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- Written by a highly-experienced International GCSE Mathematics teacher and author
- Content is mapped to the specification to provide comprehensive coverage
- Exam practice throughout, with full answers included in the back of the book
- Signposted transferable skills
- Reviewed by a language specialist to ensure the book is written in a clear and accessible style
- Glossary of key mathematics terminology
- eBook included
- Online Teacher Resource Pack (ISBN 9780435191214) also available, providing further planning, teaching and assessment support

For Pearson Edexcel International GCSE (9–1) Further Pure Mathematics (4PM1) Higher Tier for first teaching 2017.
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<table>
<thead>
<tr>
<th>CONTENTS</th>
<th>ix</th>
</tr>
</thead>
</table>

**PREFACE**

**CHAPTER 1: SURDS AND LOGARITHMIC FUNCTIONS**

**CHAPTER 2: THE QUADRATIC FUNCTION**

**CHAPTER 3: INEQUALITIES AND IDENTITIES**

**CHAPTER 4: SKETCHING POLYNOMIALS**

**CHAPTER 5: SEQUENCES AND SERIES**

**CHAPTER 6: THE BINOMIAL SERIES**

**CHAPTER 7: SCALAR AND VECTOR QUANTITIES**

**CHAPTER 8: RECTANGULAR CARTESIAN COORDINATES**

**CHAPTER 9: DIFFERENTIATION**

**CHAPTER 10: INTEGRATION**

**CHAPTER 11: TRIGONOMETRY**

**GLOSSARY**

**ANSWERS**

**INDEX**
<table>
<thead>
<tr>
<th>CHAPTER 1</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>WRITE A NUMBER EXACTLY USING A SURD</td>
<td>4</td>
</tr>
<tr>
<td>RATIONALISE THE DENOMINATOR OF A SURD</td>
<td>5</td>
</tr>
<tr>
<td>BE FAMILIAR WITH THE FUNCTIONS $a^x$ AND $\log_b x$ AND RECOGNISE THE SHAPES OF THEIR GRAPHS</td>
<td>7</td>
</tr>
<tr>
<td>BE FAMILIAR WITH EXPRESSIONS OF THE TYPE $e^x$ AND USE THEM IN GRAPHS</td>
<td>9</td>
</tr>
<tr>
<td>BE ABLE TO USE GRAPHS OF FUNCTIONS TO SOLVE EQUATIONS</td>
<td>12</td>
</tr>
<tr>
<td>WRITING AN EXPRESSION AS A LOGARITHM</td>
<td>14</td>
</tr>
<tr>
<td>UNDERSTAND AND USE THE LAWS OF LOGARITHMS</td>
<td>15</td>
</tr>
<tr>
<td>CHANGE THE BASE OF A LOGARITHM</td>
<td>17</td>
</tr>
<tr>
<td>SOLVE EQUATIONS OF THE FORM $a^x = b$</td>
<td>18</td>
</tr>
<tr>
<td>EXAM PRACTICE QUESTIONS</td>
<td>21</td>
</tr>
<tr>
<td>CHAPTER SUMMARY</td>
<td>23</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>CHAPTER 2</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>FACTORISE QUADRATIC EXPRESSIONS WHERE THE COEFFICIENT OF $x^2$ IS GREATER THAN 1</td>
<td>26</td>
</tr>
<tr>
<td>COMPLETE THE SQUARE AND USE THIS TO SOLVE QUADRATIC EQUATIONS</td>
<td>27</td>
</tr>
<tr>
<td>SOLVE QUADRATIC EQUATIONS USING THE QUADRATIC FORMULA</td>
<td>28</td>
</tr>
<tr>
<td>UNDERSTAND AND USE THE DISCRIMINANT TO IDENTIFY WHETHER THE ROOTS ARE (i) EQUAL AND REAL, (ii) UNEQUAL AND REAL OR (iii) NOT REAL</td>
<td>29</td>
</tr>
<tr>
<td>UNDERSTAND THE ROOTS $\alpha$ AND $\beta$ AND HOW TO USE THEM</td>
<td>31</td>
</tr>
<tr>
<td>EXAM PRACTICE QUESTIONS</td>
<td>34</td>
</tr>
<tr>
<td>CHAPTER SUMMARY</td>
<td>35</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>CHAPTER 3</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>SOLVE SIMULTANEOUS EQUATIONS, ONE LINEAR AND ONE QUADRATIC</td>
<td>38</td>
</tr>
<tr>
<td>SOLVE LINEAR INEQUALITIES</td>
<td>39</td>
</tr>
<tr>
<td>SOLVE QUADRATIC INEQUALITIES</td>
<td>41</td>
</tr>
<tr>
<td>GRAPH LINEAR INEQUALITIES IN TWO VARIABLES</td>
<td>44</td>
</tr>
<tr>
<td>DIVIDE A POLYNOMIAL BY $(x \pm p)$</td>
<td>50</td>
</tr>
<tr>
<td>FACTORISE A POLYNOMIAL BY USING THE FACTOR THEOREM</td>
<td>52</td>
</tr>
<tr>
<td>USING THE REMAINDER THEOREM, FIND THE REMAINDER WHEN A POLYNOMIAL IS DIVIDED BY $(ax - b)$</td>
<td>54</td>
</tr>
<tr>
<td>EXAM PRACTICE QUESTIONS</td>
<td>56</td>
</tr>
<tr>
<td>CHAPTER SUMMARY</td>
<td>58</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>CHAPTER 4</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>SKETCH CUBIC CURVES OF THE FORM $y = ax^3 + bx^2 + cx + d$ OR $y = (x + a)(x + b)(x + c)$</td>
<td>62</td>
</tr>
<tr>
<td>SKETCH AND INTERPRET GRAPHS OF CUBIC FUNCTIONS OF THE FORM $y = x^3$</td>
<td>64</td>
</tr>
<tr>
<td>SKETCH THE RECIPROCAL FUNCTION $y = \frac{k}{x}$ WHERE $k$ IS A CONSTANT</td>
<td>65</td>
</tr>
<tr>
<td>SKETCH CURVES OF DIFFERENT FUNCTIONS TO SHOW POINTS OF INTERSECTION AND SOLUTIONS TO EQUATIONS</td>
<td>67</td>
</tr>
<tr>
<td>APPLY TRANSFORMATIONS TO CURVES</td>
<td>69</td>
</tr>
<tr>
<td>EXAM PRACTICE QUESTIONS</td>
<td>71</td>
</tr>
<tr>
<td>CHAPTER SUMMARY</td>
<td>73</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>CHAPTER 5</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>IDENTIFY AN ARITHMETIC SEQUENCE</td>
<td>76</td>
</tr>
<tr>
<td>FIND THE COMMON DIFFERENCE, FIRST TERM AND nth TERM OF AN ARITHMETIC SERIES</td>
<td>78</td>
</tr>
<tr>
<td>FIND THE SUM OF AN ARITHMETIC SERIES AND BE ABLE TO USE $\Sigma$ NOTATION</td>
<td>80</td>
</tr>
<tr>
<td>IDENTIFY A GEOMETRIC SEQUENCE</td>
<td>84</td>
</tr>
<tr>
<td>FIND THE COMMON RATIO, FIRST TERM AND nth TERM OF A GEOMETRIC SEQUENCE</td>
<td>85</td>
</tr>
<tr>
<td>FIND THE SUM OF A GEOMETRIC SERIES</td>
<td>87</td>
</tr>
<tr>
<td>FIND THE SUM TO INFINITY OF A CONVERGENT GEOMETRIC SERIES</td>
<td>90</td>
</tr>
<tr>
<td>EXAM PRACTICE QUESTIONS</td>
<td>93</td>
</tr>
<tr>
<td>CHAPTER SUMMARY</td>
<td>95</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>CHAPTER 6</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>USE $\binom{n}{r}$ TO WORK OUT THE COEFFICIENTS IN THE BINOMIAL EXPANSION</td>
<td>98</td>
</tr>
</tbody>
</table>
COURSE STRUCTURE

USE THE BINOMIAL EXPANSION TO EXPAND \((1 + x)^n\) 100
Determine the range of values for which \(x\) is true and valid for an expansion 101
Exam practice questions 104
Chapter summary 105

CHAPTER 7
Vector notation and how to draw vector diagrams 108
Perform simple vector arithmetic and understand the definition of a unit vector 110
Use vectors to describe the position of a point in two dimensions 113
Use vectors to demonstrate simple properties of geometrical figures 114
Write down and use Cartesian components of a vector in two dimensions 117
Exam practice questions 120
Chapter summary 123

