Endorsed for
Pearson Edexcel
Qualifications

MATHEMATICS

eBook included

PEARSON EDEXCEL INTERNATIONAL A LEVEL

FURTHER PURE MATHEMATICS 2

STUDENT BOOK

PEARSON EDEXCEL INTERNATIONAL A LEVEL

FURTHER PURE MATHEMATICS 2

Student Book

Series Editors: Joe Skrakowski and Harry Smith

Authors: Greg Attwood, Jack Barraclough, Tom Begley, Dave Berry, Ian Bettison, Linnet Bruce, Lee Cope, Charles Garnet Cox, Keith Gallick, Tim Garry, Alistair Macpherson, Bronwen Moran, Johnny Nicholson, Laurence Pateman, Joe Petran, Keith Pledger, Joe Skrakowski, Harry Smith, Geoff Staley, Ibrahim Wazir, Dave Wilkins

Published by Pearson Education Limited, 80 Strand, London, WC2R 0RL.

www.pearsonglobalschools.com

Copies of official specifications for all Pearson qualifications may be found on the website: https://qualifications.pearson.com

Text © Pearson Education Limited 2019
Edited by Richard Hutchinson
Typeset by Tech-Set Ltd, Gateshead, UK
Original illustrations © Pearson Education Limited 2019
Illustrated by © Tech-Set Ltd, Gateshead, UK
Cover design by © Pearson Education Limited 2019

The rights of Greg Attwood, Jack Barraclough, Tom Begley, Dave Berry, Ian Bettison, Linnet Bruce, Lee Cope, Charles Garnet Cox, Keith Gallick, Tim Garry, Alistair Macpherson, Bronwen Moran, Johnny Nicholson, Laurence Pateman, Joe Petran, Keith Pledger, Joe Skrakowski, Harry Smith, Geoff Staley, Ibrahim Wazir and Dave Wilkins to be identified as the authors of this work have been asserted by them in accordance with the Copyright, Designs and Patents Act 1988.

First published 2019

22 21 20 19
10 9 8 7 6 5 4 3 2 1

British Library Cataloguing in Publication Data
A catalogue record for this book is available from the British Library

ISBN 978 1 292244 65 5

Printed in Slovakia by Neografia

Picture Credits
The authors and publisher would like to thank the following individuals and organisations for permission to reproduce photographs:

Shutterstock.com: Tatiana Shepeleva 90, Jag_cz 125; **Getty Images:** Steve DF 22, Digital Vision. 105, mbbirdy 149; **Alamy Stock Photo:** Kevin Britland 1, Science Photo Library 46; **Science Photo Library:** Andrew Brookes, National Physical Laboratory 14

Cover images: *Front*: **Getty Images:** Werner Van Steen
Inside front cover: **Shutterstock.com:** Dmitry Lobanov

Endorsement Statement
In order to ensure that this resource offers high-quality support for the associated Pearson qualification, it has been through a review process by the awarding body. This process confirms that this resource fully covers the teaching and learning content of the specification or part of a specification at which it is aimed. It also confirms that it demonstrates an appropriate balance between the development of subject skills, knowledge and understanding, in addition to preparation for assessment.

Endorsement does not cover any guidance on assessment activities or processes (e.g. practice questions or advice on how to answer assessment questions) included in the resource, nor does it prescribe any particular approach to the teaching or delivery of a related course.

While the publishers have made every attempt to ensure that advice on the qualification and its assessment is accurate, the official specification and associated assessment guidance materials are the only authoritative source of information and should always be referred to for definitive guidance.

Pearson examiners have not contributed to any sections in this resource relevant to examination papers for which they have responsibility.

Examiners will not use endorsed resources as a source of material for any assessment set by Pearson. Endorsement of a resource does not mean that the resource is required to achieve this Pearson qualification, nor does it mean that it is the only suitable material available to support the qualification, and any resource lists produced by the awarding body shall include this and other appropriate resources.

ABOUT THIS BOOK

The following three themes have been fully integrated throughout the Pearson Edexcel International Advanced Level in Mathematics series, so they can be applied alongside your learning.

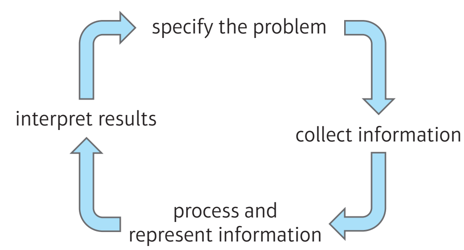
1. Mathematical argument, language and proof

- Rigorous and consistent approach throughout
- Notation boxes explain key mathematical language and symbols

2. Mathematical problem-solving

- Hundreds of problem-solving questions, fully integrated into the main exercises
- Problem-solving boxes provide tips and strategies
- Challenge questions provide extra stretch

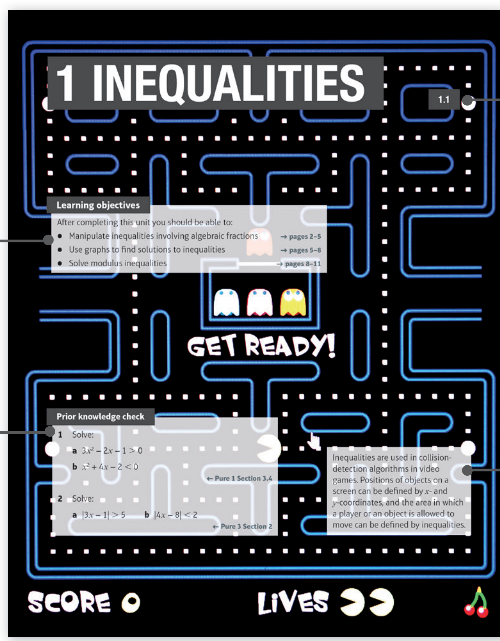
The Mathematical Problem-Solving Cycle



3. Transferable skills

- Transferable skills are embedded throughout this book, in the exercises and in some examples
- These skills are signposted to show students which skills they are using and developing

