centres are responsible for making sure that calculators used by their students meet the requirements given in the table below. Students are expected to have available a calculator with at least the following keys: +, −, ×, ÷, π, \(x^2\), \(\sqrt{x}\), \(0^x\), \(\ln x\), \(e^x\), \(x!\), sine, cosine and tangent and their inverses in degrees and decimals of a degree, and in radians; memory.

**Prohibitions**

Calculators with any of the following facilities are prohibited in all examinations:
- databanks
- retrieval of text or formulae
- built-in symbolic algebra manipulations
- symbolic differentiation and/or integration
- language translators
- communication with other machines or the internet
**Extra online content**

Whenever you see an *Online* box, it means that there is extra online content available to support you.

**SolutionBank**

SolutionBank provides worked solutions for questions in the book. Download the solutions as a PDF or quickly find the solution you need online.

**Use of technology**

Explore topics in more detail, visualise problems and consolidate your understanding. Use pre-made GeoGebra activities or Casio resources for a graphic calculator.

**GeoGebra**

GeoGebra-powered interactives

Interact with the maths you are learning using GeoGebra's easy-to-use tools

**CASIO**

Graphic calculator interactives

Explore the maths you are learning and gain confidence in using a graphic calculator

**Calculator tutorials**

Our helpful video tutorials will guide you through how to use your calculator in the exams. They cover both Casio's scientific and colour graphic calculators.

**Online**

Work out each coefficient quickly using the \(^nC_r\) and power functions on your calculator.

Step-by-step guide with audio instructions on exactly which buttons to press and what should appear on your calculator's screen.
4 BINOMIAL EXPANSION

Learning objectives

After completing this chapter you should be able to:

● Expand \((1 + x)^n\) for any rational constant \(n\) and determine the range of values of \(x\) for which the expansion is valid → pages 31–34

● Expand \((a + bx)^n\) for any rational constant \(n\) and determine the range of values of \(x\) for which the expansion is valid → pages 36–38

● Use partial fractions to expand fractional expressions → pages 40–41

Prior knowledge check

1. Expand the following expressions in ascending powers of \(x\) up to and including the term in \(x^3\):
   a. \((1 + 5x)^7\)  
   b. \((5 - 2x)^{10}\)  
   c. \((1 - x)(2 + x)^6\)  
   ← Pure 2 Section 4.3

2. Write each of the following using partial fractions:
   a. \(-14x + 7 \over (1 + 2x)(1 - 5x)\)  
   b. \(24x - 1 \over (1 + 2x)^2\)  
   c. \(24x^2 + 48x + 24 \over (1 + x)(4 - 3x)^2\)  
   ← Pure 4 Sections 2.1, 2.2

The binomial expansion can be used to find polynomial approximations for expressions involving fractional and negative indices. Medical physicists use these approximations to analyse magnetic fields in an MRI scanner.
4.1 Expanding \((1 + x)^n\)

If \(n\) is a natural number you can find the binomial expansion for \((a + bx)^n\) using the formula:

\[
(a + b)^n = a^n + \binom{n}{1} a^{n-1}b + \binom{n}{2} a^{n-2}b^2 + \ldots + \binom{n}{r} a^{n-r}b^r + \ldots + b^n, \quad (n \in \mathbb{N})
\]

If \(n\) is a fraction or a negative number you need to use a different version of the binomial expansion.

- This form of the binomial expansion can be applied to negative or fractional values of \(n\) to obtain an infinite series.

\[
(1 + x)^n = 1 + nx + \frac{n(n - 1)}{2!} x^2 + \frac{n(n - 1)(n - 2)}{3!} x^3 + \ldots + \binom{n}{r} \frac{n(n - 1)\ldots(n - r + 1)}{r!} x^r + \ldots
\]

- The expansion is valid when \(|x| < 1, n \in \mathbb{R}\)

When \(n\) is not a natural number, none of the factors in the expression \(n(n - 1) \ldots (n - r + 1)\) are equal to zero. This means that this version of the binomial expansion produces an infinite number of terms.

Example 1

Find the first four terms in the binomial expansion of \(\frac{1}{1 + x}\)

\[
\frac{1}{1 + x} = (1 + x)^{-1} = 1 + (-1)x + \frac{(-1)(-2)}{2} x^2 + \frac{(-1)(-2)(-3)}{3!} x^3 + \ldots
\]

\[
= 1 - x + x^2 - x^3 + \ldots
\]

Write in index form.

Replace \(n\) by \(-1\) in the expansion.

As \(n\) is not a positive integer, no coefficient will ever be equal to zero. Therefore, the expansion is infinite.