CHAPTER 8
Work out the gradient of a straight line 125
Find the equation of a straight line 127
Understand the relationship between perpendicular lines 128
Find the distance between two points on a line 131
Find the coordinates of a point that divides a line in a given ratio 132
Exam practice questions 133
Chapter summary 135

CHAPTER 9
Find the gradient function of a curve and differentiate a function that has multiple powers of \(x\) 138
Differentiate \(e^{ax}\), \(\sin ax\) and \(\cos ax\) 141
Use the chain rule to differentiate more complicated functions 143
Use the product rule to differentiate more complicated functions 146
Use the quotient rule to differentiate more complicated functions 148
Find the equation of the tangent and normal to the curve \(y = f(x)\) 149
Find the stationary points of a curve and calculate whether they are minimum or maximum stationary points 151
Apply what you have learnt about turning points to solve problems 153
Exam practice questions 156
Chapter summary 159

CHAPTER 10
Integration as the reverse of differentiation 162
Understand how calculus is related to problems involving kinematics 164
Use integration to find areas 166
Use integration to find a volume of revolution 171
Relate rates of change to each other 174
Exam practice questions 178
Chapter summary 181

CHAPTER 11
Measure angles in radians 184
Calculate arc length and the area of a circle using radians 185
Understand the basic trigonometrical ratios and sine, cosine and tangent graphs 190
Use sine and cosine rules 193
Use sine and cosine rules to solve problems in 3D 197
Use trigonometry identities to solve problems 199
Solve trigonometric equations 202
Use trigonometric formulae to solve equations 206
Exam practice questions 208
Chapter summary 210
Glossary 212
Answers 216
ABOUT THIS BOOK

This book is written for students following the Edexcel International GCSE (9–1) Further Pure Maths specification and covers both years of the course. The specification and sample assessment materials for Further Pure Maths can be found on the Pearson Qualifications website.

In each chapter, there are concise explanations and worked examples, plus numerous exercises that will help you build up confidence.

There are also exam practice questions and a chapter summary to help with exam preparation. Answers to all exercises are included at the back of the book as well as a glossary of Maths-specific terminology.
Examples provide a clear, instructional framework. The blue highlighted text gives further explanation of the method.

Language is graded for speakers of English as an additional language (EAL), with advanced Maths-specific terminology highlighted and defined in the glossary at the back of the book.

Key Points boxes summarise the essentials.

Exam Practice tests cover the whole chapter and provide quick, effective feedback on your progress.

Chapter Summaries state the most important points of each chapter.
ASSESSMENT OVERVIEW

The following tables give an overview of the assessment for the Edexcel International GCSE in Further Pure Mathematics.

We recommend that you study this information closely to help ensure that you are fully prepared for this course and know exactly what to expect in the assessment.

<table>
<thead>
<tr>
<th>PAPER 1</th>
<th>PERCENTAGE</th>
<th>MARK</th>
<th>TIME</th>
<th>AVAILABILITY</th>
</tr>
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<tbody>
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<td>100</td>
<td>2 hours</td>
<td>January and June examination series</td>
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<tr>
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<td>First assessment June 2019</td>
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<tr>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
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<th>PERCENTAGE</th>
<th>MARK</th>
<th>TIME</th>
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<tbody>
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<td></td>
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<td>First assessment June 2019</td>
</tr>
<tr>
<td>Externally set and assessed by Edexcel</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

CONTENT SUMMARY

- Number
- Algebra and calculus
- Geometry and calculus

ASSESSMENT

- Each paper will consist of around 11 questions with varying mark allocations per questions, which will be stated on the paper
- Each paper will contain questions from any part of the specification content, and the solution of any questions may require knowledge of more than one section of the specification content
- A formulae sheet will be included in the written examinations
- A calculator may be used in the examinations

ASSESSMENT OBJECTIVES AND WEIGHTINGS

<table>
<thead>
<tr>
<th>ASSESSMENT OBJECTIVE</th>
<th>DESCRIPTION</th>
<th>% IN INTERNATIONAL GCSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>AO1</td>
<td>Demonstrate a confident knowledge of the techniques of pure mathematics required in the specification</td>
<td>30%-40%</td>
</tr>
<tr>
<td>AO2</td>
<td>Apply a knowledge of mathematics to the solutions of problems for which an immediate method of solution is not available and which may involve knowledge of more than one topic in the specification</td>
<td>20%-30%</td>
</tr>
<tr>
<td>AO3</td>
<td>Write clear and accurate mathematical solutions</td>
<td>35%-50%</td>
</tr>
</tbody>
</table>
RELATIONSHIP OF ASSESSMENT OBJECTIVES TO UNITS

<table>
<thead>
<tr>
<th>UNIT NUMBER</th>
<th>ASSESSMENT OBJECTIVE</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>AO1</td>
</tr>
<tr>
<td>Paper 1</td>
<td>15%–20%</td>
</tr>
<tr>
<td>Paper 2</td>
<td>15%–20%</td>
</tr>
<tr>
<td>Total for International GCSE</td>
<td>30%–40%</td>
</tr>
</tbody>
</table>

ASSESSMENT SUMMARY

The Edexcel International GCSE in Further Pure Mathematics requires students to demonstrate application and understanding of the following topics.

Number
• Use numerical skills in a purely mathematical way and in real-life situations.

Algebra and calculus
• Use algebra and calculus to set up and solve problems.
• Develop competence and confidence when manipulating mathematical expressions.
• Construct and use graphs in a range of situations.

Geometry and trigonometry
• Understand the properties of shapes, angles and transformations.
• Use vectors and rates of change to model situations.
• Use coordinate geometry.
• Use trigonometry.

Students will be expected to have a thorough knowledge of the content common to the Pearson Edexcel International GCSE in Mathematics (Specification A) (Higher Tier) or Pearson Edexcel International GCSE in Mathematics (Specification B).

Questions may be set which assumes knowledge of some topics covered in these specifications, however knowledge of statistics and matrices will not be required.

Students will be expected to carry out arithmetic and algebraic manipulation, such as being able to change the subject of a formula and evaluate numerically the value of any variable in a formula, given the values of the other variables. The use and notation of set theory will be adopted where appropriate.

CALCULATORS

Students will be expected to have access to a suitable electronic calculator for all examination papers. The electronic calculator should have these functions as a minimum:
+ , − , × , ÷ , π, x², √x, 1/x, xⁿ, ln x, eˣ, sine, cosine and tangent and their inverses in degrees and decimals of a degree or radians.

Prohibitions
Calculators with any of the following facilities are prohibited in all examinations:
  databanks
  retrieval of text or formulae
  QWERTY keyboards
  built-in symbolic algebra manipulations
  symbolic differentiation or integration.
FORMULAE SHEET

These formulae will be provided for you during the examination.

MENSURATION

Surface area of sphere = $4\pi r^2$
Curved surface area of cone = $\pi r \times$ slant height
Volume of sphere = $\frac{4}{3}\pi r^3$

SERIES

Arithmetic series
Sum to $n$ terms $S_n = \frac{n}{2}[2a + (n - 1)d]$

Geometric series
Sum to $n$ terms, $S_n = \frac{a(1 - r^n)}{(1 - r)}$
Sum to infinity, $S = \frac{a}{1 - r}$ $|r| < 1$

Binomial series
$(1 + x)^n = 1 + nx + \frac{n(n - 1)}{2!}x^2 + \ldots + \frac{n(n - 1)\ldots(n - r + 1)}{r!}x^r + \ldots$ for $|x| < 1$, $n \in \mathbb{Q}$

CALCULUS

Quotient rule (differentiation)
$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

TRIGONOMETRY

Cosine rule
In triangle $ABC$: $a^2 = b^2 + c^2 - 2bc \cos A$

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

$$\sin(A + B) = \sin A \cos B + \cos A \sin B \quad \sin(A - B) = \sin A \cos B - \cos A \sin B$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B \quad \cos(A - B) = \cos A \cos B + \sin A \sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} \quad \tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

LOGARITHMS

$$\log_a x = \frac{\log_b x}{\log_b a}$$
FORMULAE TO KNOW

The following are formulae that you are expected to know and remember during the examination. These formulae will not be provided for you. Note that this list is not exhaustive.