Finding your way around the book



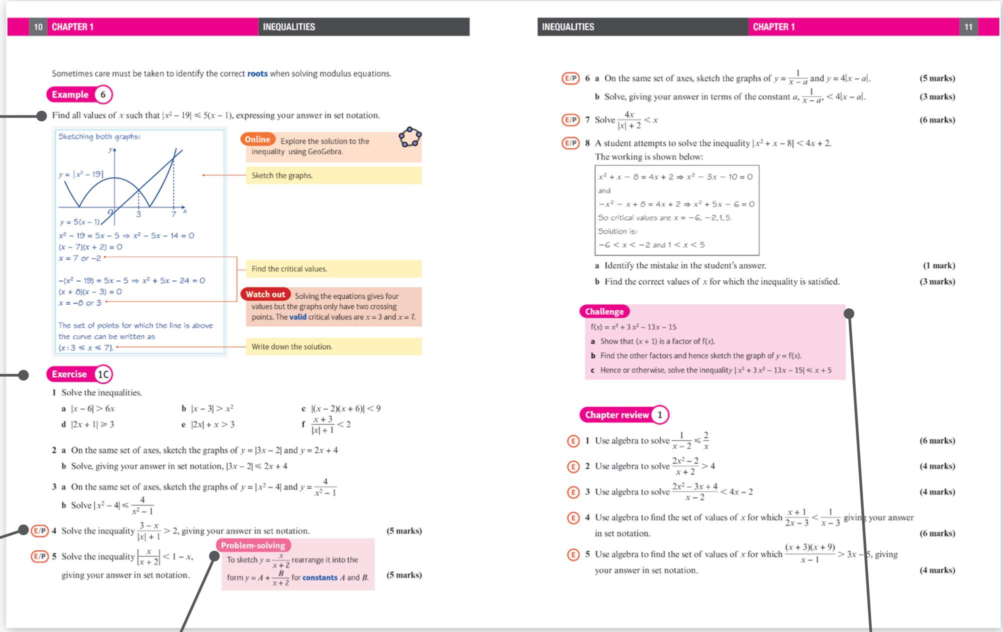
Each chapter is mapped to the specification content for easy reference

Each chapter starts with a list of Learning objectives

The *Prior knowledge check* helps make sure you are ready to start the chapter

The real world applications of the maths you are about to learn are highlighted at the start of the chapter.

Glossary terms will be identified by bold blue text on their first appearance.



Each section begins with explanation and key learning points

Exercise questions are carefully graded so they increase in difficulty and gradually bring you up to exam standard

Exercises are packed with exam-style questions to ensure you are ready for the exams

Problem-solving boxes provide hints, tips and strategies, and *Watch out* boxes highlight areas where students often lose marks in their exams

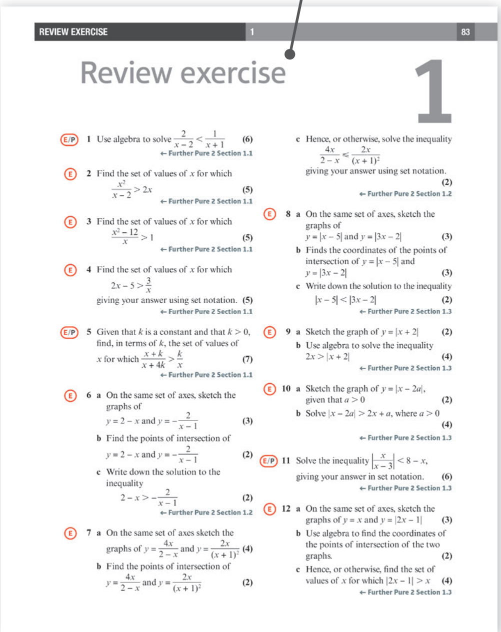
Challenge boxes give you a chance to tackle some more difficult questions

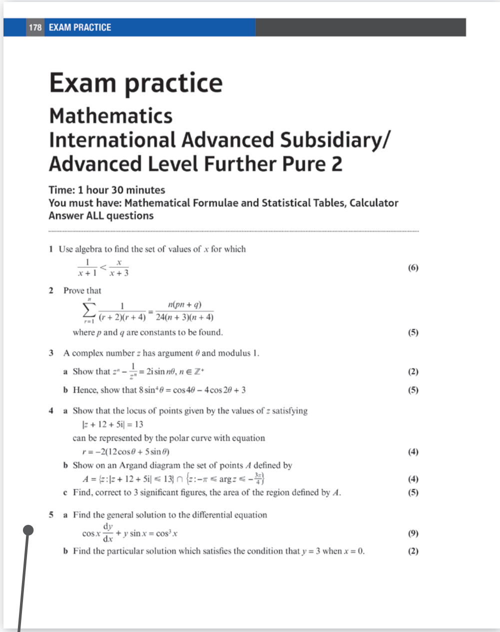
Exam-style questions are flagged with (E)

Problem-solving questions are flagged with (P)

Each chapter ends with a *Chapter review* and a *Summary of key points*

After every few chapters, a *Review exercise* helps you consolidate your learning with lots of exam-style questions





A full practice paper at the back of the book helps you prepare for the real thing

QUALIFICATION AND ASSESSMENT OVERVIEW

Qualification and content overview

Further Pure Mathematics 2 (FP2) is an **optional*** unit in the following qualifications:

International Advanced Subsidiary in Further Mathematics

International Advanced Level in Further Mathematics

*It is compulsory to study **either** FP2 **or** FP3 for the International Advanced Level in Further Mathematics.

Assessment overview

The following table gives an overview of the assessment for this unit.

We recommend that you study this information closely to help ensure that you are fully prepared for this course and know exactly what to expect in the assessment.

Unit	Percentage	Mark	Time	Availability
FP2: Further Pure Mathematics 2 Paper code WFM02/01	$33\frac{1}{3}$ % of IAS $16\frac{2}{3}$ % of IAL	75	1 hour 30 mins	January and June First assessment June 2020

IAS: International Advanced Subsidiary, IAL: International Advanced A Level.

Assessment objectives and weightings

		Minimum weighting in IAS and IAL
AO1	Recall, select and use their knowledge of mathematical facts, concepts and techniques in a variety of contexts.	30%
AO2	Construct rigorous mathematical arguments and proofs through use of precise statements, logical deduction and inference and by the manipulation of mathematical expressions, including the construction of extended arguments for handling substantial problems presented in unstructured form.	30%
AO3	Recall, select and use their knowledge of standard mathematical models to represent situations in the real world; recognise and understand given representations involving standard models; present and interpret results from such models in terms of the original situation, including discussion of the assumptions made and refinement of such models.	10%
AO4	Comprehend translations of common realistic contexts into mathematics; use the results of calculations to make predictions, or comment on the context; and, where appropriate, read critically and comprehend longer mathematical arguments or examples of applications.	5%
AO5	Use contemporary calculator technology and other permitted resources (such as formulae booklets or statistical tables) accurately and efficiently; understand when not to use such technology, and its limitations. Give answers to appropriate accuracy.	5%

Relationship of assessment objectives to units

FP2	Assessment objective				
	AO1	AO2	AO3	AO4	AO5
Marks out of 75	25–30	25–30	0–5	7–12	5–10
%	$33\frac{1}{3}$–40	$33\frac{1}{3}$–40	0–$6\frac{2}{3}$	$9\frac{1}{3}$–16	$6\frac{2}{3}$–$13\frac{1}{3}$

Calculators

Students may use a calculator in assessments for these qualifications. Centres are responsible for making sure that calculators used by their students meet the requirements given in the table below.

