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• Content is mapped to the specification to provide comprehensive coverage
• Exam practice throughout, with full answers included in the back of the book
• Signposted transferable skills
• Reviewed by a language specialist to ensure the book is written in a clear and accessible style
• Glossary of key mathematics terminology
• eBook included
• Online Teacher Resource Pack (ISBN 9780435191214) also available, providing further planning, teaching and assessment support

For Pearson Edexcel International GCSE (9–1) Further Pure Mathematics (4PM1) Higher Tier for first teaching 2017.
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• If you have any questions about accessing your ActiveBook, please contact our ActiveLearn support site at www.pearsonactivelearn.com/support
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ABOUT THIS BOOK

This book is written for students following the Edexcel International GCSE (9–1) Further Pure Maths specification and covers both years of the course. The specification and sample assessment materials for Further Pure Maths can be found on the Pearson Qualifications website.

In each chapter, there are concise explanations and worked examples, plus numerous exercises that will help you build up confidence.

There are also exam practice questions and a chapter summary to help with exam preparation. Answers to all exercises are included at the back of the book as well as a glossary of Maths-specific terminology.

Points of Interest put the maths you are about to learn in a real-world context.

Learning Objectives show what you will learn in each chapter.

Hint boxes give you tips and reminders.
Examples provide a clear, instructional framework. The blue highlighted text gives further explanation of the method.

Language is graded for speakers of English as an additional language (EAL), with advanced Maths-specific terminology highlighted and defined in the glossary at the back of the book.

Key Points boxes summarise the essentials.

Exam Practice tests cover the whole chapter and provide quick, effective feedback on your progress.

Chapter Summaries state the most important points of each chapter.
ASSESSMENT OVERVIEW

The following tables give an overview of the assessment for the Edexcel International GCSE in Further Pure Mathematics.

We recommend that you study this information closely to help ensure that you are fully prepared for this course and know exactly what to expect in the assessment.

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<th>TIME</th>
<th>AVAILABILITY</th>
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<tbody>
<tr>
<td>Written examination paper</td>
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<td>100</td>
<td>2 hours</td>
<td>January and June examination series</td>
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<tr>
<td>Paper code 4PM1/01C</td>
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CONTENT SUMMARY

- Number
- Algebra and calculus
- Geometry and calculus

ASSESSMENT

- Each paper will consist of around 11 questions with varying mark allocations per questions, which will be stated on the paper
- Each paper will contain questions from any part of the specification content, and the solution of any questions may require knowledge of more than one section of the specification content
- A formulae sheet will be included in the written examinations
- A calculator may be used in the examinations

ASSESSMENT OBJECTIVES AND WEIGHTINGS

<table>
<thead>
<tr>
<th>ASSESSMENT OBJECTIVE</th>
<th>DESCRIPTION</th>
<th>% IN INTERNATIONAL GCSE</th>
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<tbody>
<tr>
<td>AO1</td>
<td>Demonstrate a confident knowledge of the techniques of pure mathematics required in the specification</td>
<td>30%–40%</td>
</tr>
<tr>
<td>AO2</td>
<td>Apply a knowledge of mathematics to the solutions of problems for which an immediate method of solution is not available and which may involve knowledge of more than one topic in the specification</td>
<td>20%–30%</td>
</tr>
<tr>
<td>AO3</td>
<td>Write clear and accurate mathematical solutions</td>
<td>35%–50%</td>
</tr>
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<th>ASSESSMENT OBJECTIVE</th>
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<td>AO1</td>
</tr>
<tr>
<td>Paper 1</td>
<td>15%–20%</td>
</tr>
<tr>
<td>Paper 2</td>
<td>15%–20%</td>
</tr>
<tr>
<td>Total for International GCSE</td>
<td>30%–40%</td>
</tr>
</tbody>
</table>

ASSESSMENT SUMMARY

The Edexcel International GCSE in Further Pure Mathematics requires students to demonstrate application and understanding of the following topics.

Number
• Use numerical skills in a purely mathematical way and in real-life situations.

Algebra and calculus
• Use algebra and calculus to set up and solve problems.
• Develop competence and confidence when manipulating mathematical expressions.
• Construct and use graphs in a range of situations.

Geometry and trigonometry
• Understand the properties of shapes, angles and transformations.
• Use vectors and rates of change to model situations.
• Use coordinate geometry.
• Use trigonometry.

Students will be expected to have a thorough knowledge of the content common to the Pearson Edexcel International GCSE in Mathematics (Specification A) (Higher Tier) or Pearson Edexcel International GCSE in Mathematics (Specification B).

Questions may be set which assumes knowledge of some topics covered in these specifications, however knowledge of statistics and matrices will not be required.

Students will be expected to carry out arithmetic and algebraic manipulation, such as being able to change the subject of a formula and evaluate numerically the value of any variable in a formula, given the values of the other variables. The use and notation of set theory will be adopted where appropriate.

CALCULATORS

Students will be expected to have access to a suitable electronic calculator for all examination papers. The electronic calculator should have these functions as a minimum:
+ , − , × , ÷ , π , x² , √x , 1/x , x³ , ln x, eˣ, sine, cosine and tangent and their inverses in degrees and decimals of a degree or radians.

Prohibitions
Calculators with any of the following facilities are prohibited in all examinations:
databanks
retrieval of text or formulae
QWERTY keyboards
built-in symbolic algebra manipulations
symbolic differentiation or integration.
FORMULAE SHEET

These formulae will be provided for you during the examination.

MENSURATION

Surface area of sphere = $4\pi r^2$
Curved surface area of cone = $\pi r \times \text{slant height}$
Volume of sphere = $\frac{4}{3}\pi r^3$

SERIES

Arithmetic series
Sum to $n$ terms $S_n = \frac{n}{2}[2a + (n - 1)d]$

Geometric series
Sum to $n$ terms, $S_n = \frac{a(1 - r^n)}{(1 - r)}$
Sum to infinity, $S_\infty = \frac{a}{1 - r}$ \hspace{1cm} |r| < 1

Binomial series
$$(1 + x)^n = 1 + nx + \frac{n(n - 1)}{2!}x^2 + \ldots + \frac{n(n - 1)\ldots(n - r + 1)}{r!}x^r + \ldots$$ for $|x| < 1, n \in \mathbb{Q}$

CALCULUS

Quotient rule (differentiation)
$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

TRIGONOMETRY

Cosine rule
In triangle $ABC$: $a^2 = b^2 + c^2 - 2bc \cos A$

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

$$\sin(A + B) = \sin A \cos B + \cos A \sin B \hspace{1cm} \sin(A - B) = \sin A \cos B - \cos A \sin B$$
$$\cos(A + B) = \cos A \cos B - \sin A \sin B \hspace{1cm} \cos(A - B) = \cos A \cos B + \sin A \sin B$$
$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} \hspace{1cm} \tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

LOGARITHMS

$$\log_a x = \frac{\log_b x}{\log_b a}$$
FORMULAE TO KNOW

The following are formulae that you are expected to know and remember during the examination. These formulae will not be provided for you. Note that this list is not exhaustive.