LOGARITHMIC FUNCTIONS AND INDICES

\[ \log_a xy = \log_a x + \log_a y \]
\[ \log_a \frac{x}{y} = \log_a x - \log_a y \]
\[ \log_a x^k = k \log_a x \]
\[ \log_a \frac{1}{x} = -\log_a x \]
\[ \log_a a = 1 \]
\[ \log_a 1 = 0 \]
\[ \log_a b = \frac{1}{\log_b a} \]

QUADRATIC EQUATIONS

\[ ax^2 + bx + c = 0 \]

Has roots given by

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

When the roots of \( ax^2 + bx + c = 0 \) are \( \alpha \) and \( \beta \) then \( \alpha + \beta = -\frac{b}{a} \) and \( \alpha \beta = \frac{c}{a} \)

and the equation can be written \( x^2 - (\alpha + \beta)x + \alpha \beta = 0 \)

SERIES

Arithmetic series: \( n \)th term \( = l = a + (n - 1)d \)

Geometric series: \( n \)th term \( = a r^{n-1} \)

COORDINATE GEOMETRY

The gradient of the line joining two points \((x_1, y_1)\) and \((x_2, y_2)\) is \( \frac{y_2 - y_1}{x_2 - x_1} \)

The distance \( d \) between two points \((x_1, y_1)\) and \((x_2, y_2)\) is given by

\[ d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 \]

The coordinates of the point dividing the line joining \((x_1, y_1)\) and \((x_2, y_2)\) in the ratio \( m:n \) are

\[ \left( \frac{nx_1 - mx_2}{m+n}, \frac{ny_1 - my_2}{m+n} \right) \]
CALCULUS

Differentiation: function derivative
\( x^n \) \( nx^{n-1} \)
\( \sin ax \) \( a \cos ax \)
\( \cos ax \) \( -a \sin ax \)
\( e^{ax} \) \( ae^{ax} \)
\( f(x)g(x) \) \( f'(x)g(x) + f(x)g'(x) \)
\( f(g(x)) \) \( f'(g(x))g'(x) \)

Integration: function derivative
\( x^n \) \( \frac{1}{n + 1}x^{n+1} + c \quad n \neq -1 \)
\( \sin ax \) \( \frac{1}{a} \cos ax + c \)
\( \cos ax \) \( \frac{1}{a} \sin ax + c \)
\( e^{ax} \) \( \frac{1}{a} e^{ax} + c \)

AREA AND VOLUME

Area between a curve and the \( x \)-axis = \( \int_a^b y \, dx \), \( y \geq 0 \)
\( \left| \int_a^b y \, dx \right|, \ y < 0 \)

Area between a curve and the \( y \)-axis = \( \int_c^d x \, dy \), \( x \geq 0 \)
\( \left| \int_c^d x \, dy \right|, \ x < 0 \)

Area between \( g(x) \) and \( f(x) \) = \( \int_a^b |g(x) - f(x)| \, dx \)

Volume of revolution = \( \int_a^b \pi y^2 \, dx \) or \( \int_c^d \pi x^2 \, dy \)

TRIGONOMETRY

Radian measure:
- length of arc = \( r \theta \)
- area of sector = \( \frac{1}{2}r^2 \theta \)

In a triangle \( ABC \):
- \( \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \)
- \( \cos^2 \theta + \sin^2 \theta = 1 \)
- area of a triangle = \( \frac{1}{2}ab \sin C \)
CHAPTER 1
The Richter scale, which describes the energy released by an earthquake, uses the base 10 logarithm as its unit. An earthquake of magnitude 9 is 10 times as powerful as one of magnitude 8, and 100,000 times as powerful as one of magnitude 4.

The devastating 2004 earthquake in the Indian Ocean had a magnitude of 9. Thankfully, such events are rare. The most common earthquakes, which occur over 100,000 times a year, are magnitude 2 to 3, so humans can hardly feel them.

### LEARNING OBJECTIVES

- Write a number exactly using surds
- Rationalise the denominator of a surd
- Be familiar with the functions $a^x$ and $\log_b x$ and recognise the shapes of their graphs
- Be familiar with functions including $e^x$ and similar terms, and use them in graphs
- Use graphs of functions to solve equations
- Rewrite expressions including powers using logarithms instead
- Understand and use the laws of logarithms
- Change the base of a logarithm
- Solve equations of the form $a^x = b$

### STARTER ACTIVITIES

1. Simplify
   
<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>$y^6 \times y^5$</td>
<td>b</td>
</tr>
<tr>
<td>d</td>
<td>$(x^4)^4$</td>
<td>e</td>
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</tbody>
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2. Simplify
   
<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
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</thead>
<tbody>
<tr>
<td>a</td>
<td>$(m^3)^\frac{1}{3}$</td>
<td>b</td>
</tr>
<tr>
<td>d</td>
<td>$6b^{\frac{1}{2}} \times 3b^{-\frac{1}{2}}$</td>
<td>e</td>
</tr>
</tbody>
</table>

3. Evaluate
   
<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>$16^{\frac{1}{2}}$</td>
<td>b</td>
<td>$125^{\frac{1}{3}}$</td>
</tr>
<tr>
<td>e</td>
<td>$\left(\frac{6}{7}\right)^0$</td>
<td>f</td>
<td>$81^{-\frac{1}{4}}$</td>
</tr>
<tr>
<td>g</td>
<td>$\left(\frac{9}{16}\right)^{-\frac{3}{2}}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
WRITE A NUMBER EXACTLY USING A SURD

A surd is a number that cannot be simplified to remove a square root (or a cube root, fourth root etc). Surds are irrational numbers.

<table>
<thead>
<tr>
<th>NUMBER</th>
<th>DECIMAL</th>
<th>IS IT A SURD?</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\sqrt{1})</td>
<td>1</td>
<td>No</td>
</tr>
<tr>
<td>(\sqrt{2})</td>
<td>1.414213…</td>
<td>Yes</td>
</tr>
<tr>
<td>(\sqrt{4})</td>
<td>2</td>
<td>No</td>
</tr>
<tr>
<td>(\frac{\sqrt{1}}{4})</td>
<td>0.5</td>
<td>No</td>
</tr>
<tr>
<td>(\frac{\sqrt{2}}{3})</td>
<td>0.816496…</td>
<td>Yes</td>
</tr>
</tbody>
</table>

You can manipulate surds using these rules:

\[\sqrt{ab} = \sqrt{a} \times \sqrt{b}\]
\[\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}\]

**Example 1**

Simplify

**a** \(\sqrt{12}\)

\[
\begin{align*}
\sqrt{12} &= \sqrt{4 \times 3} \\
&= \sqrt{4} \times \sqrt{3} \\
&= 2\sqrt{3}
\end{align*}
\]

**b** \(\frac{\sqrt{20}}{2}\)

\[
\begin{align*}
\frac{\sqrt{20}}{2} &= \frac{\sqrt{4 \times 5}}{2} \\
&= \frac{\sqrt{4} \times \sqrt{5}}{2} \\
&= \frac{2 \times \sqrt{5}}{2} \\
&= \sqrt{5}
\end{align*}
\]

**c** \(5\sqrt{6} - 2\sqrt{24} + \sqrt{294}\)

\[
\begin{align*}
5\sqrt{6} - 2\sqrt{24} + \sqrt{294} &= 5\sqrt{6} - 2\sqrt{6\times4} + \sqrt{6\times49} \\
&= \sqrt{6}(5 - 2\sqrt{4} + \sqrt{49}) \\
&= \sqrt{6}(5 - 2 \times 2 + 7) \\
&= 8\sqrt{6}
\end{align*}
\]
EXERCISE 1

1 ▶ Simplify without using a calculator

a \( \sqrt{18} \)  b \( \sqrt{50} \)  c \( \sqrt{125} \)

a \( \sqrt{128} \)  b \( \sqrt{132} \)  c \( \sqrt{8625} \)

d \( \sqrt{60} \)  e \( \sqrt{135} \)  f \( \sqrt{128} \)

d \( \sqrt{68} \)  e \( \sqrt{96} \)  f \( \sqrt{128} \)

2 ▶ Simplify without using a calculator

a \( \frac{\sqrt{60}}{2} \)  b \( \frac{\sqrt{135}}{2} \)  c \( \frac{\sqrt{128}}{8} \)

d \( \frac{\sqrt{68}}{4} \)  e \( \sqrt{96} \)  f \( \frac{\sqrt{128}}{8} \)

3 ▶ Simplify without using a calculator

a \( 6\sqrt{3} - 2\sqrt{3} \)  b \( 7\sqrt{3} - 12 + \sqrt{48} \)  c \( \sqrt{112} + 2\sqrt{172} - \sqrt{63} \)

d \( 6\sqrt{48} - 3\sqrt{12} + 2\sqrt{27} \)  e \( 3\sqrt{578} - \sqrt{162} + 4\sqrt{32} \)  f \( 2\sqrt{5} \times 3\sqrt{5} \)

a \( 6\sqrt{7} \times 4\sqrt{7} \)  b \( 4\sqrt{8} \times 6\sqrt{8} \)

g \( 6\sqrt{7} \times 4\sqrt{7} \)  h \( 4\sqrt{8} \times 6\sqrt{8} \)