Students are expected to have available a calculator with at least the following keys: +, –, ×, ÷, π, x^2, $\sqrt{x}$, $\frac{1}{x}$, x^y, $\ln x$, e^x, $x!$, sine, cosine and tangent and their inverses in degrees and decimals of a degree, and in radians; memory.

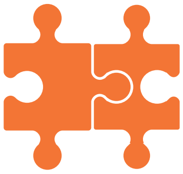
Prohibitions

Calculators with any of the following facilities are prohibited in all examinations:

- databanks
- retrieval of text or formulae
- built-in symbolic algebra manipulations
- symbolic differentiation and/or integration
- language translators
- communication with other machines or the internet

Extra online content

Whenever you see an *Online* box, it means that there is extra online content available to support you.



SolutionBank

SolutionBank provides worked solutions for questions in the book. Download the solutions as a PDF or quickly find the solution you need online.

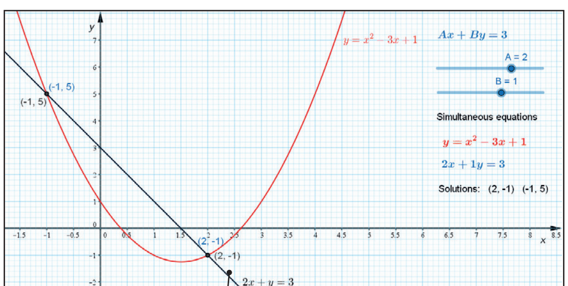
Use of technology

Explore topics in more detail, visualise problems and consolidate your understanding. Use pre-made GeoGebra activities or Casio resources for a graphic calculator.

Online Find the point of intersection graphically using technology.

GeoGebra

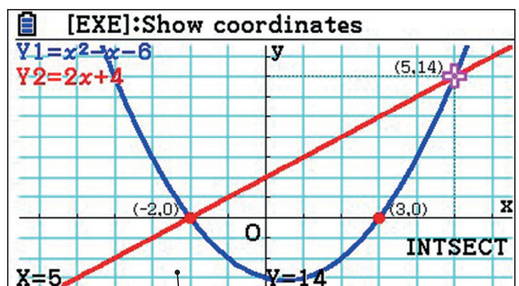
GeoGebra-powered interactives



Interact with the maths you are learning using GeoGebra's easy-to-use tools

CASIO

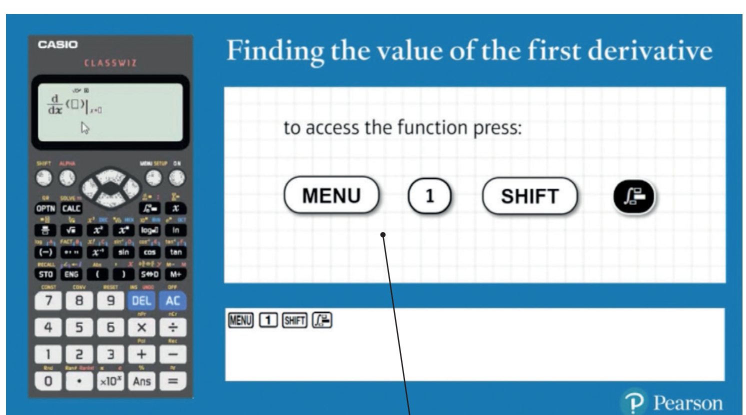
Graphic calculator interactives



Explore the maths you are learning and gain confidence in using a graphic calculator