LOGARITHMIC FUNCTIONS AND INDICES

\[
\log_a xy = \log_a x + \log_a y
\]

\[
\log_a \frac{x}{y} = \log_a x - \log_a y
\]

\[
\log_a x^k = k \log_a x
\]

\[
\log_a \frac{1}{x} = -\log_a x
\]

\[
\log_a a = 1
\]

\[
\log_a 1 = 0
\]

\[
\log_a b = \frac{1}{\log_b a}
\]

QUADRATIC EQUATIONS

\[ax^2 + bx + c = 0\] has roots given by

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

When the roots of \[ax^2 + bx + c = 0\] are \(\alpha\) and \(\beta\) then \(\alpha + \beta = \frac{-b}{a}\) and \(\alpha \beta = \frac{c}{a}\)

and the equation can be written \(x^2 - (\alpha + \beta)x + \alpha \beta = 0\)

SERIES

Arithmetic series: \(n\)th term \(= l = a + (n - 1)d\)

Geometric series: \(n\)th term \(= ar^{n-1}\)

COORDINATE GEOMETRY

The gradient of the line joining two points \((x_1, y_1)\) and \((x_2, y_2)\) is \(\frac{y_2 - y_1}{x_2 - x_1}\)

The distance \(d\) between two points \((x_1, y_1)\) and \((x_2, y_2)\) is given by

\[d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2\]

The coordinates of the point dividing the line joining \((x_1, y_1)\) and \((x_2, y_2)\) in the ratio \(m : n\) are

\[
\left(\frac{nx_1 - mx_2}{m + n}, \frac{ny_1 - my_2}{m + n}\right)
\]
CALCULUS

Differentiation:

<table>
<thead>
<tr>
<th>Function</th>
<th>Derivative</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^n$</td>
<td>$nx^{n-1}$</td>
</tr>
<tr>
<td>$\sin ax$</td>
<td>$a \cos ax$</td>
</tr>
<tr>
<td>$\cos ax$</td>
<td>$-a \sin ax$</td>
</tr>
<tr>
<td>$e^{ax}$</td>
<td>$ae^{ax}$</td>
</tr>
<tr>
<td>$f(x)g(x)$</td>
<td>$f'(x)g(x) + f(x)g'(x)$</td>
</tr>
<tr>
<td>$f(g(x))$</td>
<td>$f'(g(x))g'(x)$</td>
</tr>
</tbody>
</table>

Integration:

<table>
<thead>
<tr>
<th>Function</th>
<th>Derivative</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^n$</td>
<td>$\frac{1}{n+1}x^{n+1} + c$ $n \neq -1$</td>
</tr>
<tr>
<td>$\sin ax$</td>
<td>$\frac{1}{a}\cos ax + c$</td>
</tr>
<tr>
<td>$\cos ax$</td>
<td>$\frac{1}{a}\sin ax + c$</td>
</tr>
<tr>
<td>$e^{ax}$</td>
<td>$\frac{1}{a}e^{ax} + c$</td>
</tr>
</tbody>
</table>

AREA AND VOLUME

Area between a curve and the $x$-axis $= \int_a^b y \, dx$, $y \geq 0$

Area between a curve and the $x$-axis $= \int_a^b y \, dx$, $y < 0$

Area between a curve and the $y$-axis $= \int_c^d x \, dy$, $x \geq 0$

Area between $g(x)$ and $f(x) = \int_a^b |g(x) - f(x)| \, dx$

Volume of revolution $= \int_a^b \pi y^2 \, dx$ or $\int_c^d \pi x^2 \, dy$

TRIGONOMETRY

Radian measure:

- Length of arc $= r \theta$
- Area of sector $= \frac{1}{2} r^2 \theta$

In a triangle $ABC$:

- $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$
- $\cos^2 \theta + \sin^2 \theta = 1$
- Area of a triangle $= \frac{1}{2} ab \sin C$
The Richter scale, which describes the energy released by an earthquake, uses the base 10 logarithm as its unit. An earthquake of magnitude 9 is 10 times as powerful as one of magnitude 8, and 100 000 times as powerful as one of magnitude 4.

The devastating 2004 earthquake in the Indian Ocean had a magnitude of 9. Thankfully, such events are rare. The most common earthquakes, which occur over 100 000 times a year, are magnitude 2 to 3, so humans can hardly feel them.

LEARNING OBJECTIVES

• Write a number exactly using surds
• Rationalise the denominator of a surd
• Be familiar with the functions \(a^x\) and \(\log_b x\) and recognise the shapes of their graphs
• Be familiar with functions including \(e^x\) and similar terms, and use them in graphs
• Use graphs of functions to solve equations
• Rewrite expressions including powers using logarithms instead
• Understand and use the laws of logarithms
• Change the base of a logarithm
• Solve equations of the form \(a^x = b\)

STARTER ACTIVITIES

1 ▶ Simplify
   a \(y^6 \times y^5\)   b \(2q^3 \times 4q^4\)   c \(3k^2 \times 3k^7 \times 3k^{-3}\)
   d \((x^2)^4\)   e \((a^3)^2 + a^3\)   f \(64x^4y^6 \div 4xy^2\)

2 ▶ Simplify
   a \((m^3)^{\frac{1}{3}}\)   b \(3p^{\frac{1}{2}} \times p^{\frac{3}{2}}\)   c \(28c^{\frac{1}{2}} \div 7c^{\frac{3}{2}}\)
   d \(6b^{\frac{1}{2}} \times 3b^{-\frac{1}{2}}\)   e \(27p^{\frac{3}{2}} \div 9p^{\frac{1}{2}}\)   f \(5y^6 \times 3y^{-7}\)

3 ▶ Evaluate
   a \(16^{\frac{1}{2}}\)   b \(125^{\frac{1}{3}}\)   c \(8^{-2}\)   d \((-2)^{-3}\)
   e \(\left(\frac{6}{7}\right)^0\)   f \(81^{-4}\)   g \(\left(\frac{9}{16}\right)^{-\frac{3}{2}}\)
WRITE A NUMBER EXACTLY USING A SURD

A surd is a number that cannot be simplified to remove a square root (or a cube root, fourth root etc). Surds are irrational numbers.