4 ▶ Simplify without using a calculator

a \( 6(\sqrt{4} - \sqrt{2}) \)  b \( 9(6 - 3\sqrt{29}) \)  c \( 4(1 + \sqrt{3}) + 3\sqrt{3} + 2\sqrt{3} \)

d \( 3(\sqrt{2} - \sqrt{7}) - 5(\sqrt{2} + \sqrt{7}) \)  e \( 4 + \sqrt{3}\sqrt{4} - \sqrt{3} \)

f \( (2\sqrt{7} - \sqrt{6})(\sqrt{7} - 2\sqrt{6}) \)  g \( \sqrt{8} + \sqrt{5}\sqrt{8} - \sqrt{5} \)

5 ▶ A garden is \( \sqrt{30} \) m long and \( \sqrt{8} \) m wide.
The garden is covered in grass except for a small rectangular pond which is \( \sqrt{2} \) m long and \( \sqrt{6} \) m wide.

Express the area of the pond as a percentage of the area of the garden.

6 ▶ Find the value of \( 2p^2 - 3pq \) when \( p = \sqrt{2} + 3 \) and \( q = \sqrt{2} - 2 \)

RATIONALISE THE DENOMINATOR OF A SURD

Rationalising the denominator of a surd means removing a root from the denominator of a fraction. You will usually need to rationalise the denominator when you are asked to simplify it.

The rules for rationalising the denominator of a surd are:

- For fractions in the form \( \frac{1}{\sqrt{a}} \), multiply the numerator and denominator by \( \sqrt{a} \)
- For fractions in the form \( \frac{1}{a + \sqrt{b}} \), multiply the numerator and denominator by \( a - \sqrt{b} \)
- For fractions in the form \( \frac{1}{a - \sqrt{b}} \), multiply the numerator and denominator by \( a + \sqrt{b} \)
### EXAMPLE 2

Rationalise the denominator of

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>( \frac{1}{\sqrt{3}} )</td>
</tr>
<tr>
<td>b</td>
<td>( \frac{1}{3 + \sqrt{2}} )</td>
</tr>
<tr>
<td>c</td>
<td>( \frac{\sqrt{5} + \sqrt{2}}{\sqrt{5} - \sqrt{2}} )</td>
</tr>
</tbody>
</table>

\[
\text{a} \quad \frac{1}{\sqrt{3}} \\
= \frac{1 \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}} \\
= \frac{\sqrt{3}}{3} \\
\]

Multiply the top and bottom by \( \sqrt{3} \)

\[
\text{b} \quad \frac{1}{3 + \sqrt{2}} \\
= \frac{1 \times (3 - \sqrt{2})}{(3 + \sqrt{2})(3 - \sqrt{2})} \\
= \frac{3 - \sqrt{2}}{9 - 3\sqrt{2} + 3\sqrt{2} - 2} \\
= \frac{3 - \sqrt{2}}{7} \\
\]

Multiply the top and bottom by \( 3 - \sqrt{2} \)

\[
\text{c} \quad \frac{\sqrt{5} + \sqrt{2}}{\sqrt{5} - \sqrt{2}} \\
= \frac{(\sqrt{5} + \sqrt{2})(\sqrt{5} + \sqrt{2})}{(\sqrt{5} - \sqrt{2})(\sqrt{5} + \sqrt{2})} \\
= \frac{5 + 2\sqrt{5} + 2\sqrt{5} + 2}{5 - 2} \\
= \frac{7 + 2\sqrt{10}}{3} \\
\]

Multiply the top and bottom by \( \sqrt{5} + \sqrt{2} \)

### EXERCISE 2

#### SKILLS

**EXECUTIVE FUNCTION**

1. Rationalise

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>( \frac{1}{\sqrt{3}} )</td>
</tr>
<tr>
<td>b</td>
<td>( \frac{1}{\sqrt{7}} )</td>
</tr>
<tr>
<td>c</td>
<td>( \frac{2}{\sqrt{3}} )</td>
</tr>
<tr>
<td>d</td>
<td>( \frac{\sqrt{6}}{\sqrt{3}} )</td>
</tr>
<tr>
<td>e</td>
<td>( \frac{12}{\sqrt{3}} )</td>
</tr>
<tr>
<td>f</td>
<td>( \frac{3\sqrt{5}}{\sqrt{3}} )</td>
</tr>
<tr>
<td>g</td>
<td>( \frac{9\sqrt{12}}{2\sqrt{18}} )</td>
</tr>
<tr>
<td>h</td>
<td>( \frac{1}{2 - \sqrt{3}} )</td>
</tr>
</tbody>
</table>

2. Rationalise

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>( \frac{\sqrt{6}}{\sqrt{3} + \sqrt{6}} )</td>
</tr>
<tr>
<td>b</td>
<td>( \frac{2 + \sqrt{3}}{2 - \sqrt{3}} )</td>
</tr>
<tr>
<td>c</td>
<td>( \frac{\sqrt{2} - \sqrt{3}}{\sqrt{2} + \sqrt{3}} )</td>
</tr>
<tr>
<td>d</td>
<td>( \frac{4\sqrt{2} - 2\sqrt{3}}{\sqrt{2} + \sqrt{3}} )</td>
</tr>
<tr>
<td>e</td>
<td>( \frac{\sqrt{2} + 2\sqrt{5}}{\sqrt{5} - \sqrt{2}} )</td>
</tr>
<tr>
<td>f</td>
<td>( \frac{\sqrt{7} + \sqrt{3}}{\sqrt{7} - \sqrt{3}} )</td>
</tr>
<tr>
<td>g</td>
<td>( \frac{\sqrt{11} + 2\sqrt{5}}{\sqrt{11} + 3\sqrt{5}} )</td>
</tr>
<tr>
<td>h</td>
<td>( \frac{2\sqrt{5} - 3\sqrt{7}}{5\sqrt{6} + 4\sqrt{2}} )</td>
</tr>
<tr>
<td>i</td>
<td>( \frac{2 + \sqrt{10}}{\sqrt{2} + \sqrt{5}} )</td>
</tr>
<tr>
<td>j</td>
<td>( \frac{ab}{a - b} )</td>
</tr>
<tr>
<td>k</td>
<td>( \frac{a - b}{a - b} )</td>
</tr>
</tbody>
</table>
BE FAMILIAR WITH THE FUNCTIONS $a^x$ AND $\log_a x$ AND RECOGNISE THE SHAPES OF THEIR GRAPHS

You need to be familiar with functions in the form $y = a^x$ where $a > 0$

Look at a table of values for $y = 2^x$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$-3$</th>
<th>$-2$</th>
<th>$-1$</th>
<th>$0$</th>
<th>$1$</th>
<th>$2$</th>
<th>$3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>$\frac{1}{8}$</td>
<td>$\frac{1}{4}$</td>
<td>$\frac{1}{2}$</td>
<td>$1$</td>
<td>$2$</td>
<td>$4$</td>
<td>$8$</td>
</tr>
</tbody>
</table>

**Note:** $2^0 = 1$

In fact $a^0$ is always equal to 1 if $a$ is positive and

$$2^{-3} = \frac{1}{2^3} = \frac{1}{8}$$

a negative index turns the number into its **reciprocal**

The graph of $y = 2^x$ looks like this:

![Graph of $y = 2^x$]

**Note:** the $x$-axis is an **asymptote** to the curve.

Other graphs of the type $y = a^x$ have similar shapes, always passing through (0, 1).