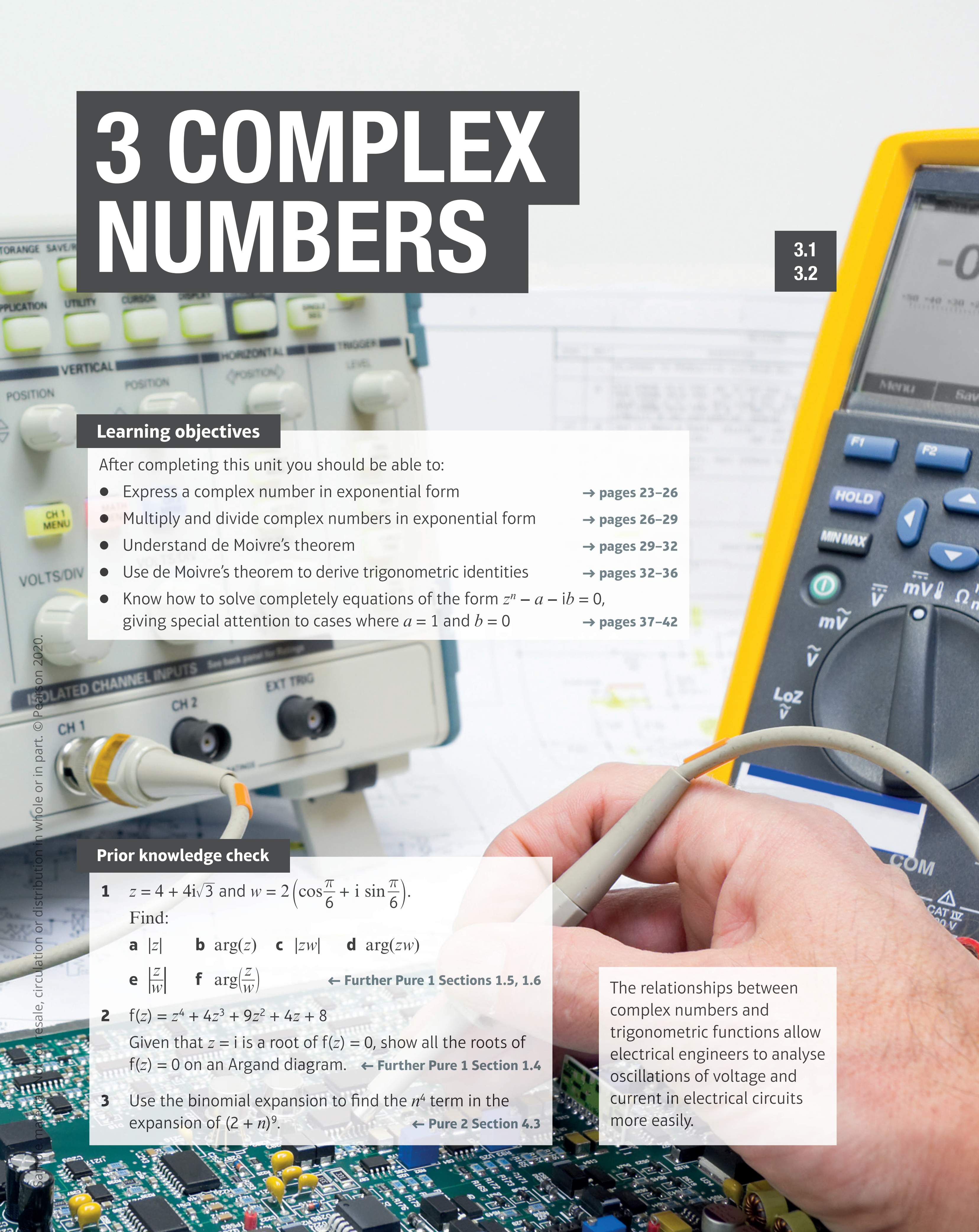
Calculator tutorials

Our helpful video tutorials will guide you through how to use your calculator in the exams. They cover both Casio's scientific and colour graphic calculators.



Step-by-step guide with audio instructions on exactly which buttons to press and what should appear on your calculator's screen

Online Work out each coefficient quickly using the ${}^nC_r$ and power functions on your calculator.



3 COMPLEX NUMBERS

3.1
3.2

Learning objectives

After completing this unit you should be able to:

- Express a complex number in exponential form → pages 23–26
- Multiply and divide complex numbers in exponential form → pages 26–29
- Understand de Moivre's theorem → pages 29–32
- Use de Moivre's theorem to derive trigonometric identities → pages 32–36
- Know how to solve completely equations of the form $z^n - a - \text{i}b = 0$, giving special attention to cases where $a = 1$ and $b = 0$ → pages 37–42

Prior knowledge check

1 $z = 4 + 4\text{i}\sqrt{3}$ and $w = 2\left(\cos\frac{\pi}{6} + \text{i}\sin\frac{\pi}{6}\right)$.

Find:

a $|z|$ **b** $\arg(z)$ **c** $|zw|$ **d** $\arg(zw)$

e $\left|\frac{z}{w}\right|$ **f** $\arg\left(\frac{z}{w}\right)$ ← Further Pure 1 Sections 1.5, 1.6

2 $\text{f}(z) = z^4 + 4z^3 + 9z^2 + 4z + 8$

Given that $z = \text{i}$ is a root of $\text{f}(z) = 0$, show all the roots of $\text{f}(z) = 0$ on an Argand diagram. ← Further Pure 1 Section 1.4

3 Use the binomial expansion to find the n^4 term in the expansion of $(2 + n)^9$. ← Pure 2 Section 4.3

The relationships between complex numbers and trigonometric functions allow electrical engineers to analyse oscillations of voltage and current in electrical circuits more easily.

3.1 Exponential form of complex numbers

You can use the **modulus–argument form** of a **complex number** to express it in the **exponential form**: $z = re^{i\theta}$.

> **Links** The **modulus–argument** form of a complex number is $z = r(\cos\theta + i\sin\theta)$, where $r = |z|$ and $\theta = \arg z$.
> ← Further Pure 1 Section 1.6

You can write $\cos\theta$ and $\sin\theta$ as infinite series of powers of θ:

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots + \frac{(-1)^r\,\theta^{2r}}{(2r)!} + \dots \qquad \textbf{(1)}$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots + \frac{(-1)^r\,\theta^{2r+1}}{(2r+1)!} + \dots \qquad \textbf{(2)}$$

You can also write e^x, $x \in \mathbb{R}$, as a series expansion in powers of x.

> **Links** These are the Maclaurin series expansions of $\sin\theta$, $\cos\theta$ and e^x.
> → Further Pure 2 Section 7.2

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots + \frac{x^r}{r!} + \dots$$

You can use this expansion to define the exponential function for complex powers, by replacing x with a complex number. In particular, if you replace x with the **imaginary number** $i\theta$, you get

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \dots$$

$$= 1 + i\theta + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \frac{i^5\theta^5}{5!} + \frac{i^6\theta^6}{6!} + \dots$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} + \dots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\right)$$

By comparing this series expansion with **(1)** and **(2)**, you can write $e^{i\theta}$ as

$$e^{i\theta} = \cos\theta + i\sin\theta$$

This formula is known as **Euler's relation**. It is important for you to remember this result.

- **You can use Euler's relation, $e^{i\theta} = \cos\theta + i\sin\theta$, to write a complex number z in exponential form:**

  $$z = re^{i\theta}$$

  where $r = |z|$ and $\theta = \arg z$

> **Notation** Substituting $\theta = \pi$ into Euler's relation yields **Euler's identity**:
>
> $$e^{i\pi} + 1 = 0$$
>
> This equation links the five fundamental constants 0, 1, π, e and i, and is considered an example of mathematical beauty.

Example 1

Express in the form $re^{i\theta}$, where $-\pi < \theta \leqslant \pi$.

a $z = \sqrt{2}\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)$ **b** $z = 5\left(\cos\frac{\pi}{8} - i\sin\frac{\pi}{8}\right)$

a $z = \sqrt{2}\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)$

So $r = \sqrt{2}$ and $\theta = \frac{\pi}{10}$

Therefore, $z = \sqrt{2}e^{\frac{\pi i}{10}}$

b $z = 5\left(\cos\frac{\pi}{8} - i\sin\frac{\pi}{8}\right)$

$z = 5\left(\cos\left(-\frac{\pi}{8}\right) + i\sin\left(-\frac{\pi}{8}\right)\right)$

So $r = 5$ and $\theta = -\frac{\pi}{8}$

Therefore, $z = 5e^{-\frac{\pi i}{8}}$

Compare with $r(\cos\theta + i\sin\theta)$.

$z = re^{i\theta}$

Problem-solving

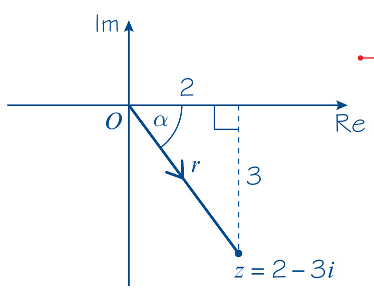
Use $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$.

Compare with $r(\cos\theta + i\sin\theta)$.