For the series to be convergent, \(|x| < 1\)

- The expansion of \((1 + bx)^n\), where \(n\) is negative or a fraction, is valid for \(|bx| < 1\), or \(|x| < \frac{1}{|b|}\).
Find the binomial expansions of

a $\left(1 - \frac{x}{3}\right)^{\frac{1}{3}}$

b $\frac{1}{(1 + 4x)^2}$

up to and including the term in $x^3$. State the range of values of $x$ for which each expansion is valid.

**Example 2**

**a** $\left(1 - \frac{x}{3}\right)^{\frac{1}{3}}$

$= 1 + \left(\frac{1}{3}\right)(-x) + \frac{1}{3!}\left(\frac{1}{3} - 1\right)(-x)^2 + \frac{1}{3!}\left(\frac{1}{3} - 1\right)^2(-x)^3 + ...$

$= 1 + \left(\frac{1}{3}\right)(-x) + \frac{1}{6}\left(\frac{1}{3} - 1\right)(-x)^2 + \frac{1}{6}\left(\frac{1}{3} - 1\right)^2(-x)^3 + ...

= 1 - \frac{1}{3}x - \frac{1}{9}x^2 - \frac{5}{81}x^3 + ...

Expansion is valid as long as $|x| < 1$.

Replace $n$ by $\frac{1}{3}$ and $x$ by $(-x)$.

Simplify brackets.

**Watch out** Be careful working out whether each term should be positive or negative:
- even number of negative signs means term is positive
- odd number of negative signs means term is negative

The $x^3$ term here has 5 negative signs in total, so it is negative.

Simplify coefficients.

Terms in expansion are $(-x), (-x)^2, (-x)^3$

**b** $\frac{1}{(1 + 4x)^2}$ = $(1 + 4x)^{-2}$

$= 1 + (-2)(4x) + \frac{(-2)(-2 - 1)(4x)^2}{2!} + \frac{(-2)(-2 - 1)(-2 - 2)(4x)^3}{3!} + ...$

$= 1 + (-2)(4x) + \frac{(-2)(-3)(16)x^2}{2} + \frac{(-2)(-3)(-4)(64)x^3}{6} + ...

= 1 - 8x + 48x^2 - 256x^3 + ...

Expansion is valid as long as $|4x| < 1$.

⇒ $|x| < \frac{1}{4}$

Replace $n$ by $-2$, $x$ by $4x$.

Simplify brackets.

Simplify coefficients.

Terms in expansion are $(4x), (4x)^2, (4x)^3$

**Online** Use technology to explore why the expansions are only valid for certain values of $x$. 

Simplify brackets.

Simplify coefficients.

Terms in expansion are $(4x), (4x)^2, (4x)^3$
a Find the expansion of \( \sqrt{1 - 2x} \) up to and including the term in \( x^3 \).

b By substituting in \( x = 0.01 \), find a decimal approximation to \( \sqrt{2} \).

\[
\begin{align*}
\text{a } \sqrt{1 - 2x} &= (1 - 2x)^{\frac{1}{2}} \\
&= 1 + \left( \frac{1}{2} \right)(-2x) \\
&\quad + \frac{\left( \frac{1}{2} \right)(1 - 1)(-2x)^2}{2!} \\
&\quad + \frac{\left( \frac{1}{2} \right)(1 - 1)(1 - 2)(-2x)^3}{3!} + \ldots \\
&= 1 + \left( \frac{1}{2} \right)(-2x) \\
&\quad + \frac{\left( \frac{1}{2} \right)(-1)(4x^2)}{2!} \\
&\quad + \frac{\left( \frac{1}{2} \right)(-1)(-3)(8x^3)}{6} + \ldots \\
&= 1 - x - \frac{x^2}{2} - \frac{x^3}{2} + \ldots \\
\text{Expansion is valid if } |\ -2x\ | < 1 \\
\Rightarrow |x| < \frac{1}{2}
\end{align*}
\]

\[
\begin{align*}
\text{b } \sqrt{1 - 2 \times 0.01} &\approx 1 - 0.01 - \frac{0.01^2}{2} \\
&\quad - \frac{0.01^3}{6} \\
&\quad + \ldots \\
&\approx 0.9899495
\end{align*}
\]

This approximation is accurate to 7 decimal places.