**EXAMPLE 3**

**SKILLS**

**ANALYSIS**

**a** On the same axes, **sketch** the graphs of $y = 3^x$, $y = 2^x$ and $y = 1.5^x$

**b** On another set of axes, sketch the graphs of $y = \left(\frac{1}{2}\right)^x$ and $y = 2^x$

**a** For all three graphs, $y = 1$ when $x = 0$

When $x > 0$, $3^x > 2^x > 1.5^x$

$3^0 = 1$

When $x < 0$, $3^x < 2^x < 1.5^x$

Work out the relative positions of the graphs
\[ b \quad \frac{1}{2} = 2^{-1} \]

so, \( y = \left(\frac{1}{2}\right)^x \) is the same as \( y = (2^{-1})^x = 2^{-x} \)

Therefore the graph of \( y = \left(\frac{1}{2}\right)^x \) is a reflection in the y-axis of the graph of \( y = 2^x \)

If you compare the graphs of \( y = 2^x \) and \( y = \log_2 x \) you see the following relationship:

On the same set of axes sketch the graphs \( y = \log_2 x \) and \( y = \log_5 x \)

Note:
For both graphs \( y = 0 \) when \( x = 1 \), since \( \log_a 1 = 0 \) for every value of \( a \).
\( \log_2 2 = 1 \) so \( y = \log_2 x \) passes through \((2, 1)\)
and \( \log_5 5 = 1 \) so \( y = \log_5 x \) passes through \((5, 1)\)

1 ▶ On the same set of axes sketch the graphs of

\[ \text{a} \quad y = 5^x \quad \text{b} \quad y = 7^x \quad \text{c} \quad y = \left(\frac{1}{3}\right)^x \]
2 ▶ On the same set of axes sketch the graphs of
   a  \( y = \log_5 x \)  
   b  \( y = \log_7 x \)  
   c  Write down the coordinates of the point of intersection of these two graphs.

3 ▶ On the same set of axes sketch the graphs of
   a  \( y = 3^x \)  
   b  \( y = \log_3 x \)  

4 ▶ On the same set of axes sketch the graphs of
   a  \( y = \log_3 x \)  
   b  \( y = \log_5 x \)  
   c  \( y = \log_{0.5} x \)  
   d  \( y = \log_{0.25} x \)

**BE FAMILIAR WITH EXPRESSIONS OF THE TYPE \( e^x \) AND USE THEM IN GRAPHS**

Consider this example: Zainab opens an account with $1.00. The account pays 100% interest per year. If the interest is credited once, at the end of the year, her account will contain $2.00. How much will it contain after a year if the interest is calculated and credited more frequently? Let us investigate this more thoroughly.

<table>
<thead>
<tr>
<th>HOW OFTEN INTEREST IS CREDITED INTO THE ACCOUNT</th>
<th>VALUE OF ACCOUNT AFTER 1 YEAR ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yearly</td>
<td>( \left(1 + \frac{1}{1}\right)^1 = 2 )</td>
</tr>
<tr>
<td>Semi-annually</td>
<td>( \left(1 + \frac{1}{2}\right)^2 = 2.25 )</td>
</tr>
<tr>
<td>Quarterly</td>
<td>( \left(1 + \frac{1}{4}\right)^4 = 2.441406... )</td>
</tr>
<tr>
<td>Monthly</td>
<td>( \left(1 + \frac{1}{12}\right)^{12} = 2.6130529... )</td>
</tr>
<tr>
<td>Weekly</td>
<td>( \left(1 + \frac{1}{52}\right)^{52} = 2.6925695... )</td>
</tr>
<tr>
<td>Daily</td>
<td>( \left(1 + \frac{1}{365}\right)^{365} = 2.71456748... )</td>
</tr>
<tr>
<td>Hourly</td>
<td>( \left(1 + \frac{1}{8760}\right)^{8760} = 2.71812669... )</td>
</tr>
<tr>
<td>Every minute</td>
<td>( \left(1 + \frac{1}{525600}\right)^{525600} = 2.7182154... )</td>
</tr>
<tr>
<td>Every second</td>
<td>( \left(1 + \frac{1}{31536000}\right)^{31536000} = 2.71828247... )</td>
</tr>
</tbody>
</table>
The amount in her account gets bigger and bigger the more often the interest is compounded, but the rate of growth slows. As the number of compounds increases, the calculated value appears to be approaching a fixed value. This value gets closer and closer to a fixed value of 2.71828247254…….This number is called ‘e’.

The number e is called a natural exponential because it arises naturally in mathematics and has numerous real life applications.

Draw the graphs of $e^x$ and $e^{-x}$

<table>
<thead>
<tr>
<th>$x$</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e^x$</td>
<td>0.14</td>
<td>0.37</td>
<td>1</td>
<td>2.7</td>
<td>7.4</td>
<td>20</td>
<td>55</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$x$</th>
<th>-4</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e^{-x}$</td>
<td>55</td>
<td>20</td>
<td>7.4</td>
<td>2.7</td>
<td>1</td>
<td>0.37</td>
<td>0.14</td>
</tr>
</tbody>
</table>

Draw the graphs of these exponential functions.

a $y = e^{2x}$

b $y = 10e^{-x}$

c $y = 3 + 4e^{\frac{x}{2}}$

<table>
<thead>
<tr>
<th>$x$</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e^{2x}$</td>
<td>0.02</td>
<td>0.1</td>
<td>1</td>
<td>7.4</td>
<td>55</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$x$</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10e^{-x}$</td>
<td>73</td>
<td>27</td>
<td>10</td>
<td>3.7</td>
<td>1.4</td>
</tr>
</tbody>
</table>
On pages 7–9 you saw the connection between $y = \log_a x$ and and $y = a^x$. The function $y = \log_e x$ is particularly important in mathematics and so it has a special notation:

$$\log_e x \equiv \ln x$$

Your calculator should have a special button for evaluating $\ln x$.

**EXAMPLE 8**

Solve these equations.

- **a** \(e^x = 3\)
- **b** \(\ln x = 4\)

- **a** When \(e^x = 3\) \(x = \ln 3\)
- **b** When \(\ln x = 4\) \(x = e^4\)

As you can see, the inverse of \(e^x\) is \(\ln x\) (and vice versa)

**EXAMPLE 9**

Sketch these graphs on the same set of axes.

- **a** \(y = \ln x\)
- **b** \(y = \ln(3 - x)\)
- **c** \(y = 3 + \ln(2x)\)

**EXERCISE 4**

1. Sketch these graphs.
   - **a** \(y = e^x + 1\)
   - **b** \(y = 4e^{-2x}\)
   - **c** \(y = 2e^x - 3\)
   - **d** \(y = 6 + 10^{\frac{1}{x}}\)
   - **e** \(y = 100e^{-x} + 10\)

2. Sketch these graphs, stating any asymptotes and intersections with the axes.
   - **a** \(y = \ln(x + 1)\)
   - **b** \(y = 2 \ln x\)
   - **c** \(y = \ln(2x)\)
   - **d** \(y = \ln(4 - x)\)
   - **e** \(y = 4 + \ln(x + 2)\)
BE ABLE TO USE GRAPHS OF FUNCTIONS TO SOLVE EQUATIONS

**Example 10**

a. Complete the table of values for: \( y = e^{\frac{1}{x}} - 2 \)
   Giving your answers to two decimal places where appropriate.

b. Draw the graph of \( y = e^{\frac{1}{x}} - 2 \) for \( 0 \leq x \leq 5 \)

c. Use your graph to estimate, to 2 significant figures, the solution of the equation \( e^{\frac{1}{x}} = 8 \)
   Show your method clearly.

d. By drawing a suitable line on your graph, estimate to 2 significant figures the solution to the equation \( x = 2 \ln(7 - 2x) \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>(-1)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>-1.39</td>
<td>-1</td>
<td>-0.35</td>
<td>0.72</td>
<td>2.48</td>
<td>5.39</td>
<td>10.18</td>
</tr>
</tbody>
</table>

**HINT**
Make the LHS = \( e^{\frac{1}{x}} - 2 \)
   i.e. the equation of the graph.
To do this, you need to subtract 2 from 8 and draw the line \( y = 6 \) (as shown in the diagram).

**HINT**
Make the LHS equal to the given equation
   i.e \( e^{\frac{1}{x}} - 2 \).
Draw the line \( y = 5 - 2x \) (as shown in the diagram) on your graph and find points of intersection.

**Example 11**

a. Complete the table below of values of \( y = 2 + \ln x \), giving your values of \( y \) to decimal places.