<table>
<thead>
<tr>
<th>NUMBER</th>
<th>DECIMAL</th>
<th>IS IT A SURD?</th>
</tr>
</thead>
<tbody>
<tr>
<td>√1</td>
<td>1</td>
<td>No</td>
</tr>
<tr>
<td>√2</td>
<td>1.414213...</td>
<td>Yes</td>
</tr>
<tr>
<td>√4</td>
<td>2</td>
<td>No</td>
</tr>
<tr>
<td>1/√4</td>
<td>0.5</td>
<td>No</td>
</tr>
<tr>
<td>2/√3</td>
<td>0.816496...</td>
<td>Yes</td>
</tr>
</tbody>
</table>

You can manipulate surds using these rules:

\[ \sqrt{ab} = \sqrt{a} \times \sqrt{b} \]

\[ \sqrt{a} \div \sqrt{b} = \frac{\sqrt{a}}{\sqrt{b}} \]

**EXAMPLE 1**

**SKILLS CRITICAL THINKING**

Simplify

\[ a \quad \sqrt{12} \quad b \quad \frac{\sqrt{20}}{2} \quad c \quad 5\sqrt{6} - 2\sqrt{24} + \sqrt{294} \]

\[ a \quad \sqrt{12} \]

\[ = \sqrt{4 \times 3} \]

\[ = \sqrt{4} \times \sqrt{3} \]

\[ = 2\sqrt{3} \]

\[ b \quad \frac{\sqrt{20}}{2} \]

\[ = \frac{\sqrt{4 \times 5}}{2} \]

\[ = \frac{2 \times \sqrt{5}}{2} \]

\[ = \sqrt{5} \]

\[ c \quad 5\sqrt{6} - 2\sqrt{24} + \sqrt{294} \]

\[ = 5\sqrt{6} - 2\sqrt{6 \times 4} + \sqrt{6 \times 49} \]

\[ = \sqrt{6}(5 - 2) + \sqrt{49} \]

\[ = \sqrt{6}(5 - 2 \times 2 + 7) \]

\[ = 8\sqrt{6} \]

**HINT**
An irrational number is a number that cannot be expressed as a fraction, for example \(\pi\).
EXERCISE 1

1. Simplify without using a calculator
   a) $\sqrt{18}$
   b) $\sqrt{50}$
   c) $\sqrt{125}$
   d) $\sqrt{128}$
   e) $\sqrt{132}$
   f) $\sqrt{8625}$

2. Simplify without using a calculator
   a) $\frac{\sqrt{60}}{2}$
   b) $\frac{\sqrt{135}}{2}$
   c) $\frac{\sqrt{128}}{8}$
   d) $\frac{\sqrt{68}}{4}$
   e) $\frac{\sqrt{96}}{6}$

3. Simplify without using a calculator
   a) $6\sqrt{3} - 2\sqrt{3}$
   b) $7\sqrt{3} - 12 + \sqrt{48}$
   c) $\sqrt{112} + 2\sqrt{172} - \sqrt{63}$
   d) $6\sqrt{48} - 3\sqrt{12} + 2\sqrt{27}$
   e) $3\sqrt{578} - \sqrt{162} + 4\sqrt{32}$
   f) $2\sqrt{5} \times 3\sqrt{5}$
   g) $6\sqrt{7} \times 4\sqrt{7}$
   h) $4\sqrt{8} \times 6\sqrt{8}$

4. Simplify without using a calculator
   a) $6(\sqrt{4} - \sqrt{12})$
   b) $9(\sqrt{6} - \sqrt{29})$
   c) $4(1 + \sqrt{3}) + 3\sqrt{3} + 2\sqrt{3}$
   d) $3(\sqrt{2} - \sqrt{7}) - 5(\sqrt{2} + \sqrt{7})$
   e) $(4 + \sqrt{3})\sqrt{4 - \sqrt{3}}$
   f) $(\sqrt{2} - \sqrt{6})\sqrt{7} - 2\sqrt{6}$
   g) $\sqrt{8} + \sqrt{5}\sqrt{8 - \sqrt{5}}$

5. A garden is $30\sqrt{3}$ m long and $8\sqrt{8}$ m wide. The garden is covered in grass except for a small rectangular pond which is $2\sqrt{2}$ m long and $6\sqrt{m}$ wide. Express the area of the pond as a percentage of the area of the garden.

6. Find the value of $2p^2 - 3pq$ when $p = \sqrt{2} + 3$ and $q = \sqrt{2} - 2$

HINT
In the denominator, the multiplication gives the difference of two squares, with the result $a^2 - b^2$, which means the surd disappears.

RATIONALISE THE DENOMINATOR OF A SURD

Rationalising the denominator of a surd means removing a root from the denominator of a fraction. You will usually need to rationalise the denominator when you are asked to simplify it.

The rules for rationalising the denominator of a surd are:
- For fractions in the form $\frac{1}{\sqrt{a}}$, multiply the numerator and denominator by $\sqrt{a}$
- For fractions in the form $\frac{1}{a + \sqrt{b}}$, multiply the numerator and denominator by $a - \sqrt{b}$
- For fractions in the form $\frac{1}{a - \sqrt{b}}$, multiply the numerator and denominator by $a + \sqrt{b}$
EXAMPLE 2

Rationalise the denominator of

\[
\begin{align*}
\text{a) } & \quad \frac{1}{\sqrt{3}} \\
\text{b) } & \quad \frac{1}{3 + \sqrt{2}} \\
\text{c) } & \quad \frac{\sqrt{5} + \sqrt{2}}{\sqrt{5} - \sqrt{2}}
\end{align*}
\]

\[
\begin{align*}
\text{a) } & \quad \frac{1}{\sqrt{3}} \\
& = \frac{1 \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}} \\
& = \frac{\sqrt{3}}{3} \\
& \text{Multiply the top and bottom by } \sqrt{3}
\end{align*}
\]
\[
\begin{align*}
\text{b) } & \quad \frac{1}{3 + \sqrt{2}} \\
& = \frac{1 \times (3 - \sqrt{2})}{(3 + \sqrt{2})(3 - \sqrt{2})} \\
& = \frac{3 - \sqrt{2}}{9 - 3\sqrt{2} + 3\sqrt{2} - 2} \\
& = \frac{3 - \sqrt{2}}{7} \\
& \text{Multiply the top and bottom by } 3 - \sqrt{2}
\end{align*}
\]
\[
\begin{align*}
\text{c) } & \quad \frac{\sqrt{5} + \sqrt{2}}{\sqrt{5} - \sqrt{2}} \\
& = \frac{(\sqrt{5} + \sqrt{2})(\sqrt{5} + \sqrt{2})}{(\sqrt{5} - \sqrt{2})(\sqrt{5} + \sqrt{2})} \\
& = \frac{5 + \sqrt{2} + \sqrt{2} + 2}{5 - 2} \\
& = \frac{7 + 2\sqrt{10}}{3} \\
& \text{Multiply the top and bottom by } \sqrt{5} + \sqrt{2}
\end{align*}
\]

\[
\begin{align*}
\text{EXERCISE 2} \\
\text{SKILLS} \\
\text{EXECUTIVE FUNCTION}
\end{align*}
\]