\[
\begin{align*}
\sqrt{\frac{49 \times 2}{100}} &\approx 0.9899495 \\
\sqrt{\frac{7}{10}} &\approx 0.9899495 \\
\sqrt{2} &\approx 0.9899495 \times 10 \\
&\approx 1.414213571
\end{align*}
\]
Example 4  CRITICAL THINKING

\[ f(x) = \frac{2 + x}{\sqrt{1 + 5x}} \]

a Find the \( x^2 \) term in the series expansion of \( f(x) \).

b State the range of values of \( x \) for which the expansion is valid.

\[
\begin{align*}
(1 + 5x)^{-\frac{1}{2}} &= 1 + \left(-\frac{\frac{1}{2}}{2!}\right)(5x) \\
&+ \left(-\frac{\frac{1}{2}}{2!}\right)(\frac{-3}{2})(5x)^2 \\
&+ \left(-\frac{\frac{1}{2}}{2!}\right)(\frac{-3}{2})(\frac{-5}{2})(5x)^3 + \ldots \\
&= 1 - \frac{5}{2}x + \frac{75}{8}x^2 - \frac{625}{16}x^3 + \ldots \\
f(x) &= (2 + x) \left(1 - \frac{5}{2}x + \frac{75}{8}x^2 - \frac{625}{16}x^3 + \ldots\right) \\
2 \times \frac{75}{8} + 1 \times -\frac{5}{2} &= \frac{65}{4} \\
x^2 \text{ term is } \frac{65}{4}x^2 \\
\end{align*}
\]

b The expansion is valid if \(|5x| < 1\)
\[ |x| < \frac{1}{5} \]

Example 5  PROBLEM-SOLVING

In the expansion of \((1 + kx)^{-4}\) the coefficient of \( x \) is 20.

a Find the value of \( k \).

b Find the corresponding coefficient of the \( x^2 \) term.

\[
\begin{align*}
(1 + kx)^{-4} &= 1 + (-4)(kx) + \frac{(-4)(-5)(kx)^2}{2!} + \ldots \\
&= 1 - 4kx + 10k^2x^2 + \ldots \\
-4k &= 20 \\
k &= -5 \\
\end{align*}
\]

b Coefficient of \( x^2 \) = \( 10k^2 \) = \( 10(-5)^2 \) = 250
1 For each of the following:
   i find the binomial expansion up to and including the $x^3$ term
   ii state the range of values of $x$ for which the expansion is valid.
   a $(1 + x)^{-4}$  
   b $(1 + x)^{-6}$  
   c $(1 + x)^{\frac{1}{2}}$  
   d $(1 + x)^{\frac{3}{2}}$  
   e $(1 + x)^{-\frac{1}{4}}$  
   f $(1 + x)^{-\frac{3}{4}}$

2 For each of the following:
   i find the binomial expansion up to and including the $x^3$ term
   ii state the range of values of $x$ for which the expansion is valid.
   a $(1 + 3x)^{-3}$  
   b $(1 + \frac{1}{x})^{-5}$  
   c $(1 + 2x)^{\frac{1}{2}}$  
   d $(1 - 5x)^{\frac{1}{2}}$  
   e $(1 + 6x)^{-\frac{1}{3}}$  
   f $(1 - \frac{3}{4}x)^{\frac{1}{3}}$

3 For each of the following:
   i find the binomial expansion up to and including the $x^3$ term
   ii state the range of values of $x$ for which the expansion is valid.
   a $\frac{1}{(1 + x)^2}$  
   b $\frac{1}{(1 + 3x)^4}$  
   c $\sqrt{1 - x}$  
   d $\sqrt[3]{1 - 3x}$  
   e $\frac{1}{\sqrt{1 + \frac{1}{2}x}}$  
   f $\frac{\sqrt[3]{1 - 2x}}{1 - 2x}$

4 \( f(x) = \frac{1 + x}{1 - 2x} \)
   a Show that the series expansion of $f(x)$ up to and including the $x^3$ term is $1 + 3x + 6x^2 + 12x^3$ \( (4 \text{ marks}) \)
   b State the range of values of $x$ for which the expansion is valid. \( (1 \text{ mark}) \)

5 \( f(x) = \sqrt{1 + 3x}, \quad -\frac{1}{3} < x < \frac{1}{3} \)
   a Find the series expansion of $f(x)$, in ascending powers of $x$, up to and including the $x^3$ term. Simplify each term. \( (4 \text{ marks}) \)
   b Show that, when $x = \frac{1}{100}$, the exact value of $f(x)$ is $\sqrt{\frac{103}{10}}$. \( (2 \text{ marks}) \)
   c Find the percentage error made in using the series expansion in part a to estimate the value of $f(0.01)$. Give your answer to 2 significant figures. \( (3 \text{ marks}) \)