<table>
<thead>
<tr>
<th>( x )</th>
<th>0.1</th>
<th>0.5</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>-0.3</td>
<td>1.31</td>
<td>2.41</td>
<td>2.69</td>
<td>3.10</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
b) Draw the graph of \( y = 2 + \ln x \) for \( 0.1 \leq x \leq 4 \)

c) Use your graph to estimate, to 2 significant figures, the solution of the equation \( \ln x = 0.5 \)

d) By drawing a suitable line on your graph estimate, to 1 significant figure, the solution of the equation \( x = e^{x - 2} \)

\[ \begin{array}{|c|c|c|c|c|c|c|}
\hline
x & 0.1 & 0.5 & 1 & 1.5 & 2 & 3 & 4 \\
\hline
y & -0.3 & 1.31 & 2 & 2.41 & 2.69 & 3.10 & 3.39 \\
\hline
\end{array} \]

\[ y = 2 + \ln x \]

\[ y = 2.5 \]

The solution is the intersection of the curve and the line \( y = 2.5 \). From the graph this is approximately 1.6. In the exam you will be given a small range of answers.

d) \( x = e^{x - 2} \)

Using the properties of logs

\( \ln x = x - 2 \)

\( \ln x + 2 = x - 2 + 2 \)

Make the LHS equal to the given equation i.e. \( \ln x + 2 \).
1. a. Draw the graph \( y = 3 + 2e^{-\frac{1}{2}x} \) for \( 0 \leq x \leq 6 \)
   b. Use your graph to estimate, to 2 significant figures, the solution to the equation \( e^{-\frac{1}{2}x} = 0.5 \), showing your method clearly.
   c. By drawing a suitable line, estimate, to 2 significant figures, the solution of the equation
   \[ x = -2 \ln\left(\frac{x - 2}{2}\right) \]

2. a. Draw the graph \( y = 2 + \frac{1}{3}e^x \) for \(-1 \leq x \leq 3\)
   b. Use your graph to estimate, to 2 significant figures, the solution to the equation \( e^x = 12 \) showing your method clearly.
   c. By drawing a suitable line, estimate, to 2 significant figures, the solution of the equation
   \[ x = \ln(6 - 6x) \]

3. a. Complete the table below of values of \( y = 5 \sin 2x - 2 \cos x \), giving your values of \( y \) to 2 decimal places.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( 0 )</th>
<th>15</th>
<th>30</th>
<th>45</th>
<th>60</th>
<th>75</th>
<th>90</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>-2</td>
<td>0.57</td>
<td>3.59</td>
<td>3.33</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b. Draw the graph of \( y = 5 \sin 2x - 2 \cos x \) for \( \theta \leq x \leq 90^\circ \)

c. Use your graph to estimate, to 2 significant figures, the solution of the equation
   \[ 2(1 + \cos x) = 5 \sin 2x \] showing your method clearly.

**Writing an Expression as a Logarithm**

\( \log_a n = x \) means that \( a^x = n \), where \( a \) is called the base of the logarithm.

- \( \log_a 1 = 0 \) \((a > 0)\), because \( a^0 = 1 \)
- \( \log_a a = 1 \) \((a > 0)\), because \( a^1 = a \)

Write as a logarithm \( 2^5 = 32 \)

\[ 2^5 = 32 \]

So \( \log_2 32 = 5 \) \( \quad \) Here \( a = 2, x = 5, n = 32 \)

Here 2 is the base, 5 is the logarithm. In words, you would say ‘2 to the power of 5 equals 32’. You would also say ‘the logarithm of 32, to base 2, is 5’.

Rewrite using a logarithm

\[
\begin{align*}
\text{a} & \quad 10^3 = 1000 & \text{b} & \quad 5^4 = 625 & \text{c} & \quad 2^{10} = 1024 \\
\text{a} & \quad \log_{10} 1000 = 3 & \text{b} & \quad \log_5 625 = 4 & \text{c} & \quad \log_2 1024 = 10
\end{align*}
\]

Find the value of

\[
\begin{align*}
\text{a} & \quad \log_3 81 & \text{b} & \quad \log_4 0.25 & \text{c} & \quad \log_{0.5} 4 & \text{d} & \quad \log_a (a^5)
\end{align*}
\]
1 ▶ Rewrite these exponentials as logarithms.

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>( \log_3 81 = 4 )</td>
</tr>
<tr>
<td>b</td>
<td>( \log_4 0.25 = -1 )</td>
</tr>
<tr>
<td>c</td>
<td>( \log_{0.5} 4 = -2 )</td>
</tr>
<tr>
<td>d</td>
<td>( \log_a (a^5) = 5 )</td>
</tr>
</tbody>
</table>

2 ▶ Write these logarithms in exponential form.

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>( \log_3 81 = 4 )</td>
</tr>
<tr>
<td>b</td>
<td>( \log_4 729 = 6 )</td>
</tr>
<tr>
<td>c</td>
<td>( \log_{0.5} 4 = -2 )</td>
</tr>
<tr>
<td>d</td>
<td>( \log_4 \left(\frac{1}{25}\right) = -3 )</td>
</tr>
<tr>
<td>e</td>
<td>( \log_5 1000 = 3 )</td>
</tr>
<tr>
<td>f</td>
<td>( \log_{1000} 0.01 = -2 )</td>
</tr>
</tbody>
</table>

3 ▶ Without a calculator find the value of

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>( \log_2 4 = 2 )</td>
</tr>
<tr>
<td>b</td>
<td>( \log_3 27 = 3 )</td>
</tr>
<tr>
<td>c</td>
<td>( \log_5 125 = 3 )</td>
</tr>
<tr>
<td>d</td>
<td>( \log_4 \left(\frac{1}{125}\right) = -3 )</td>
</tr>
<tr>
<td>e</td>
<td>( \log_{\sqrt{10}} 10 = 2 )</td>
</tr>
<tr>
<td>f</td>
<td>( \log_3 \sqrt{27} = 3 )</td>
</tr>
<tr>
<td>g</td>
<td>( \log_5 \sqrt{\frac{1}{3}} = \frac{1}{2} )</td>
</tr>
<tr>
<td>h</td>
<td>( \log_3 \sqrt[3]{\frac{1}{3}} = \frac{1}{3} )</td>
</tr>
</tbody>
</table>

4 ▶ Find the value of \( x \) for which

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>( \log_3 x = 4 )</td>
</tr>
<tr>
<td>b</td>
<td>( \log_6 x = 3 )</td>
</tr>
<tr>
<td>c</td>
<td>( \log_4 x = 3 )</td>
</tr>
<tr>
<td>d</td>
<td>( \log_{16} \frac{4}{3} = \frac{4}{3} )</td>
</tr>
<tr>
<td>e</td>
<td>( \log_4 \frac{2}{3} = \frac{2}{3} )</td>
</tr>
</tbody>
</table>

5 ▶ Find using your calculator

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>( \log 20 = 1.30103 )</td>
</tr>
<tr>
<td>b</td>
<td>( \log 14 = 1.14614 )</td>
</tr>
<tr>
<td>c</td>
<td>( \log 0.25 = -0.60206 )</td>
</tr>
<tr>
<td>d</td>
<td>( \log 0.3 = -0.52293 )</td>
</tr>
<tr>
<td>e</td>
<td>( \log 54.6 = 1.74004 )</td>
</tr>
</tbody>
</table>

UNDERSTAND AND USE THE LAWS OF LOGARITHMS

\( 2^5 = 32 \) and \( \log_2 32 = 5 \)

The rules of logarithms follow the rules of indices.

<table>
<thead>
<tr>
<th>EXponent (PoWerS)</th>
<th>loGAriThMs</th>
<th>laW</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c^x \times c^y = c^{x+y} )</td>
<td>( \log_c xy = \log_c x + \log_c y )</td>
<td>Multiplication Law</td>
</tr>
<tr>
<td>( c^x \div c^y = c^{x-y} )</td>
<td>( \log_c \frac{x}{y} = \log_c x - \log_c y )</td>
<td>Division Law</td>
</tr>
<tr>
<td>( (c)^q )</td>
<td>( \log_c (c^q) = q \log_c x )</td>
<td>Power Law</td>
</tr>
<tr>
<td>( \frac{1}{c} = c^{-1} )</td>
<td>( \log_c \left(\frac{1}{x}\right) = -\log_c x )</td>
<td></td>
</tr>
<tr>
<td>( c^1 = c )</td>
<td>( \log_c c = 1 )</td>
<td></td>
</tr>
<tr>
<td>( c^0 = 1 )</td>
<td>( \log_c 1 = 0 )</td>
<td></td>
</tr>
</tbody>
</table>
EXAMPLE 15

Write as a single logarithm

\[ a \quad \log_3 6 + \log_3 7 \quad b \quad \log_2 15 - \log_2 3 \quad c \quad 2 \log_5 3 + 3 \log_5 2 \quad d \quad \lg 3 - 4 \lg \left( \frac{1}{2} \right) \]

\[ a \quad \log_3 (6 \times 7) = \log_3 (42) \quad \text{Use the multiplication law} \]

\[ b \quad \log_2 (15 \div 3) = \log_2 5 \quad \text{Use the division law} \]

\[ c \quad 2 \log_5 3 = 3 \log_5 2 \]

\[ = \log_5 (3^2) = \log_5 (2^3) \]

\[ = \log_5 9 + \log_5 8 \]

\[ = \log_5 72 \quad \text{Apply the power law to both expressions} \]

\[ d \quad \lg 3 - 4 \lg \left( \frac{1}{2} \right) \]

\[ = \lg 3 - \lg \left( \frac{1}{2}^4 \right) \]

\[ = \lg \left( 3 + \frac{1}{16} \right) \]

\[ = \lg 48 \quad \text{Use the division law} \]