1 ▶ Rationalise

\[
\begin{align*}
\text{a) } & \quad \frac{1}{\sqrt{3}} \\
\text{b) } & \quad \frac{1}{\sqrt{7}} \\
\text{c) } & \quad \frac{2}{\sqrt{3}} \\
\text{d) } & \quad \frac{\sqrt{6}}{\sqrt{3}} \\
\text{e) } & \quad \frac{12}{\sqrt{3}} \\
\text{f) } & \quad \frac{3\sqrt{5}}{\sqrt{3}} \\
\text{g) } & \quad \frac{9\sqrt{12}}{2\sqrt{18}} \\
\text{h) } & \quad \frac{1}{2 - \sqrt{3}}
\end{align*}
\]

2 ▶ Rationalise

\[
\begin{align*}
\text{a) } & \quad \frac{\sqrt{6}}{\sqrt{3} + \sqrt{6}} \\
\text{b) } & \quad \frac{2 + \sqrt{3}}{2 - \sqrt{3}} \\
\text{c) } & \quad \frac{\sqrt{2} - \sqrt{3}}{\sqrt{2} + \sqrt{3}} \\
\text{d) } & \quad \frac{4\sqrt{2} - 2\sqrt{3}}{\sqrt{2} + \sqrt{3}} \\
\text{e) } & \quad \frac{\sqrt{2} + 2\sqrt{5}}{\sqrt{5} - \sqrt{2}} \\
\text{f) } & \quad \frac{\sqrt{7} + \sqrt{3}}{\sqrt{7} - \sqrt{3}} \\
\text{g) } & \quad \frac{\sqrt{11} + 2\sqrt{5}}{\sqrt{11} + 3\sqrt{5}} \\
\text{h) } & \quad \frac{2\sqrt{5} - 3\sqrt{7}}{5\sqrt{6} + 4\sqrt{2}} \\
\text{i) } & \quad \frac{2 + \sqrt{10}}{\sqrt{2} + \sqrt{5}} \\
\text{j) } & \quad \frac{ab}{a\sqrt{b} - b\sqrt{a}} \\
\text{k) } & \quad \frac{a - b}{a\sqrt{b} - b\sqrt{a}}
\end{align*}
\]
BE FAMILIAR WITH THE FUNCTIONS $a^x$ AND $\log_a x$ AND RECOGNISE THE SHAPES OF THEIR GRAPHS

You need to be familiar with functions in the form $y = a^x$ where $a > 0$

Look at a table of values for $y = 2^x$

<table>
<thead>
<tr>
<th>$x$</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>$\frac{1}{8}$</td>
<td>$\frac{1}{4}$</td>
<td>$\frac{1}{2}$</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>8</td>
</tr>
</tbody>
</table>

**Note:** $2^0 = 1$

In fact $a^0$ is always equal to 1 if $a$ is positive and $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$ a negative index turns the number into its reciprocal

The graph of $y = 2^x$ looks like this:

![Graph of $y = 2^x$]

**Note:** the $x$-axis is an asymptote to the curve.

Other graphs of the type $y = a^x$ have similar shapes, always passing through (0, 1).

**EXAMPLE 3**

**SKILLS ANALYSIS**

**a** On the same axes, sketch the graphs of $y = 3^x$, $y = 2^x$ and $y = 1.5^x$

**b** On another set of axes, sketch the graphs of $y = \left(\frac{1}{2}\right)^x$ and $y = 2^x$

**a** For all three graphs, $y = 1$ when $x = 0$

When $x > 0$, $3^x > 2^x > 1.5^x$

When $x < 0$, $3^x < 2^x < 1.5^x$

Work out the relative positions of the graphs
b \( \frac{1}{2} = 2^{-1} \)

so, \( y = \left( \frac{1}{2} \right)^x \) is the same as \( y = (2^{-1})^x = 2^{-x} \)

Therefore the graph of \( y = \left( \frac{1}{2} \right)^x \) is a reflection in the y-axis of the graph of \( y = 2^x \)

If you compare the graphs of \( y = 2^x \) and \( y = \log_2 x \) you see the following relationship:

On the same set of axes sketch the graphs \( y = \log_2 x \) and \( y = \log_5 x \)

Note:
For both graphs \( y = 0 \) when \( x = 1 \), since \( \log_a 1 = 0 \) for every value of \( a \).

\( \log_2 2 = 1 \) so \( y = \log_2 x \) passes through (2, 1)

and \( \log_5 5 = 1 \) so \( y = \log_5 x \) passes through (5, 1)

1. On the same set of axes sketch the graphs of
   a. \( y = 5^x \)
   b. \( y = 7^x \)
   c. \( y = \left( \frac{1}{3} \right)^x \)
2 ▶ On the same set of axes sketch the graphs of
   a  \( y = \log_5 x \)  
   b  \( y = \log_7 x \)  
   c  Write down the coordinates of the point of intersection of these two graphs.

3 ▶ On the same set of axes sketch the graphs of
   a  \( y = 3^x \)  
   b  \( y = \log_3 x \)  

4 ▶ On the same set of axes sketch the graphs of
   a  \( y = \log_3 x \)  
   b  \( y = \log_5 x \)  
   c  \( y = \log_{0.5} x \)  
   d  \( y = \log_{0.25} x \)  

**BE FAMILIAR WITH EXPRESSIONS OF THE TYPE \( e^x \) AND USE THEM IN GRAPHS**

Consider this example: Zainab opens an account with $1.00. The account pays 100% interest per year. If the interest is credited once, at the end of the year, her account will contain $2.00. How much will it contain after a year if the interest is calculated and credited more frequently? Let us investigate this more thoroughly.

<table>
<thead>
<tr>
<th>HOW OFTEN INTEREST IS CREDITED INTO THE ACCOUNT</th>
<th>VALUE OF ACCOUNT AFTER 1 YEAR ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yearly</td>
<td>( (1 + \frac{1}{1}) ) = 2</td>
</tr>
<tr>
<td>Semi-annually</td>
<td>( (1 + \frac{1}{2}) ) = 2.25</td>
</tr>
<tr>
<td>Quarterly</td>
<td>( (1 + \frac{1}{4}) ) = 2.441406...</td>
</tr>
<tr>
<td>Monthly</td>
<td>( (1 + \frac{1}{12}) ) = 2.61303529...</td>
</tr>
<tr>
<td>Weekly</td>
<td>( (1 + \frac{1}{52}) ) = 2.69259695...</td>
</tr>
<tr>
<td>Daily</td>
<td>( (1 + \frac{1}{365}) ) = 2.71456748...</td>
</tr>
<tr>
<td>Hourly</td>
<td>( (1 + \frac{1}{8760}) ) = 2.71812669...</td>
</tr>
<tr>
<td>Every minute</td>
<td>( (1 + \frac{1}{525 600}) ) = 2.7182154...</td>
</tr>
<tr>
<td>Every second</td>
<td>( (1 + \frac{1}{31 536 000}) ) = 2.71828247...</td>
</tr>
</tbody>
</table>
The amount in her account gets bigger and bigger the more often the interest is compounded, but the rate of growth slows. As the number of compounds increases, the calculated value appears to be approaching a fixed value. This value gets closer and closer to a fixed value of 2.71828247254…….This number is called ‘e’.