6 In the expansion of $(1 + ax)^{-\frac{1}{2}}$ the coefficient of $x^2$ is 24.
   a Find the possible values of $a$.
   b Find the corresponding coefficient of the $x^3$ term.
Show that if \( x \) is small, the expression \( \sqrt{\frac{1 + x}{1 - x}} \) is approximated by \( 1 + x + \frac{1}{2}x^2 \).

\[ h(x) = \frac{6}{1 + 5x} - \frac{4}{1 - 3x} \]

a. Find the series expansion of \( h(x) \), in ascending powers of \( x \), up to and including the \( x^2 \) term. Simplify each term. (6 marks)
b. Find the percentage error made in using the series expansion in part a to estimate the value of \( h(0.01) \). Give your answer to 2 significant figures. (3 marks)
c. Explain why it is not valid to use the expansion to find \( h(0.5) \). (1 mark)

Find the binomial expansion of \( (1 - 3x)^{\frac{1}{2}} \) in ascending powers of \( x \) up to and including the \( x^3 \) term, simplifying each term. (4 marks)

b. Show that, when \( x = \frac{9}{100} \), the exact value of \( (1 - 3x)^{\frac{1}{2}} \) is \( \frac{\sqrt{97}}{10} \). (2 marks)
c. Substitute \( x = \frac{1}{100} \) into the binomial expansion in part a and hence obtain an approximation to \( \sqrt{97} \). Give your answer to 5 decimal places. (3 marks)

**Challenge**

\( h(x) = \left( 1 + \frac{1}{x} \right)^{-\frac{3}{2}}, \quad |x| > 1 \)

a. Find the binomial expansion of \( h(x) \) in ascending powers of \( x \) up to and including the \( x^3 \) term, simplifying each term.

b. Show that, when \( x = 9 \), the exact value of \( h(x) \) is \( \frac{3\sqrt{10}}{10} \).

c. Use the expansion in part a to find an approximate value of \( \sqrt{10} \). Write your answer to 2 decimal places.

4.2 Expanding \((a + bx)^n\)

The binomial expansion of \((1 + x)^n\) can be used to expand \((a + bx)^n\) for any constants \( a \) and \( b \).

You need to take a factor of \( a^n \) out of the expression:

\[ (a + bx)^n = \left( a \left( 1 + \frac{b}{a}x \right) \right)^n = a^n \left( 1 + \frac{b}{a}x \right)^n \]

Make sure you multiply \( a^n \) by every term in the expansion of \( \left( 1 + \frac{b}{a}x \right)^n \).
The expansion of \((a + bx)^n\), where \(n\) is negative or a fraction, is valid for \(\left|\frac{b}{a}\right| < 1\) or \(|x| < \left|\frac{a}{b}\right|\).

Example 6

Find the first four terms in the binomial expansion of \(a \sqrt[4]{4 + x} \quad \text{and} \quad \frac{1}{(2 + 3x)^2}\)

State the range of values of \(x\) for which each of these expansions is valid.

\[
a \sqrt[4]{4 + x} = (4 + x)^{\frac{1}{4}}
\]

\[
= \left(4\left(1 + \frac{x}{4}\right)\right)^{\frac{1}{4}}
\]

\[
= 4^{\frac{1}{4}}\left(1 + \frac{x}{4}\right)^{\frac{1}{4}}
\]

\[
= 2\left(1 + \frac{x}{4}\right)^{\frac{1}{4}}
\]

\[
= 2\left(1 + \frac{1}{2}\left(x\right) + \frac{\frac{1}{2}\left(1 - 1\right)}{2!}\frac{x^2}{16} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}\frac{x^3}{64} + \ldots\right)
\]

\[
= 2\left(1 + \frac{x^2}{8} + \frac{x^3}{128} + \ldots\right)
\]

\[
= 2 + \frac{x}{4} - \frac{x^2}{64} + \frac{x^3}{512} + \ldots
\]

Expansion is valid if \(\left|\frac{x}{4}\right| < 1\)

\[
\Rightarrow |x| < 4
\]

Write in index form.

Take out a factor of \(4^{\frac{1}{4}}\)

Write \(4^{\frac{1}{4}}\) as 2.

Expand \(1 + \frac{x^2}{8} + \ldots\) using the binomial expansion with \(n = \frac{1}{2}\) and \(x = \frac{x}{4}\)

Simplify coefficients.

Multiply every term in the expansion by 2.