EXAMPLE 16

Find the value of

Find the value of

Write in terms of \( \log_a x \), \( \log_a y \) and \( \log_a z \)

\[ a \quad \log_a (x^2 y z^3) \quad b \quad \log_a \left( \frac{x}{y^3} \right) \quad c \quad \log_a \left( \frac{x \sqrt{y}}{z} \right) \quad d \quad \log_a \left( \frac{x}{a^4} \right) \]

\[ a \quad \log_a (x^2 y z^3) \]

\[ = \log_a (x^2) + \log_a (y) + 3 \log_a (z) \]

\[ = 2 \log_a (x) + \log_a (y) + 3 \log_a (z) \]

\[ b \quad \log_a \left( \frac{x}{y^3} \right) \]

\[ = \log_a (x) - \log_a (y^3) \]

\[ = \log_a (x) - 3 \log_a (y) \]

\[ c \quad \log_a \left( \frac{x \sqrt{y}}{z} \right) \]

\[ = \log_a (x \sqrt{y}) - \log_a (z) \]

\[ = \log_a (x \sqrt{y}) - \log_a (\sqrt{y}) - \log_a (z) \]

\[ = \log_a (x) + \frac{1}{2} \log_a (y) - \log_a (z) \quad \text{Use the power law } \sqrt{y} = y^{\frac{1}{2}} \]

\[ d \quad \log_a \left( \frac{x}{a^4} \right) \]

\[ = \log_a (x) - \log_a (a^4) \]
= \log_a(x) - 4 \log_a(a)
= \log_a(x) - 4
\log_a a = 1

1. Write as a single logarithm

\hspace{1cm} a \quad \log_4 8 + \log_4 8
\hspace{1cm} b \quad \log_3 3 + \log_2 2
\hspace{1cm} c \quad \log_5 27 + \log_5 3
\hspace{1cm} d \quad \log_4 24 + \log_4 15 - \log_5 3
\hspace{1cm} e \quad 2 \log_6 9 - 10 \log_6 81
\hspace{1cm} f \quad \frac{1}{2} \log_2 25 + 2 \log_2 3
\hspace{1cm} g \quad \log_9 25 + \log_9 10 - 3 \log_9 5
\hspace{1cm} h \quad 2 \log_{12} 3 + 4 \log_{12} 2
\hspace{1cm} i \quad 2 \log_{20} 20 - (\log 5 + \log 8)

2. Write in terms of \( \log_a x \), \( \log_a y \), \( \log_a z \)

\hspace{1cm} a \quad \log_a x^2 y^3 z
\hspace{1cm} b \quad \log_a \frac{x^6}{y^3}
\hspace{1cm} c \quad \log_a (xz)^2
\hspace{1cm} d \quad \log_a \frac{1}{xyz}
\hspace{1cm} e \quad \log_a \sqrt[3]{xy}
\hspace{1cm} f \quad \log_a \sqrt[2]{x^2 y^3}
\hspace{1cm} g \quad \log_a \frac{x^3 y^7}{z^3}

CHANGE THE BASE OF A LOGARITHM

Working in base \( a \), suppose that \( \log_a x = m \)
Writing this as a power
\( a^m = x \)
Taking logs to a different base \( b \)
\( \log_b (a^m) = \log_b (x) \)
Using the power law
\( m \log_b a = \log_b x \)
Writing \( m \) as \( \log_a x \)
\( \log_b x = \log_a x \times \log_b a \)
This can be written as
\( \log_a x = \frac{\log_b x}{\log_b a} \)
Using this rule, notice in particular that \( \log_a b = \frac{\log_b b}{\log_b a} \)
but \( \log_b b = 1 \)
so, \( \log_a b = \frac{1}{\log_b a} \)

Example 17

Find, to 3 significant figures, the value of \( \log_8 (11) \)

One method is to use the change of base rule
\( \log_8 11 = \frac{\log_8 11}{\log_8 8} \)
\( = 1.15 \)

Another method is to solve \( 8^x = 11 \)
Let \( x = \log_8 (11) \)
\( 8^x = 11 \)
\[ \lg (8^1) = \lg 11 \]  
Take logs to base 10 of each side

\[ x \lg 8 = \lg 11 \]  
Use the power law

\[ x = \frac{\lg 11}{\lg 8} \]  
Divide by \( \lg 8 \)

\[ x = 1.15 \text{ (3 s.f.)} \]

**EXAMPLE 18**

Solve the equation \( \log_5 x + 6 \log_5 x = 5 \)

Use change of rule, special case

\[ \log_5 x + \frac{6}{\log_5 x} = 5 \]

Let \( \log_5 (x) = y \)

\[ y + \frac{6}{y} = 5 \]

\[ y^2 + 6 = 5y \]

\[ y^2 - 5y + 6 = 0 \]

\[ (y - 3)(y - 2) = 0 \]

So \( y = 3 \) or \( y = 2 \)

\[ \log_5 x = 3 \text{ or } \log_5 x = 2 \]

\[ x = 5^3 \text{ or } x = 5^2 \]

\[ x = 125 \text{ or } x = 25 \]

**EXERCISE 8**

**SKILLS**

**EXECUTIVE FUNCTION**

1. Find, to 3 significant figures
   - a. \( \log_6 785 \)
   - b. \( \log_4 15 \)
   - c. \( \log_6 32 \)
   - d. \( \log_{12} 4 \)
   - e. \( \log_{15} \frac{1}{7} \)

2. Solve, giving your answer to 3 significant figures
   - a. \( 6^x = 15 \)
   - b. \( 9^x = 751 \)
   - c. \( 15^x = 3 \)
   - d. \( 3^x = 17.3 \)
   - e. \( 3^{2x} = 25 \)
   - f. \( 4^{3x} = 64 \)
   - g. \( 7^{3x} = 152 \)

3. Solve, giving your answer to 3 significant figures
   - a. \( \log_2 x = 8 + 9 \log_2 2 \)
   - b. \( \log_5 x + 3 \log_5 6 = 4 \)
   - c. \( \lg x + 5 \log_{10} 10 = -6 \)
   - d. \( \log_2 x + \log_4 x = 2 \)

**EXAMPLE 19**

**PROBLEM SOLVING ANALYSIS**

**SKILLS**

Solve equations of the form \( a^x = b \)

You need to be able to solve equations of the form \( a^x = b \)

Solve the equation \( 3^x = 20 \), giving your answer to 3 significant figures.

\[ 3^x = 20 \]

\[ \lg (3^x) = \lg 20 \]  
Take logs to base 10 on each side
\[ x \log 3 = \log 20 \]

Use the power law

\[ x = \frac{\log 20}{\log 3} \]

Divide by \( \log 3 \)

\[ x = \frac{1.3010...}{0.4771...} = 2.73 \text{ (3 s.f.)} \]

Or, a simpler version

\[ 3^x = 20 \]

\[ x = \log_3 20 \]

\[ x = 2.73 \]

---

**EXAMPLE 20**

Solve the equation \( 7^{x+1} = 3^{x+2} \)

\[ (x + 1) \log 7 = (x + 2) \log 3 \]

Use the power law

\[ x \log 7 + \log 7 = x \log 3 + 2 \log 3 \]

Multiply out

\[ x \log 7 - x \log 3 = 2 \log 3 - \log 7 \]

Collect \( x \) terms on left and numerical terms on right

\[ x(\log 7 - \log 3) = 2 \log 3 - \log 7 \]

Factorise

\[ x = \frac{2 \log 3 - \log 7}{\log 7 - \log 3} \]

Divide by \( \log 7 - \log 3 \)

\[ x = 0.297 \text{ (3 s.f.)} \]

---

**EXAMPLE 21**

Solve the equation \( 5^{2x} + 7(5^x) - 30 = 0 \), giving your answer to 2 decimal places.