The number e is called a natural exponential because it arises naturally in mathematics and has numerous real life applications.

Draw the graphs of the exponential functions.

\( y = e^x \)

\[
\begin{array}{c|c|c|c|c|c|c}
 x & -2 & -1 & 0 & 1 & 2 & 3 \\
 e^x & 0.14 & 0.37 & 1 & 2.7 & 7.4 & 20 \\
\end{array}
\]

\( y = e^{-x} \)

\[
\begin{array}{c|c|c|c|c|c|c}
 x & -4 & -3 & -2 & -1 & 0 & 1 \\
 e^{-x} & 55 & 20 & 7.4 & 2.7 & 1 & 0.37 \\
\end{array}
\]

Draw the graphs of these exponential functions.

a \( y = e^{2x} \)

\[
\begin{array}{c|c|c|c|c|c|c}
 x & -2 & -1 & 0 & 1 & 2 \\
 e^{2x} & 0.02 & 0.1 & 1 & 7.4 & 55 \\
\end{array}
\]

b \( y = 10e^{-x} \)

\[
\begin{array}{c|c|c|c|c|c|c}
 x & -2 & -1 & 0 & 1 & 2 \\
 10e^{-x} & 73 & 27 & 10 & 3.7 & 1.4 \\
\end{array}
\]
On pages 7–9 you saw the connection between $y = \log_a x$ and $y = a^x$. The function $y = \log_e x$ is particularly important in mathematics and so it has a special notation:

$$\log_e x \equiv \ln x$$

Your calculator should have a special button for evaluating $\ln x$.

**EXAMPLE 8**

Solve these equations.

\[
\begin{align*}
\text{a} & \quad e^x = 3 \\
\text{b} & \quad \ln x = 4
\end{align*}
\]

\[
\begin{align*}
\text{a} & \quad \text{When } e^x = 3 & \quad \text{b} & \quad \text{When } \ln x = 4 \\
& \quad x = \ln 3 & \quad & \quad x = e^4
\end{align*}
\]

As you can see, the inverse of $e^x$ is $\ln x$ (and vice versa).

**EXAMPLE 9**

Sketch these graphs on the same set of axes.

\[
\begin{align*}
\text{a} & \quad y = \ln x \\
\text{b} & \quad y = \ln(3 - x) \\
\text{c} & \quad y = 3 + \ln(2x)
\end{align*}
\]

**EXERCISE 4**

1. Sketch these graphs.

\[
\begin{align*}
\text{a} & \quad y = e^x + 1 \\
\text{b} & \quad y = 4e^{-2x} \\
\text{c} & \quad y = 2e^x - 3 \\
\text{d} & \quad y = 6 + 10^{\frac{1}{2}x} \\
\text{e} & \quad y = 100e^{-x} + 10
\end{align*}
\]

2. Sketch these graphs, stating any asymptotes and intersections with the axes.

\[
\begin{align*}
\text{a} & \quad y = \ln(x + 1) \\
\text{b} & \quad y = 2 \ln x \\
\text{c} & \quad y = \ln(2x) \\
\text{d} & \quad y = \ln(4 - x) \\
\text{e} & \quad y = 4 + \ln(x + 2)
\end{align*}
\]
**EXAMPLE 10**

a Complete the table of values for: \( y = e^{1x} - 2 \)

Giving your answers to two decimal places where appropriate.

b Draw the graph of \( y = e^{1x} - 2 \) for \( 0 \leq x \leq 5 \)

c Use your graph to estimate, to 2 significant figures, the solution of the equation \( e^{1x} = 8 \)

Show your method clearly.

d By drawing a suitable line on your graph, estimate to 2 significant figures the solution to the equation \( x = 2 \ln(7 - 2x) \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>0.1</th>
<th>0.5</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>-0.3</td>
<td>1.31</td>
<td>2.41</td>
<td>2.69</td>
<td>3.10</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**HINT**

Make the LHS = \( e^{1x} - 2 \)

e.g. the equation of the graph.

To do this, you need to subtract 2 from 8 and draw the line \( y = 6 \) (as shown in the diagram).

**HINT**

Make the LHS equal to the given equation i.e \( e^{1x} - 2 \).

Draw the line \( y = 5 - 2x \) (as shown in the diagram) on your graph and find points of intersection.

**EXAMPLE 11**

Complete the table below of values of \( y = 2 + \ln x \), giving your values of \( y \) to decimal places.

<table>
<thead>
<tr>
<th>( x )</th>
<th>0.1</th>
<th>0.5</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>-0.3</td>
<td>1.31</td>
<td>2.41</td>
<td>2.69</td>
<td>3.10</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
**b** Draw the graph of \( y = 2 + \ln x \) for \( 0.1 \leq x \leq 4 \)

**c** Use your graph to estimate, to 2 significant figures, the solution of the equation \( \ln x = 0.5 \)

**d** By drawing a suitable line on your graph estimate, to 1 significant figure, the solution of the equation \( x = e^{x - 2} \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>0.1</th>
<th>0.5</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>−0.3</td>
<td>1.31</td>
<td>2</td>
<td>2.41</td>
<td>2.69</td>
<td>3.10</td>
<td>3.39</td>
</tr>
</tbody>
</table>

**c** \( 2 + \ln x = 0.5 \)  
Make the LHS \( = 2 + \ln x \) i.e the equation of the graph.  
Therefore you need to add \( \ln x \) from 0.5 and draw the line \( y = 2.5 \)

**d** \( x = e^{x - 2} \)  
Using the properties of logs

\[
\ln x = x - 2
\]

\[
\ln x + 2 = x - 2 + 2
\]

Make the LHS equal to the given equation i.e. \( \ln x + 2 \).  