The expansion is infinite, and converges when \(|\frac{x}{4}| < 1\), or \(|x| < 4\).
**Exercise 4B**

**SKILLS ANALYSIS**

**Exercise 4B**

**ANALYSIS**

1. For each of the following:
   - find the binomial expansion up to and including the $x^3$ term
   - state the range of values of $x$ for which the expansion is valid.

   a. $\sqrt{4 + 2x}$
   b. $\frac{1}{2 + x}$
   c. $\frac{1}{(4 - x)^2}$
   d. $\sqrt{9 + x}$
   e. $\frac{1}{\sqrt{2 + x}}$
   f. $\frac{5}{3 + 2x}$
   g. $\frac{1 + x}{2 + x}$
   h. $\sqrt{\frac{2 + x}{1 - x}}$

**Hint**

- Write part g as $1 - \frac{1}{x + 2}$

---

Write in index form.

Take out a factor of $2^{-2}$

Write $2^{-2} = \frac{1}{2^2} = \frac{1}{4}$

Expand $\left(1 + \frac{3x}{2}\right)^2$ using the binomial expansion with $n = -2$ and $x = \frac{3x}{2}$

Simplify coefficients.

Multiply every term by $\frac{1}{4}$

The expansion is infinite, and converges when $|\frac{3x}{2}| < 1$, $|x| < \frac{2}{3}$
2. \( f(x) = (5 + 4x)^{-2}, \ |x| < \frac{5}{4} \)

Find the binomial expansion of \( f(x) \) in ascending powers of \( x \), up to and including the term in \( x^3 \). Give each coefficient as a simplified fraction. \( (5 \text{ marks}) \)

3. \( m(x) = \sqrt{4 - x}, \ |x| < 4 \)

a. Find the series expansion of \( m(x) \), in ascending powers of \( x \), up to and including the \( x^2 \) term. Simplify each term. \( (4 \text{ marks}) \)

b. Show that, when \( x = \frac{1}{9} \), the exact value of \( m(x) \) is \( \frac{\sqrt{35}}{3} \). \( (2 \text{ marks}) \)

c. Use your answer to part a to find an approximate value for \( \sqrt{35} \), and calculate the percentage error in your approximation. \( (4 \text{ marks}) \)

4. The first three terms in the binomial expansion of \( \frac{1}{a + bx} \) are \( 3 + \frac{1}{3}x + \frac{1}{18}x^2 + \ldots \)

a. Find the values of the constants \( a \) and \( b \).

b. Find the coefficient of the \( x^3 \) term in the expansion. \( (5 \text{ marks}) \)

5. \( f(x) = \frac{3 + 2x - x^2}{4 - x} \)

Prove that if \( x \) is sufficiently small, \( f(x) \) may be approximated by \( \frac{3}{4} + \frac{11}{16}x - \frac{5}{64}x^2 \). \( (4 \text{ marks}) \)

6. a. Expand \( \frac{1}{\sqrt{5 + 2x}} \), where \( |x| < \frac{5}{2} \), in ascending powers of \( x \) up to and including the term in \( x^3 \), giving each coefficient in simplified surd form. \( (5 \text{ marks}) \)

b. Hence or otherwise, find the first 3 terms in the expansion of \( \frac{2x - 1}{\sqrt{5 + 2x}} \) as a series in ascending powers of \( x \). \( (4 \text{ marks}) \)

7. a. Use the binomial theorem to expand \( (16 - 3x)^{\frac{1}{2}}, |x| < \frac{16}{3} \) in ascending powers of \( x \), up to and including the term in \( x^2 \), giving each term as a simplified fraction. \( (4 \text{ marks}) \)

b. Use your expansion, with a suitable value of \( x \), to obtain an approximation to \( 4\sqrt{15.7} \). Give your answer to 3 decimal places. \( (2 \text{ marks}) \)

8. \( g(x) = -\frac{3}{4 - 2x} - \frac{2}{3 + 5x}, \ |x| < \frac{1}{2} \)

a. Show that the first three terms in the series expansion of \( g(x) \) can be written as \( \frac{1}{12} + \frac{107}{72}x - \frac{719}{432}x^2 \). \( (5 \text{ marks}) \)

b. Find the exact value of \( g(0.01) \). Round your answer to 7 decimal places. \( (2 \text{ marks}) \)

c. Find the percentage error made in using the series expansion in part a to estimate the value of \( g(0.01) \). Give your answer to 2 significant figures. \( (3 \text{ marks}) \)
4.3 Using partial fractions

Partial fractions can be used to simplify the expansions of more difficult expressions.