$z = re^{i\theta}$

Example 2

Express $z = 2 - 3i$ in the form $re^{i\theta}$, where $-\pi < \theta \leqslant \pi$.



$r = |z| = \sqrt{2^2 + (-3)^2} = \sqrt{13}$

$\theta = \arg z = -\arctan\left(\frac{3}{2}\right) = -0.983$ (3 s.f.)

Therefore, $z = \sqrt{13}e^{-0.983i}$

Sketch the **Argand diagram**, showing the position of the complex number.

Here z is in the fourth **quadrant** so the required argument is $-\alpha$.

Find r and θ.

$z = re^{i\theta}$

Example 3

Express $z = \sqrt{2}e^{\frac{3\pi i}{4}}$ in the form $x + iy$, where $x, y \in \mathbb{R}$.

$z = \sqrt{2}e^{\frac{3\pi i}{4}}$, so $r = \sqrt{2}$ and $\theta = \frac{3\pi}{4}$ — Compare with $re^{i\theta}$.

$z = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$ — Write z in modulus–argument form.

$= \sqrt{2}\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$

Therefore, $z = -1 + i$ — Simplify.

Example 4

Express $z = 2e^{\frac{23\pi i}{5}}$ in the form $r(\cos\theta + i\sin\theta)$, where $-\pi < \theta \leqslant \pi$.

$z = 2e^{\frac{23\pi i}{5}}$, so $r = 2$ and $\theta = \frac{23\pi}{5}$ — Compare with $re^{i\theta}$.

$\frac{23\pi}{5} - 2\pi = \frac{13\pi}{5}, \frac{13\pi}{5} - 2\pi = \frac{3\pi}{5}$

$\frac{3\pi}{5}$ is in the range $-\pi < \theta \leqslant \pi$

So $z = 2\left(\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}\right)$ — Write z in the form $r(\cos\theta + i\sin\theta)$.

Problem-solving

$\cos\theta = \cos(\theta + 2\pi)$ and $\sin\theta = \sin(\theta + 2\pi)$.

Subtract multiples of 2π from $\frac{23\pi}{5}$ until you find a value in the range $-\pi < \theta \leqslant \pi$.

Example 5

Use $e^{i\theta} = \cos\theta + i\sin\theta$ to show that $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$.

$e^{i\theta} = \cos\theta + i\sin\theta$ **(1)**

$e^{-i\theta} = e^{i(-\theta)} = \cos(-\theta) + i\sin(-\theta)$

So $e^{-i\theta} = \cos\theta - i\sin\theta$ **(2)** — Use $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$.

$e^{i\theta} + e^{-i\theta} = 2\cos\theta$ — Add **(1)** and **(2)**.

$\Rightarrow \frac{e^{i\theta} + e^{-i\theta}}{2} = \cos\theta$ — Divide both sides by 2.

Hence, $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$, as required.

Exercise 3A

1 Express in the form $re^{i\theta}$, where $-\pi < \theta \leqslant \pi$. Use exact values of r and θ where possible, or values to 3 significant figures otherwise.

a -3 **b** $6i$ **c** $-2\sqrt{3} - 2i$

d $-8 + i$ **e** $2 - 5i$ **f** $-2\sqrt{3} + 2i\sqrt{3}$

g $\sqrt{8}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$ **h** $8\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)$ **i** $2\left(\cos\frac{\pi}{5} - i\sin\frac{\pi}{5}\right)$

2 Express in the form $x + iy$ where $x, y \in \mathbb{R}$.

a $e^{\frac{\pi i}{3}}$ **b** $4e^{\pi i}$ **c** $3\sqrt{2}\,e^{\frac{\pi i}{4}}$

d $8e^{\frac{\pi i}{6}}$ **e** $3e^{-\frac{\pi i}{2}}$ **f** $e^{\frac{5\pi i}{6}}$

g $e^{-\pi i}$ **h** $3\sqrt{2}e^{-\frac{3\pi i}{4}}$ **i** $8e^{-\frac{4\pi i}{3}}$

3 Express in the form $r(\cos\theta + i\sin\theta)$, where $-\pi < \theta \leqslant \pi$.

a $e^{\frac{16\pi i}{13}}$ **b** $4e^{\frac{17\pi i}{5}}$ **c** $5e^{-\frac{9\pi i}{8}}$

(P) 4 Use $e^{i\theta} = \cos\theta + i\sin\theta$ to show that $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$

3.2 Multiplying and dividing complex numbers

You can apply the modulus–argument rules for multiplying and dividing complex numbers to numbers written in exponential form.

Recall that, for any two complex numbers z_1 and z_2,

- $|z_1 z_2| = |z_1||z_2|$
- $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$
- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
- $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

Links These results can be proved by considering the numbers z_1 and z_2 in the form $r(\cos\theta + i\sin\theta)$ and using the addition formulae for cos and sin. ← Further Pure 1 Section 1.6

Applying these results to numbers in exponential form gives the following result:

■ **If $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$, then:**

- $z_1z_2 = r_1r_2e^{i(\theta_1 + \theta_2)}$
- $\frac{z_1}{z_2} = \frac{r_1}{r_2}e^{i(\theta_1 - \theta_2)}$

Watch out You cannot automatically assume the laws of indices work the same way with complex numbers as with **real numbers**. This result only shows that they can be applied in these specific cases.

Example 6

a Express $2e^{\frac{\pi i}{6}} \times \sqrt{3}e^{\frac{\pi i}{3}}$ in the form $x + iy$.

b $z = 2 + 2i$, $\text{Im}(zw) = 0$ and $|zw| = 3|z|$
Use geometrical reasoning to find the two possibilities for w, giving your answers in exponential form.

a $2e^{\frac{\pi i}{6}} \times \sqrt{3}e^{\frac{\pi i}{3}} = (2 \times \sqrt{3})e^{i\left(\frac{\pi}{6} + \frac{\pi}{3}\right)}$

$= 2\sqrt{3}e^{\frac{\pi i}{2}}$

$= 2\sqrt{3}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

$= 2\sqrt{3}(0 + i)$

$= 2i\sqrt{3}$

$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$

Convert the complex number to modulus–argument form.

b $|zw| = 3|z| \Rightarrow |w| = 3$

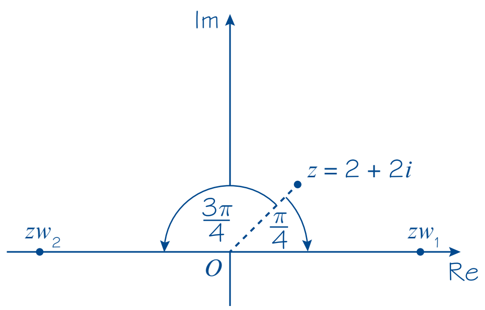
$\arg z = \arctan\left(\frac{2}{2}\right) = \frac{\pi}{4}$

$\text{Im}(zw) = 0$ so $\arg(zw) = 0$ or π

So $\arg w = \frac{3\pi}{4}$ or $-\frac{\pi}{4}$

$|zw| = |z||w| = 3|z|$.

wz lies on the real axis, so z is rotated $\frac{3\pi}{4}$ clockwise or $\frac{\pi}{4}$ anticlockwise when multiplied by w.