The solution is the intersection of the curve and the line \( y = 2.5 \). From the graph this is approximately 1.6. In the exam you will be given a small range of answers.
1 ▶ a Draw the graph \( y = 3 + 2e^{-\frac{1}{2}x} \) for \( 0 \leq x \leq 6 \)

b Use your graph to estimate, to 2 significant figures, the solution to the equation \( e^{-\frac{1}{2}x} = 0.5 \), showing your method clearly.

c By drawing a suitable line, estimate, to 2 significant figures, the solution of the equation \( x = -2 \ln\left(\frac{x-2}{2}\right) \)

2 ▶ a Draw the graph \( y = 2 + \frac{1}{3}e^x \) for \(-1 \leq x \leq 3\)

b Use your graph to estimate, to 2 significant figures, the solution to the equation \( e^x = 12 \) showing your method clearly.

c By drawing a suitable line, estimate, to 2 significant figures, the solution to the equation \( x = \ln(6 - 6x) \)

3 ▶ a Complete the table below of values of \( y = 5 \sin 2x - 2 \cos x \), giving your values of \( y \) to 2 decimal places.

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>15</th>
<th>30</th>
<th>45</th>
<th>60</th>
<th>75</th>
<th>90</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>-2</td>
<td>0.57</td>
<td>3.59</td>
<td>3.33</td>
<td></td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

b Draw the graph of \( y = 5 \sin 2x - 2 \cos x \) for \( \theta \leq x \leq 90^\circ \)

c Use your graph to estimate, to 2 significant figures, the solution of the equation \( 2(1 + \cos x) = 5 \sin 2x \) showing your method clearly.

**WRITING AN EXPRESSION AS A LOGARITHM**

\( \log_a n = x \) means that \( a^x = n \), where \( a \) is called the base of the logarithm.

- \( \log_a 1 = 0 \) (\( a > 0 \)), because \( a^0 = 1 \)
- \( \log_a a = 1 \) (\( a > 0 \)), because \( a^1 = a \)

Write as a logarithm \( 2^5 = 32 \)

\( 2^5 = 32 \)

So \( \log_2 32 = 5 \) \quad \text{Here} \quad a = 2, \ x = 5, \ n = 32 \)

Here 2 is the base, 5 is the logarithm. In words, you would say ‘2 to the power of 5 equals 32’. You would also say ‘the logarithm of 32, to base 2, is 5’.

**EXAMPLE 13**

Rewrite using a logarithm

\[
\begin{align*}
\text{a} & \quad 10^3 = 1000 & \quad \text{b} & \quad 5^4 = 625 & \quad \text{c} & \quad 2^{10} = 1024 \\
\text{a} & \quad \lg 1000 = 3 & \quad \text{b} & \quad \log_5 625 = 4 & \quad \text{c} & \quad \log_2 1024 = 10
\end{align*}
\]

**EXAMPLE 14**

Find the value of

\[
\begin{align*}
\text{a} & \quad \log_3 81 & \quad \text{b} & \quad \log_4 0.25 & \quad \text{c} & \quad \log_{0.5} 4 & \quad \text{d} & \quad \log_n (a^5)
\end{align*}
\]
a \ log_3 81 = 4 \quad 3^4 = 81

b \ \log_{0.25} 0.25 = -1 \quad 4^{-1} = \frac{1}{4} = 0.25

c \ \log_{0.5} 4 = -2 \quad 0.5^{-2} = \left(\frac{1}{2}\right)^{-2} = 2^2 = 4

d \ \log_a (a^5) = 5 \quad a^5 = a^5

EXERCISE 6

1 ▶ Rewrite these exponentials as logarithms.
   a \ 4^3 = 64
   b \ 5^{-2} = \frac{1}{25}
   c \ 8^6 = 262,144
   d \ 3^x = 9
   e \ 8^x = 1
   f \ 2^x = \frac{1}{4}

2 ▶ Write these logarithms in exponential form.
   a \ \log_3 81 = 4
   b \ \log_{3} 729 = 6
   c \ \log_{3} 625 = 4
   d \ \log_4 4 = \frac{1}{2}
   e \ \log_3 \left(\frac{1}{27}\right) = -3
   f \ \log_{10} 0.01 = -2

3 ▶ Without a calculator find the value of
   a \ \log_4 4
   b \ \log_3 27 = 3
   c \ \log_3 81
   d \ \log_5 625
   e \ \log_3 \left(\frac{1}{125}\right)
   f \ \log_3 \sqrt{10}
   g \ \log_3 \sqrt{27}
   h \ \log_3 \sqrt[3]{3}

4 ▶ Find the value of \(x\) for which
   a \ \log_3 x = 4
   b \ \log_6 x = 3
   c \ \log_{64} 64 = 3
   d \ \log_{16} 16 = \frac{4}{3}
   e \ \log_{64} 64 = \frac{2}{3}

5 ▶ Find using your calculator
   a \ \lg 20
   b \ \lg 14
   c \ \lg 0.25
   d \ \lg 0.3
   e \ \lg 54.6

UNDERSTAND AND USE THE LAWS OF LOGARITHMS

2^5 = 32 and \log_2 32 = 5

The rules of logarithms follow the rules of indices.

<table>
<thead>
<tr>
<th>EXponent (Powers)</th>
<th>Logarithms</th>
<th>Law</th>
</tr>
</thead>
<tbody>
<tr>
<td>(c^x \times c^y = c^{x+y})</td>
<td>(\log_c (xy) = \log_c x + \log_c y)</td>
<td>Multiplication Law</td>
</tr>
<tr>
<td>(c^x \div c^y = c^{x-y})</td>
<td>(\log_c \left(\frac{x}{y}\right) = \log_c x - \log_c y)</td>
<td>Division Law</td>
</tr>
<tr>
<td>(c^q)</td>
<td>(\log_c \left(c^q\right) = q \log_c x)</td>
<td>Power Law</td>
</tr>
<tr>
<td>(\frac{1}{c} = c^{-1})</td>
<td>(\log_c \left(\frac{1}{x}\right) = - \log_c x)</td>
<td></td>
</tr>
<tr>
<td>(c^1 = c)</td>
<td>(\log_c (c) = 1)</td>
<td></td>
</tr>
<tr>
<td>(c^0 = 1)</td>
<td>(\log_c (1) = 0)</td>
<td></td>
</tr>
</tbody>
</table>
EXAMPLE 15

Write as a single logarithm

\[ \text{a } \log_3 6 + \log_3 7 \quad \text{b } \log_2 15 - \log_2 3 \quad \text{c } 2 \log_5 3 + 3 \log_5 2 \quad \text{d } \lg 3 - 4 \lg \left( \frac{1}{2} \right) \]

\[ \text{a } \log_3 (6 \times 7) = \log_3 (42) \quad \text{Use the multiplication law} \]

\[ \text{b } \log_2 (15 \div 3) = \log_2 5 \quad \text{Use the division law} \]

\[ \text{c } 2 \log_5 3 = +3 \log_5 2 \quad \text{Apply the power law to both expressions} \]

\[ = \log_5 (3^2) = +\log_5 (2^3) \]

\[ = \log_5 9 + \log_5 8 \]

\[ = \log_5 72 \quad \text{Use the multiplication law} \]

\[ \text{d } \lg 3 - 4 \lg \left( \frac{1}{2} \right) \quad \text{Use the power law} \]

\[ = \lg 3 - \lg \left( \frac{1}{2}^4 \right) \]

\[ = \lg \left( 3 + \frac{1}{16} \right) \]

\[ = \lg 48 \quad \text{Use the division law} \]