### Example 7

**a** Express \( \frac{4 - 5x}{(1 + x)(2 - x)} \) as partial fractions.

**b** Hence show that the cubic approximation of \( \frac{4 - 5x}{(1 + x)(2 - x)} \) is \( 2 - \frac{7x}{2} + \frac{11x^2 - 25x^3}{8} \).

**c** State the range of values of \( x \) for which the expansion is valid.

#### Solution

**a**

\[
\frac{4 - 5x}{(1 + x)(2 - x)} \equiv \frac{A}{1 + x} + \frac{B}{2 - x}
\]

\[
\equiv \frac{A(2 - x) + B(1 + x)}{(1 + x)(2 - x)}
\]

\[
4 - 5x \equiv A(2 - x) + B(1 + x)
\]

Substitute \( x = 2 \):

\[
4 - 10 = A \times 0 + B \times 3
\]

\[
-6 = 3B
\]

\[
B = -2
\]

Substitute \( x = -1 \):

\[
4 + 5 = A \times 3 + B \times 0
\]

\[
9 = 3A
\]

\[
A = 3
\]

\[
\frac{4 - 5x}{(1 + x)(2 - x)} = \frac{3}{1 + x} - \frac{2}{2 - x}
\]

**b**

\[
\frac{4 - 5x}{(1 + x)(2 - x)} = \frac{3}{1 + x} - \frac{2}{2 - x}
\]

\[
= 3(1 + x)^{-1} - 2(2 - x)^{-1}
\]

The expansion of \( 3(1 + x)^{-1} \):

\[
= 3 \left(1 + (-1)x + \frac{(-1)(-2)x^2}{2!} + \; \frac{(-1)(-2)(-3)x^3}{3!} + \; \ldots\right)
\]

\[
= 3(1 - x + x^2 - x^3 + \ldots)
\]

\[
= 3 - 3x + 3x^2 - 3x^3 + \ldots
\]

**Problem-solving**

Use headings to keep track of your working. This will help you stay organised and check your answers.

Expand \( 3(1 + x)^{-1} \) using the binomial expansion with \( n = -1 \).
The expansion of $2(2 - x)^{-1}$

$= 2 \left( 1 - \frac{x}{2} \right)^{-1}$

$= 2 \times 2^{-1} \left( 1 - \frac{x}{2} \right)^{-1}$

$= 1 \times \left( 1 + \frac{(-1)(-2)(-x)}{2!} + \frac{(-1)(-2)(-3)(-x)^3}{3!} + \ldots \right)$

$= 1 \times \left( 1 + \frac{x^2}{2} + \frac{x^3}{4} + \ldots \right)$

$= 1 + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{8} + \ldots$

Hence $\frac{4 - 5x}{(1 + x)(2 - x)}$

$= 3(1 + x)^{-1} - 2(2 - x)^{-1}$

$= \left( 3 - 3x + 3x^2 - 3x^3 \right)$

$= \left( 1 + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{8} \right)$

$= 2 - \frac{7}{2}x + \frac{11}{4}x^2 - \frac{25}{8}x^3$

$\frac{3}{1 + x}$ is valid if $|x| < 1$

$\frac{2}{2 - x}$ is valid if $\left| \frac{x}{2} \right| < 1 \Rightarrow |x| < 2$

The expansion is valid when $|x| < 1$

### Exercise 4C

1. **a** Express $\frac{8x + 4}{(1 - x)(2 + x)}$ as partial fractions.
   
   **b** Hence or otherwise expand $\frac{8x + 4}{(1 - x)(2 + x)}$ in ascending powers of $x$ as far as the term in $x^2$.
   
   **c** State the set of values of $x$ for which the expansion is valid.
2 a Express \(-\frac{2x}{(2+x)^2}\) as partial fractions.

b Hence prove that \(-\frac{2x}{(2+x)^2}\) can be expressed in the form \(-\frac{1}{2}x + Bx^2 + Cx^3\) where constants \(B\) and \(C\) are to be determined.

c State the set of values of \(x\) for which the expansion is valid.

3 a Express \(\frac{6 + 7x + 5x^2}{(1+x)(1-x)(2+x)}\) as partial fractions.

b Hence or otherwise expand \(\frac{6 + 7x + 5x^2}{(1+x)(1-x)(2+x)}\) in ascending powers of \(x\) as far as the term in \(x^3\).

c State the set of values of \(x\) for which the expansion is valid.