$w_1 = 3e^{-\frac{\pi i}{4}}$ and $w_2 = 3e^{\frac{3\pi i}{4}}$

Example 7

Express $\dfrac{2\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)}{\sqrt{2}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)}$ in the form $re^{i\theta}$.

$\dfrac{2\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)}{\sqrt{2}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)} = \dfrac{2e^{\frac{\pi i}{12}}}{\sqrt{2}e^{\frac{5\pi i}{6}}}$

$= \frac{2}{\sqrt{2}}e^{i\left(\frac{\pi}{12} - \frac{5\pi}{6}\right)}$

$= \sqrt{2}e^{-\frac{3\pi i}{4}}$

Convert the numerator and denominator to exponential form.

$\frac{z_1}{z_2} = \frac{r_1}{r_2}e^{i(\theta_1 - \theta_2)}$

Simplify.

Exercise 3B

1 Express in the form $x + iy$, where $x, y \in \mathbb{R}$.

a $e^{\frac{\pi i}{3}} \times e^{\frac{\pi i}{4}}$ b $\sqrt{5}e^{i\theta} \times 3e^{3i\theta}$ c $\sqrt{2}e^{\frac{2\pi i}{3}} \times e^{-\frac{7\pi i}{3}} \times 3e^{\frac{\pi i}{6}}$

2 Express in the form $x + iy$ where $x, y \in \mathbb{R}$.

a $\dfrac{2e^{\frac{7\pi i}{2}}}{8e^{\frac{9\pi i}{2}}}$ b $\dfrac{\sqrt{3}e^{\frac{3\pi i}{7}}}{4e^{-\frac{2\pi i}{7}}}$ c $\dfrac{\sqrt{2}e^{-\frac{15\pi i}{6}}}{2e^{\frac{\pi i}{3}}} \times \sqrt{2}\,e^{\frac{19\pi i}{3}}$

3 Express in the form $re^{i\theta}$

a $(\cos 2\theta + i\sin 2\theta)(\cos 3\theta + i\sin 3\theta)$ b $\left(\cos\frac{3\pi}{11} + i\sin\frac{3\pi}{11}\right)\left(\cos\frac{8\pi}{11} + i\sin\frac{8\pi}{11}\right)$

c $3\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \times 2\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$

d $\sqrt{6}\left(\cos\left(-\frac{\pi}{12}\right) + i\sin\left(-\frac{\pi}{12}\right)\right) \times \sqrt{3}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

4 Express in the form $re^{i\theta}$

a $\dfrac{\cos 5\theta + i\sin 5\theta}{\cos 2\theta + i\sin 2\theta}$ b $\dfrac{\sqrt{2}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)}{\frac{1}{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)}$ c $\dfrac{3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)}{4\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)}$

5 z and w are two complex numbers where $z = -9 + 3i\sqrt{3}$, $|w| = \sqrt{3}$ and $\arg w = \frac{7\pi}{12}$

Express in the form $re^{i\theta}$, where $-\pi < \theta \leqslant \pi$.

a z b w c zw d $\frac{z}{w}$

P 6 Use the exponential form for a complex number to show that

$$\frac{(\cos 9\theta + i\sin 9\theta)(\cos 4\theta + i\sin 4\theta)}{\cos 7\theta + i\sin 7\theta} \equiv \cos 6\theta + i\sin 6\theta$$

E/P 7 $z = 1 + i\sqrt{3}$, $\text{Re}\left(\frac{z^2}{w}\right) = 0$ and $\left|\frac{z^2}{w}\right| = |z|$

Use geometrical reasoning to find the two possibilities for w, giving your answers in exponential form. **(4 marks)**

E/P 8 a Evaluate $(1 + i)^2$, giving your answer in exponential form. **(2 marks)**

b Use mathematical induction to prove that $(1 + i)^n = 2^{\frac{n}{2}}e^{\frac{n\pi i}{4}}$ for $n \in \mathbb{Z}^+$. **(4 marks)**

c Hence find $(1 + i)^{16}$. **(1 mark)**

P 9 Use Euler's relation for $e^{i\theta}$ and $e^{-i\theta}$ to verify that $\cos^2\theta + \sin^2\theta \equiv 1$.

Challenge

a Given that n is a positive integer, prove by induction that $(re^{i\theta})^n = r^n e^{in\theta}$

b Given further that $z^{-n} = \frac{1}{z^n}$ for all $z \in \mathbb{C}$, show that $(re^{i\theta})^{-n} = r^{-n}e^{-in\theta}$

Watch out You cannot assume that the laws of indices will apply to complex numbers. Prove these results using only the properties

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}$$

3.3 De Moivre's theorem

You can use Euler's relation to find powers of complex numbers given in modulus–argument form.

$$\begin{aligned}(r(\cos\theta + i\sin\theta))^2 &= (re^{i\theta})^2\\ &= re^{i\theta} \times re^{i\theta}\\ &= r^2 e^{i2\theta}\\ &= r^2(\cos 2\theta + i\sin 2\theta)\end{aligned}$$

Similarly, $(r(\cos\theta + i\sin\theta))^3 = r^3(\cos 3\theta + i\sin 3\theta)$, and so on.

The generalisation of this result is known as **de Moivre's theorem**:

- **For any integer n,**

$$(r(\cos\theta + i\sin\theta))^n = r^n(\cos n\theta + i\sin n\theta)$$

You can prove de Moivre's theorem quickly using Euler's relation.

$$\begin{aligned}(r(\cos\theta + i\sin\theta))^n &= (re^{i\theta})^n\\ &= r^n e^{in\theta}\\ &= r^n(\cos n\theta + i\sin n\theta)\end{aligned}$$

This step is valid for any integer exponent n. ← Exercise 3B Challenge

You can also prove de Moivre's theorem for **positive integer exponents** directly from the modulus–argument form of a complex number using the addition formulae for sin and cos.