EXAMPLE 16

Find the value of

Write in terms of \( \log_a x, \log_a y \) and \( \log_a z \)

\[ \text{a } \log_a (x^2yz^3) \quad \text{b } \log_a \left( \frac{x}{y^3} \right) \quad \text{c } \log_a \left( \frac{x\sqrt{y}}{z} \right) \quad \text{d } \log_a \left( \frac{x}{a^4} \right) \]

\[ \text{a } \log_a (x^2yz^3) \]

\[ = \log_a (x^2) + \log_a (y) + \log_a (z^3) \]

\[ = 2 \log_a (x) + \log_a (y) + 3 \log_a (z) \]

\[ \text{b } \log_a \left( \frac{x}{y^3} \right) \]

\[ = \log_a (x) - \log_a (y^3) \]

\[ = \log_a (x) - 3 \log_a (y) \]

\[ \text{c } \log_a \left( \frac{x\sqrt{y}}{z} \right) \]

\[ = \log_a (x\sqrt{y}) - \log_a (z) \]

\[ = \log_a (x) + \log_a \left( \sqrt{y} \right) - \log_a (z) \]

\[ = \log_a (x) + \frac{1}{2} \log_a (y) - \log_a (z) \quad \text{Use the power law } \sqrt{y} = y^{1/2} \]

\[ \text{d } \log_a \left( \frac{x}{a^4} \right) \]

\[ = \log_a (x) - \log_a (a^4) \]
1 Write as a single logarithm

a \( \log_a 8 + \log_a 8 \)

b \( \log_6 3 + \log_6 2 \)

c \( \log_5 27 + \log_5 3 \)

d \( \log_4 24 + \log_4 15 - \log_5 3 \)

e \( 2 \log_6 9 - 10 \log_6 81 \)

f \( \frac{1}{2} \log_2 25 + 2 \log_2 3 \)

g \( \log_9 25 + \log_9 10 - 3 \log_9 5 \)

h \( 2 \log_{12} 3 + 4 \log_{12} 2 \)

i \( 2 \log_{20} (\log_5 1 + \log_8) \)

2 Write in terms of \( \log_a x \), \( \log_a y \), \( \log_a z \)

a \( \log_a x^6 \)

b \( \log_a \frac{x^6}{y^3} \)

c \( \log_a ((xz)^2) \)

d \( \log_a \frac{1}{xyz} \)

e \( \log_a \sqrt{xy} \)

f \( \log_a \sqrt{x^2 y^3 z^3} \)

g \( \log_a \sqrt[3]{x^2 y^7} \)

CHANGE THE BASE OF A LOGARITHM

Working in base \( a \), suppose that \( \log_a x = m \)

Writing this as a power \( a^m = x \)

Taking logs to a different base \( b \) \( \log_b (a^m) = \log_b (x) \)

Using the power law \( m \log_b a = \log_b x \)

Writing \( m \) as \( \log_a x \) \( \log_b x = \log_a x \times \log_b a \)

This can be written as \( \log_a x = \frac{\log_b x}{\log_b a} \)

Using this rule, notice in particular that \( \log_a b = \frac{\log_a b}{\log_a a} \)

but \( \log_b b = 1 \)

so, \( \log_a b = \frac{1}{\log_b a} \)

Example 17

Find, to 3 significant figures, the value of \( \log_8 (11) \)

One method is to use the change of base rule

\[ \log_8 11 = \frac{\lg 11}{\lg 8} \]

= 1.15

Another method is to solve \( 8^x = 11 \)

Let \( x = \log_8 (11) \)

\( 8^x = 11 \)
EXAMPLE 18
Solve the equation \( \log_5 x + 6 \log_5 x = 5 \)

\[
\log_5 x + \frac{6}{\log_5 x} = 5
\]

Let \( \log_5 (x) = y \)

\[
y + \frac{6}{y} = 5
\]

\[
y^2 + 6 = 5y
\]

\[
y^2 - 5y + 6 = 0
\]

\[
(y - 3)(y - 2) = 0
\]

So \( y = 3 \) or \( y = 2 \)

\[
\log_5 x = 3 \text{ or } \log_5 x = 2
\]

\[
x = 5^3 \text{ or } x = 5^2
\]

\[
x = 125 \text{ or } x = 25
\]

EXERCISE 8

1 ▶ Find, to 3 significant figures
   a \( \log_6 785 \)
   b \( \log_4 15 \)
   c \( \log_6 32 \)
   d \( \log_{12} 4 \)
   e \( \log_{15} \frac{1}{7} \)

2 ▶ Solve, giving your answer to 3 significant figures
   a \( 6^x = 15 \)
   b \( 9^x = 751 \)
   c \( 15^x = 3 \)
   d \( 3^x = 17.3 \)
   e \( 3^2x = 25 \)
   f \( 4^3x = 64 \)
   g \( 7^3x = 152 \)

3 ▶ Solve, giving your answer to 3 significant figures
   a \( \log_2 x = 8 + 9 \log_2 2 \)
   b \( \log_5 x + 3 \log_5 6 = 4 \)
   c \( \log x + 5 \log_{10} 10 = -6 \)
   d \( \log_2 x + \log_4 x = 2 \)

SOLVE EQUATIONS OF THE FORM \( a^x = b \)

You need to be able to solve equations of the form \( a^x = b \)

Solve the equation \( 3^x = 20 \), giving your answer to 3 significant figures.

\[
3^x = 20
\]

\[
\lg (3^x) = \lg 20
\]

Take logs to base 10 on each side.
Example 20

Solve the equation \(7^{x+1} = 3^{x+2}\)

\[(x + 1) \lg 7 = (x + 2) \lg 3\]

Use the power law

\[x \lg 7 + \lg 7 = x \lg 3 + 2 \lg 3\]

Multiply out

\[x \lg 7 - x \lg 3 = 2 \lg 3 - \lg 7\]

Collect \(x\) terms on left and numerical terms on right

\[x(\lg 7 - \lg 3) = 2 \lg 3 - \lg 7\]

Factorise

\[x = \frac{2 \lg 3 - \lg 7}{\lg 7 - \lg 3}\]

Divide by \(\lg 7 - \lg 3\)

\[x = 0.297 \text{ (3 s.f.)}\]

Example 21

Solve the equation \(5^{2x} + 7(5^x) - 30 = 0\), giving your answer to 2 decimal places.