4 \(g(x) = \frac{12x - 1}{(1+2x)(1-3x)}, |x| < \frac{1}{3}\)

Given that \(g(x)\) can be expressed in the form \(g(x) = \frac{A}{1+2x} + \frac{B}{1-3x}\)

a Find the values of \(A\) and \(B\). (3 marks)

b Hence, or otherwise, find the series expansion of \(g(x)\), in ascending powers of \(x\), up to and including the \(x^2\) term. Simplify each term. (6 marks)

5 a Express \(\frac{2x^2 + 7x - 6}{(x + 5)(x - 4)}\) in partial fractions.

b Hence, or otherwise, expand \(\frac{2x^2 + 7x - 6}{(x + 5)(x - 4)}\) in ascending powers of \(x\) as far as the term in \(x^3\).

c State the set of values of \(x\) for which the expansion is valid.

6 \(\frac{3x^2 + 4x - 5}{(x + 3)(x - 2)} = A + \frac{B}{x + 3} + \frac{C}{x - 2}\)

a Find the values of the constants \(A\), \(B\) and \(C\). (4 marks)

b Hence, or otherwise, expand \(\frac{3x^2 + 4x - 5}{(x + 3)(x - 2)}\) in ascending powers of \(x\), as far as the term in \(x^2\).

Give each coefficient as a simplified fraction. (7 marks)

7 \(f(x) = \frac{2x^2 + 5x + 11}{(2x - 1)^2(x + 1)}, |x| < \frac{1}{2}\)

\(f(x)\) can be expressed in the form \(f(x) = \frac{A}{2x - 1} + \frac{B}{(2x - 1)^2} + \frac{C}{x + 1}\)

a Find the values of \(A\), \(B\) and \(C\). (4 marks)

b Hence or otherwise, find the series expansion of \(f(x)\), in ascending powers of \(x\), up to and including the term in \(x^2\). Simplify each term. (6 marks)

c Find the percentage error made in using the series expansion in part b to estimate the value of \(f(0.05)\). Give your answer to 2 significant figures. (4 marks)
1. For each of the following
   i. find the binomial expansion up to and including the $x^3$ term
   ii. state the range of values of $x$ for which the expansion is valid.

   a. $(1 - 4x)^3$
   b. $\sqrt{16 + x}$
   c. $\frac{1}{1 - 2x}$
   d. $\frac{4}{2 + 3x}$
   e. $\frac{4}{\sqrt{4 - x}}$
   f. $\frac{1 + x}{1 + 3x}$
   g. $\left(\frac{1 + x}{1 - x}\right)^2$
   h. $\frac{x - 3}{(1 - x)(1 - 2x)}$

2. Use the binomial expansion to expand $\left(1 - \frac{1}{2}x\right)^{\frac{1}{2}}$, $|x| < 2$ in ascending powers of $x$, up to and including the term in $x^3$, simplifying each term. (5 marks)

3. a. Give the binomial expansion of $(1 + x)^\frac{1}{3}$ up to and including the term in $x^3$.
   b. By substituting $x = \frac{1}{4}$, find an approximation to $\sqrt{5}$ as a fraction.

4. The binomial expansion of $(1 + 9x)^\frac{1}{3}$ in ascending powers of $x$, up to and including the term in $x^3$ is $1 + 6x + cx^2 + dx^3$, $|x| < \frac{1}{9}$
   a. Find the value of $c$ and the value of $d$. (4 marks)
   b. Use this expansion with your values of $c$ and $d$ together with an appropriate value of $x$ to obtain an estimate of $(1.45)^{\frac{1}{3}}$. (2 marks)
   c. Obtain $(1.45)^{\frac{1}{3}}$ from your calculator and hence make a comment on the accuracy of the estimate you obtained in part b. (1 mark)

5. In the expansion of $(1 + ax)^n$ the coefficient of $x^2$ is $-2$.
   a. Find the possible values of $a$.
   b. Find the corresponding coefficients of the $x^3$ term.

6. $f(x) = (1 + 3x)^{-1}$, $|x| < \frac{1}{3}$
   a. Expand $f(x)$ in ascending powers of $x$, up to and including the term in $x^3$. (5 marks)
   b. Hence show that, for small $x$:
      \[ \frac{1 + x}{1 + 3x} \approx 1 - 2x + 6x^2 - 18x^3 \] 
      (4 marks)
   c. Taking a suitable value for $x$, which should be stated, use the series expansion in part b to find an approximate value for $\frac{101}{103}$, giving your answer to 5 decimal places. (3 marks)