Let \( y = 5^x \)

\[ y^2 + 7y - 30 = 0 \]

So \( (y + 10)(y - 3) = 0 \)

So \( y = -10 \) or \( y = 3 \)

If \( y = -10 \), \( 5^x = -10 \), has no solution

\[ 5^x \] cannot be negative

If \( y = 3 \), \( 5^x = 3 \)

\[ \log (5^x) = \log 3 \]

\[ x \log 5 = \log 3 \]

\[ x = \frac{\log 3}{\log 5} \]

\[ x = 0.683 \text{ (3 s.f.)} \]
1 ▶ Solve, giving your answer to 3 significant figures
   a. $4^x = 12$
   b. $5^x = 20$
   c. $15^x = 175$
   d. $7^x = \frac{1}{4}$
   e. $4^{x+1} = 30$
   f. $7^{2x+1} = 36$
   g. $4^{x+1} = 8^{x+1}$
   h. $2^{3y-2} = 3^{2y+5}$
   i. $7^{2x+6} = 11^{3x-2}$
   j. $3^{4-3x} = 4^{x+5}$

2 ▶ Solve, giving your answer to 3 significant figures
   a. $4^{2x} + 4^x - 12 = 0$
   b. $6^{2x} - 10(6^x) + 8 = 0$
   c. $5^{2x} - 6(5^x) - 7 = 0$
   d. $4^{2x+1} + 7(4^x) - 15 = 0$
   e. $2^{2x} + 3^{2x} = 4$
   f. $3^{2x-1} = 26(3^x) + 9$
EXAM PRACTICE: CHAPTER 1

1. Simplify \(\sqrt{32} + \sqrt{18}\), giving your answer in the form \(p\sqrt{2}\), where \(p\) is an integer. [2]

2. Simplify \(\frac{\sqrt{32} + \sqrt{18}}{3 + \sqrt{2}}\) giving your answer in the form \(a\sqrt{2} + b\), where \(a\) and \(b\) are integers. [3]

3. a. Expand and simplify \((7 + \sqrt{5})(3 - \sqrt{5})\) [2]
   b. Express \(\frac{7 + \sqrt{5}}{3 + \sqrt{5}}\) in the form \(a + b\sqrt{5}\), where \(a\) and \(b\) are integers. [2]

4. Write \(\sqrt{75} - \sqrt{27}\) in the form \(k\sqrt{x}\), where \(k\) and \(x\) are integers. [2]

5. A rectangle \(A\) has a length of \((1 + \sqrt{5})\) cm and an area of \(\sqrt{80}\) cm\(^2\). Calculate the width of \(A\) in cm, giving your answer in the form \(a + b\sqrt{5}\), where \(a\) and \(b\) are integers to be found. [3]

6. Sketch the graph of \(y = 8^x\), showing the coordinates of any points at which the graph crosses the axes. [3]

7. Solve the equation \(8^{2x} - 4(8^x) = 3\), giving your answer to 3 significant figures. [3]

8. a. Given that \(y = 6x^2\), Show that \(\log_b y = 1 + 2 \log_b x\) [2]
   b. Hence, or otherwise, solve the equation \(1 + 2 \log_b x = \log_3(28x - 9)\), giving your answer to 3 significant figures. [3]

9. Find the values of \(x\) such that: \(2 \log_3 x - \log_3(x - 2) - 2 = 0\) [2]

10. Find the values of \(y\) such that: \(\frac{\log_2 32 + \log_2 16}{\log_2 y} - \log_2 y = 0\) [3]

11. Given that \(\log_b y + 3 \log_b 2 = 5\), express \(y\) in terms of \(b\) in its simplest form. [2]

12. Solve \(5^{2x} = 12(5^x) - 35\) [4]

13. Find, giving your answer to 3 significant figures where appropriate, the value of \(x\) for which \(5^x = 10\) [3]

14. Given that \(\log_3(3b + 1) - \log_3(a - 2) = -1\), \(a > 2\), express \(b\) in terms of \(a\). [2]

15. Solve \(3^{3x - 2} = 3^3\) [4]

16. Solve \(25^x + 5^{x+1} = 24\), giving your answer to 3 significant figures. [3]
17 Given that \( \log_2 x = p \), find, in terms of \( p \), the simplest form of

a \( \log_2 (16x) \). [1]

b \( \log_2 \left( \frac{x^4}{2} \right) \). [1]

c Hence, or otherwise, solve

\[ \log_2 (16x) - \log_2 \left( \frac{x^4}{2} \right) = \frac{1}{2} \]

give your answer in its simplest surd form. [3]

18 Solve \( \log_3 t + \log_3 5 = \log_3 (2t + 3) \). [3]

19 a Draw the graph \( y = 2 + \ln x \) for \( 0.1 \leq x \leq 4 \). [2]

b Use your graph to estimate, to 2 significant figures, the solution to \( \ln x = 0.5 \), showing clearly your method. [2]

c By drawing a suitable line, estimate to 2 significant figures, the solution of the equation \( x = e^{x^2} \). [2]
**CHAPTER SUMMARY: CHAPTER 1**

- You can simplify expressions by using the power (indices) laws.

<table>
<thead>
<tr>
<th>Expression</th>
<th>Rule</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c^x \times c^y = c^{x+y}$</td>
<td></td>
</tr>
<tr>
<td>$c^x \div c^y = c^{x-y}$</td>
<td></td>
</tr>
<tr>
<td>$(c^p)^q = c^{pq}$</td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{c} = c^{-1}$</td>
<td></td>
</tr>
<tr>
<td>$c^1 = c$</td>
<td></td>
</tr>
<tr>
<td>$c^0 = 1$</td>
<td></td>
</tr>
</tbody>
</table>

- You can manipulate surds using these rules:

<table>
<thead>
<tr>
<th>Rule</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sqrt{ab} = \sqrt{a} \times \sqrt{b}$</td>
<td>$\sqrt{8} = \sqrt{4} \times \sqrt{2}$</td>
</tr>
<tr>
<td>$\sqrt[3]{\frac{a}{b}} = \frac{\sqrt[3]{a}}{\sqrt[3]{b}}$</td>
<td>$\sqrt[3]{\frac{8}{2}} = \frac{\sqrt[3]{8}}{\sqrt[3]{2}}$</td>
</tr>
</tbody>
</table>

- The rules for rationalising surds are:

<table>
<thead>
<tr>
<th>Rule</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>If you have a fraction in the form $\frac{1}{\sqrt{a}}$ then multiply top and bottom by $\sqrt{a}$</td>
<td>$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$</td>
</tr>
<tr>
<td>If you have a fraction in the form $\frac{1}{1 + \sqrt{a}}$ then multiply top and bottom by $(1 - \sqrt{a})$</td>
<td>$\frac{1}{1 + \sqrt{2}} = \frac{1 - \sqrt{2}}{-1}$</td>
</tr>
<tr>
<td>If you have a fraction in the form $\frac{1}{1 - \sqrt{a}}$ then multiply top and bottom by $(1 + \sqrt{a})$</td>
<td>$\frac{1}{1 - \sqrt{2}} = \frac{1 + \sqrt{2}}{1}$</td>
</tr>
</tbody>
</table>

- $\log_a{n} = x$ can be rewritten as $a^x = n$ where $a$ is the base of the logarithm.

- The laws of logarithms are:

<table>
<thead>
<tr>
<th>Rule</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\log_a{xy} = \log_a{x} + \log_a{y}$</td>
<td>$\log_2{8\times4} = \log_2{8} + \log_2{4}$</td>
</tr>
<tr>
<td>$\log_a{\frac{x}{y}} = \log_a{x} - \log_a{y}$</td>
<td>$\log_2{\frac{8}{4}} = \log_2{8} - \log_2{4}$</td>
</tr>
<tr>
<td>$\log_a{(x^q)} = q\log_a{x}$</td>
<td>$\log_2{(2^3)} = 3\log_2{2}$</td>
</tr>
<tr>
<td>$\log_a{\frac{1}{x}} = -\log_a{x}$</td>
<td>$\log_2{\frac{1}{2}} = -\log_2{2}$</td>
</tr>
<tr>
<td>$\log_a{c} = 1$</td>
<td>$\log_2{2} = 1$</td>
</tr>
<tr>
<td>$\log_a{1} = 0$</td>
<td>$\log_2{1} = 0$</td>
</tr>
</tbody>
</table>

- The change of base rule for logarithms can be written as $\log_a{x} = \frac{\log_b{x}}{\log_b{a}}$

- From the change of base you can derive $\log_a{b} = \frac{1}{\log_b{a}}$

- The natural logarithm is defined as: $\log_e{x} \equiv \ln{x}$

- The graph of $y = e^x$ is shown below.

- The graph of $y = \ln{x}$ is shown below.