Let \(y = 5^x\)

\[y^2 + 7y - 30 = 0\]

So \((y + 10)(y - 3) = 0\)

So \(y = -10\) or \(y = 3\)

If \(y = -10\), \(5^x = -10\), has no solution

If \(y = 3\), \(5^x = 3\)

\[\lg (5^x) = \lg 3\]

\[x \lg (5) = \lg 3\]

\[x = \frac{\lg 3}{\lg 5}\]

\[x = 0.683 \text{ (3 s.f.)}\]
1. Solve, giving your answer to 3 significant figures
   
   a. $4^x = 12$
   b. $5^x = 20$
   c. $15^x = 175$
   d. $7^x = \frac{1}{4}$
   e. $4^{x+1} = 30$
   f. $7^{2x+1} = 36$
   g. $4^{x+1} = 8^{x+1}$
   h. $2^{3y-2} = 3^{2y+5}$
   i. $7^{2x+6} = 11^{3x-2}$
   j. $3^{4-3x} = 4^{x+5}$

2. Solve, giving your answer to 3 significant figures
   
   a. $4^{2x} + 4^x - 12 = 0$
   b. $6^{2x} - 10(6^t) + 8 = 0$
   c. $5^{2x} - 6(5^t) - 7 = 0$
   d. $4^{2x+1} + 7(4^t) - 15 = 0$
   e. $2^{2x} + 3^{2x} = 4$
   f. $3^{2x+1} = 26(3^t) + 9$
1. Simplify \( \sqrt{32} + \sqrt{18} \), giving your answer in the form \( p\sqrt{2} \), where \( p \) is an integer. [2]

2. Simplify \( \frac{\sqrt{32} + \sqrt{18}}{3 + \sqrt{2}} \) giving your answer in the form \( a\sqrt{2} + b \), where \( a \) and \( b \) are integers. [3]

3. a. Expand and simplify \( (7 + \sqrt{5})(3 - \sqrt{5}) \) [2]
   
b. Express \( \frac{7 + \sqrt{5}}{3 + \sqrt{5}} \) in the form \( a + b\sqrt{2} \), where \( a \) and \( b \) are integers. [2]

4. Write \( \sqrt{75} - \sqrt{27} \) in the form \( k\sqrt{2} \), where \( k \) and \( x \) are integers. [2]

5. A rectangle \( A \) has a length of \( (1 + \sqrt{5}) \) cm and an area of \( \sqrt{80} \) cm\(^2\). Calculate the width of \( A \) in cm, giving your answer in the form \( a + b\sqrt{5} \), where \( a \) and \( b \) are integers to be found. [3]

6. Sketch the graph of \( y = 8^x \), showing the coordinates of any points at which the graph crosses the axes. [3]

7. Solve the equation \( 8^{2x} - 4(8^x) = 3 \), giving your answer to 3 significant figures. [3]

8. a. Given that \( y = 6x^2 \), show that \( \log_b y = 1 + 2 \log_b x \) [2]
   
b. Hence, or otherwise, solve the equation \( 1 + 2 \log_3 x = \log_3 (28x - 9) \), giving your answer to 3 significant figures. [3]

9. Find the values of \( x \) such that: \( 2 \log_3 x - \log_3 (x - 2) - 2 = 0 \) [2]

10. Find the values of \( y \) such that: \( \frac{\log_2 32 + \log_2 16}{\log_2 y} - \log_2 y = 0 \) [3]

11. Given that \( \log_b y + 3 \log_b 2 = 5 \), express \( y \) in terms of \( b \) in its simplest form. [2]

12. Solve \( 5^{2x} = 12(5^x) - 35 \) [4]

13. Find, giving your answer to 3 significant figures where appropriate, the value of \( x \) for which \( 5^x = 10 \) [3]

14. Given that \( \log_3 (3b + 1) - \log_3 (a - 2) = -1 \), \( a > 2 \), express \( b \) in terms of \( a \). [2]

15. Solve \( 3^{x-2} = 3^{\sqrt{3}} \) [4]

16. Solve \( 25^x + 5^{x+1} = 24 \), giving your answer to 3 significant figures. [3]
17 Given that \( \log_2 x = p \), find, in terms of \( p \), the simplest form of

a \( \log_2 (16x) \).  

b \( \log_2 \left( \frac{x^4}{2} \right) \).  

c Hence, or otherwise, solve

\[ \log_2 (16x) - \log_2 \left( \frac{x^4}{2} \right) = \frac{1}{2} \]

Give your answer in its simplest surd form.

18 Solve \( \log_3 t + \log_3 5 = \log_3 (2t + 3) \)

19 a Draw the graph \( y = 2 + \ln x \) for \( 0.1 < x < 4 \)

b Use your graph to estimate, to 2 significant figures, the solution to \( \ln x = 0.5 \), showing clearly your method.

c By drawing a suitable line, estimate to 2 significant figures, the solution of the equation \( x = e^{-2} \)
CHAPTER SUMMARY: CHAPTER 1

- You can simplify expressions by using the power (indices) laws.
  
  \[
  c^x \cdot c^y = c^{x+y} \\
  c^x \div c^y = c^{x-y} \\
  (c^p)^q = c^{pq} \\
  \frac{1}{c} = c^{-1} \\
  c^1 = c \\
  c^0 = 1
  \]

- You can manipulate surds using these rules:
  
  \[
  \sqrt{ab} = \sqrt{a} \times \sqrt{b} \\
  \sqrt[5]{\frac{a}{b}} = \frac{\sqrt[5]{a}}{\sqrt[5]{b}}
  \]

- The rules for rationalising surds are:
  
  - If you have a fraction in the form \( \frac{1}{\sqrt{a}} \) then multiply top and bottom by \( \sqrt{a} \)
  - If you have a fraction in the form \( \frac{1}{1 + \sqrt{a}} \) then multiply top and bottom by \( (1 - \sqrt{a}) \)
  - If you have a fraction in the form \( \frac{1}{1 - \sqrt{a}} \) then multiply top and bottom by \( (1 + \sqrt{a}) \)

- \( \log_a n = x \) can be rewritten as \( a^x = n \) where \( a \) is the base of the logarithm.

- The laws of logarithms are:
  
  \[
  \log_a xy = \log_a x + \log_a y \\
  \log_a \frac{x}{y} = \log_a x - \log_a y \\
  \log_a (x^q) = q \log_a x \\
  \log_a \left(\frac{1}{x}\right) = -\log_a x \\
  \log_a (c) = 1 \\
  \log_a (1) = 0
  \]

- The change of base rule for logarithms can be written as \( \log_a x = \frac{\log_b x}{\log_b a} \)

- From the change of base you can derive \( \log_b a = \frac{1}{\log_b a} \)

- The natural logarithm is defined as: \( \log_e x \equiv \ln x \)

- The graph of \( y = e^x \) is shown below.

- The graph of \( y = \ln x \) is shown below.