Links This proof uses the method of proof by induction. ← Further Pure 1 Section 8.1

1. Basis step

$n = 1$; LHS $= (r(\cos\theta + i\sin\theta))^1 = r(\cos\theta + i\sin\theta)$

RHS $= r^1(\cos 1\theta + i\sin 1\theta) = r(\cos\theta + i\sin\theta)$

As LHS = RHS, de Moivre's theorem is true for $n = 1$.

2. Assumption step

Assume that de Moivre's theorem is true for $n = k$, $k \in \mathbb{Z}^+$:

$(r(\cos\theta + i\sin\theta))^k = r^k(\cos k\theta + i\sin k\theta)$

3. Inductive step

When $n = k + 1$,

$(r(\cos\theta + \mathrm{i}\sin\theta))^{k+1} = (r(\cos\theta + \mathrm{i}\sin\theta))^k \times r(\cos\theta + \mathrm{i}\sin\theta)$

$= r^k(\cos k\theta + \mathrm{i}\sin k\theta) \times r(\cos\theta + \mathrm{i}\sin\theta)$ — By assumption step

$= r^{k+1}(\cos k\theta + \mathrm{i}\sin k\theta)(\cos\theta + \mathrm{i}\sin\theta)$

$= r^{k+1}((\cos k\theta\cos\theta - \sin k\theta\sin\theta) + \mathrm{i}(\sin k\theta\cos\theta + \cos k\theta\sin\theta))$

$= r^{k+1}(\cos(k\theta + \theta) + \mathrm{i}\sin(k\theta + \theta))$ — By addition formulae

$= r^{k+1}(\cos((k+1)\theta) + \mathrm{i}\sin((k+1)\theta))$

Therefore, de Moivre's theorem is true when $n = k + 1$.

4. Conclusion step

If de Moivre's theorem is true for $n = k$, then it has been shown to be true for $n = k + 1$.

As de Moivre's theorem is true for $n = 1$, it is now proven to be true for all $n \in \mathbb{Z}^+$ by mathematical induction.

Links The **corresponding** proof for negative integer exponents is left as an exercise.
→ Exercise 3C Challenge

Example 8

Simplify $\dfrac{\left(\cos\frac{9\pi}{17} + \mathrm{i}\sin\frac{9\pi}{17}\right)^5}{\left(\cos\frac{2\pi}{17} - \mathrm{i}\sin\frac{2\pi}{17}\right)^3}$

$$\frac{\left(\cos\frac{9\pi}{17} + \mathrm{i}\sin\frac{9\pi}{17}\right)^5}{\left(\cos\frac{2\pi}{17} - \mathrm{i}\sin\frac{2\pi}{17}\right)^3}$$

$$= \frac{\left(\cos\frac{9\pi}{17} + \mathrm{i}\sin\frac{9\pi}{17}\right)^5}{\left(\cos\left(-\frac{2\pi}{17}\right) + \mathrm{i}\sin\left(-\frac{2\pi}{17}\right)\right)^3}$$

$\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$

$$= \frac{\cos\frac{45\pi}{17} + \mathrm{i}\sin\frac{45\pi}{17}}{\cos\left(-\frac{6\pi}{17}\right) + \mathrm{i}\sin\left(-\frac{6\pi}{17}\right)}$$

Apply de Moivre's theorem to both the numerator and the denominator.

$$= \cos\left(\frac{45\pi}{17} - \left(-\frac{6\pi}{17}\right)\right) + \mathrm{i}\sin\left(\frac{45\pi}{17} - \left(-\frac{6\pi}{17}\right)\right)$$

$\frac{z_1}{z_2} = \cos(\theta_1 - \theta_2) + \mathrm{i}\sin(\theta_1 - \theta_2)$

$$= \cos\frac{51\pi}{17} + \mathrm{i}\sin\frac{51\pi}{17}$$

$$= \cos 3\pi + \mathrm{i}\sin 3\pi$$

Simplify.

$$= \cos\pi + \mathrm{i}\sin\pi$$

Subtract 2π from the argument.

$$= -1 + \mathrm{i}(0)$$

So $\dfrac{\left(\cos\frac{9\pi}{17} + \mathrm{i}\sin\frac{9\pi}{17}\right)^5}{\left(\cos\frac{2\pi}{17} - \mathrm{i}\sin\frac{2\pi}{17}\right)^3} = -1$

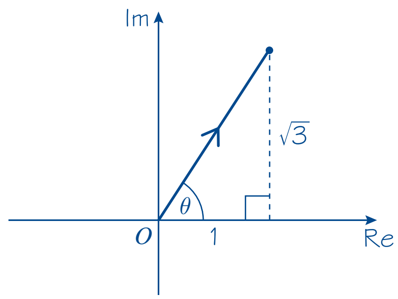
Problem-solving

You could also show this result by writing both numbers in exponential form:

$$\frac{\left(\mathrm{e}^{\frac{9\pi\mathrm{i}}{17}}\right)^5}{\left(\mathrm{e}^{-\frac{2\pi\mathrm{i}}{17}}\right)^3} = \frac{\mathrm{e}^{\frac{45\pi\mathrm{i}}{17}}}{\mathrm{e}^{-\frac{6\pi\mathrm{i}}{17}}} = \mathrm{e}^{\mathrm{i}\left(\frac{45\pi}{17} - \left(-\frac{6\pi}{17}\right)\right)} = \mathrm{e}^{3\pi\mathrm{i}} = \mathrm{e}^{\pi\mathrm{i}} = -1$$

Example 9

Express $(1 + i\sqrt{3})^7$ in the form $x + iy$ where $x, y \in \mathbb{R}$.



First, you need to find the modulus and argument of $1 + i\sqrt{3}$. You may want to draw an Argand diagram to help you.

$r = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{4} = 2$

$\theta = \arctan\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$

Find r and θ.

So $1 + i\sqrt{3} = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

Write $1 + i\sqrt{3}$ in modulus–argument form.

$(1 + i\sqrt{3})^7 = \left(2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right)^7$

$= 2^7\left(\cos\frac{7\pi}{3} + i\sin\frac{7\pi}{3}\right)$

Apply de Moivre's theorem.

$= 128\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

Subtract 2π from the argument.