7. When $(1 + ax)^n$ is expanded as a series in ascending powers of $x$, the coefficients of $x$ and $x^2$ are $-6$ and $27$ respectively.
   a. Find the values of $a$ and $n$. (4 marks)
   b. Find the coefficient of $x^3$. (3 marks)
   c. State the values of $x$ for which the expansion is valid. (1 mark)
8 Show that if $x$ is sufficiently small then $\frac{3}{\sqrt{4 + x}}$ can be approximated by $\frac{3}{2} - \frac{3}{16}x + \frac{9}{256}x^2$

9 a Expand $\frac{1}{\sqrt{4-x}}$, where $|x| < 4$, in ascending powers of $x$ up to and including the term in $x^2$. Simplify each term. (5 marks)

b Hence, or otherwise, find the first 3 terms in the expansion of $\frac{1 + 2x}{\sqrt{4 - x}}$ as a series in ascending powers of $x$. (4 marks)

10 a Find the first four terms of the expansion, in ascending powers of $x$, of $(2 + 3x)^{-1}$, $|x| < \frac{2}{3}$. (4 marks)

b Hence or otherwise, find the first four non-zero terms of the expansion, in ascending powers of $x$, of $\frac{1 + x}{2 + 3x^3}$, $|x| < \frac{2}{3}$. (3 marks)

11 a Use the binomial theorem to expand $(4 + x)^{-\frac{1}{2}}$, $|x| < 4$, in ascending powers of $x$, up to and including the $x^3$ term, giving each answer as a simplified fraction. (5 marks)

b Use your expansion, together with a suitable value of $x$, to obtain an approximation to $\sqrt{2}$. Give your answer to 4 decimal places. (3 marks)

12 $q(x) = (3 + 4x)^{-3}$, $|x| < \frac{3}{4}$

Find the binomial expansion of $q(x)$ in ascending powers of $x$, up to and including the term in the $x^2$. Give each coefficient as a simplified fraction. (5 marks)

13 $g(x) = \frac{39x + 12}{(x + 1)(x + 4)(x - 8)}$, $|x| < 1$

$g(x)$ can be expressed in the form $g(x) = \frac{A}{x + 1} + \frac{B}{x + 4} + \frac{C}{x - 8}$

a Find the values of $A$, $B$ and $C$. (4 marks)

b Hence, or otherwise, find the series expansion of $g(x)$, in ascending powers of $x$, up to and including the $x^2$ term. Simplify each term. (7 marks)

14 $f(x) = \frac{12x + 5}{(1 + 4x)^3}$, $|x| < \frac{1}{4}$

For $x \neq -\frac{1}{4}$, $\frac{12x + 5}{(1 + 4x)^3} = \frac{A}{1 + 4x} + \frac{B}{(1 + 4x)^2}$, where $A$ and $B$ are constants.

a Find the values of $A$ and $B$. (3 marks)

b Hence, or otherwise, find the series expansion of $f(x)$, in ascending powers of $x$, up to and including the term $x^2$, simplifying each term. (6 marks)
15 \( q(x) = \frac{9x^2 + 26x + 20}{(1 + x)(2 + x)}, |x| < 1 \)

a  Show that the expansion of \( q(x) \) in ascending powers of \( x \) can be approximated to \( 10 - 2x + Bx^2 + Cx^3 \) where \( B \) and \( C \) are constants to be found. (7 marks)

b  Find the percentage error made in using the series expansion in part a to estimate the value of \( q(0.1) \). Give your answer to 2 significant figures. (4 marks)

**Challenge**

Obtain the first four non-zero terms in the expansion, in ascending powers of \( x \), of the function \( f(x) \) where \( f(x) = \frac{1}{\sqrt{1 + 3x^2}}, 3x^2 < 1 \)

**Summary of key points**

1  This form of the binomial expansion can be applied to negative or fractional values of \( n \) to obtain an infinite series:

\[
(1 + x)^n = 1 + nx + \frac{n(n - 1)x^2}{2!} + \frac{n(n - 1)(n - 2)x^3}{3!} + \ldots + \frac{n(n - 1)\ldots(n - r + 1)x^r}{r!} + \ldots
\]

The expansion is valid when \(|x| < 1, n \in \mathbb{R}\).

2  The expansion of \((1 + bx)^n\), where \( n \) is negative or a fraction, is valid for \(|bx| < 1, \text{ or } |x| < \frac{1}{|b|}\).

3  The expansion of \((a + bx)^n\), where \( n \) is negative or a fraction, is valid for \(|\frac{b}{a}x| < 1 \text{ or } |x| < \frac{|a|}{|b|}\).

4  If an expression is of the form \( \frac{f(x)}{g(x)} \) where \( g(x) \) can be split into linear factors, then split \( \frac{f(x)}{g(x)} \) into partial fractions before expanding each part of the new expression.