$= 128\left(\frac{1}{2} + i\left(\frac{\sqrt{3}}{2}\right)\right)$

Therefore, $(1 + i\sqrt{3})^7 = 64 + 64i\sqrt{3}$

Exercise 3C

1 Use de Moivre's theorem to express in the form $x + iy$, where $x, y \in \mathbb{R}$.

a $(\cos\theta + i\sin\theta)^6$ **b** $(\cos 3\theta + i\sin 3\theta)^4$ **c** $\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^5$

d $\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^8$ **e** $\left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)^5$ **f** $\left(\cos\frac{\pi}{10} - i\sin\frac{\pi}{10}\right)^{15}$

2 Express in the form $e^{in\theta}$

a $\frac{\cos 5\theta + i\sin 5\theta}{(\cos 2\theta + i\sin 2\theta)^2}$ **b** $\frac{(\cos 2\theta + i\sin 2\theta)^7}{(\cos 4\theta + i\sin 4\theta)^3}$ **c** $\frac{1}{(\cos 2\theta + i\sin 2\theta)^3}$

d $\frac{(\cos 2\theta + i\sin 2\theta)^4}{(\cos 3\theta + i\sin 3\theta)^3}$ **e** $\frac{\cos 5\theta + i\sin 5\theta}{(\cos 3\theta - i\sin 3\theta)^2}$ **f** $\frac{\cos\theta - i\sin\theta}{(\cos 2\theta - i\sin 2\theta)^3}$

3 Evaluate, giving your answers in the form $x + \mathrm{i}y$, where $x, y \in \mathbb{R}$.

a $\dfrac{\left(\cos\frac{7\pi}{13} - \mathrm{i}\sin\frac{7\pi}{13}\right)^4}{\left(\cos\frac{4\pi}{13} + \mathrm{i}\sin\frac{4\pi}{13}\right)^6}$ **b** $\dfrac{\left(\cos\frac{3\pi}{7} - \mathrm{i}\sin\frac{11\pi}{7}\right)^3}{\left(\cos\frac{15\pi}{7} + \mathrm{i}\sin\frac{\pi}{7}\right)^2}$ **c** $\dfrac{\left(\cos\frac{4\pi}{3} - \mathrm{i}\sin\frac{2\pi}{3}\right)^7}{\left(\cos\frac{10\pi}{3} + \mathrm{i}\sin\frac{4\pi}{3}\right)^4}$

4 Express in the form $x + \mathrm{i}y$ where $x, y \in \mathbb{R}$.

a $(1 + \mathrm{i})^5$ **b** $(-2 + 2\mathrm{i})^8$ **c** $(1 - \mathrm{i})^6$

d $(1 - \mathrm{i}\sqrt{3})^6$ **e** $\left(\frac{3}{2} - \frac{1}{2}\mathrm{i}\sqrt{3}\right)^9$ **f** $(-2\sqrt{3} - 2\mathrm{i})^5$

E **5** Express $(3 + \mathrm{i}\sqrt{3})^5$ in the form $a + b\mathrm{i}\sqrt{3}$ where a and b are integers. **(2 marks)**

E **6** $w = 2\left(\cos\frac{\pi}{6} + \mathrm{i}\sin\frac{\pi}{6}\right)$

Find the exact value of w^4, giving your answer in the form $a + \mathrm{i}b$ where $a, b \in \mathbb{R}$. **(2 marks)**

E **7** $z = \sqrt{3}\left(\cos\frac{3\pi}{4} - \mathrm{i}\sin\frac{3\pi}{4}\right)$

Find the exact value of z^6, giving your answer in the form $a + \mathrm{i}b$ where $a, b \in \mathbb{R}$. **(3 marks)**

E/P **8** **a** Express $\dfrac{1 + \mathrm{i}\sqrt{3}}{1 - \mathrm{i}\sqrt{3}}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$ where $r > 0$ and $-\pi < \theta \leqslant \pi$. **(3 marks)**

b Hence find the smallest positive integer value of n for which $\left(\dfrac{1 + \mathrm{i}\sqrt{3}}{1 - \mathrm{i}\sqrt{3}}\right)^n$ is real and positive. **(2 marks)**

E/P **9** Use de Moivre's theorem to show that $(a + b\mathrm{i})^n + (a - b\mathrm{i})^n$ is real for all integers n. **(5 marks)**

Challenge

Without using Euler's relation, prove that if n is a positive integer,

$$(r(\cos\theta + \mathrm{i}\sin\theta))^{-n} = r^{-n}(\cos(-n\theta) + \mathrm{i}\sin(-n\theta))$$

Problem-solving

You may assume de Moivre's theorem for positive integer exponents, but do not write any complex numbers in exponential form.

3.4 Trigonometric identities

You can use de Moivre's theorem to derive trigonometric identities.

Applying the binomial expansion to $(\cos\theta + \mathrm{i}\sin\theta)^n$ allows you to express $\cos n\theta$ in terms of powers of $\cos\theta$, and $\sin n\theta$ in terms of powers of $\sin\theta$.

Links $(a + b)^n = a^n + {}^nC_1 a^{n-1}b + {}^nC_2 a^{n-2}b^2 + \ldots + {}^nC_r a^{n-r}b^r + \ldots + b^n, n \in \mathbb{N}$

where ${}^nC_r = \binom{n}{r} = \dfrac{n!}{r!(n-r)!}$

← Pure 2 Section 4.3

Example 10

Use de Moivre's theorem to show that

$$\cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$$

$(\cos\theta + i\sin\theta)^6 = \cos 6\theta + i\sin 6\theta$ — Apply de Moivre's theorem.

$$= \cos^6\theta + {}^6C_1\cos^5\theta(i\sin\theta) + {}^6C_2\cos^4\theta(i\sin\theta)^2 + {}^6C_3\cos^3\theta(i\sin\theta)^3 + {}^6C_4\cos^2\theta(i\sin\theta)^4 + {}^6C_5\cos\theta(i\sin\theta)^5 + (i\sin\theta)^6$$

Apply the binomial expansion to $(\cos\theta + i\sin\theta)^6$.

$$= \cos^6\theta + 6i\cos^5\theta\sin\theta + 15i^2\cos^4\theta\sin^2\theta + 20i^3\cos^3\theta\sin^3\theta + 15i^4\cos^2\theta\sin^4\theta + 6i^5\cos\theta\sin^5\theta + i^6\sin^6\theta$$

Simplify.

$$= \cos^6\theta + 6i\cos^5\theta\sin\theta - 15\cos^4\theta\sin^2\theta - 20i\cos^3\theta\sin^3\theta + 15\cos^2\theta\sin^4\theta + 6i\cos\theta\sin^5\theta - \sin^6\theta$$

Simplify the powers of i.

Equating the real parts gives

The real part of $\cos 6\theta + i\sin 6\theta$ is $\cos 6\theta$.

$$\cos 6\theta = \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta$$

$$= \cos^6\theta - 15\cos^4\theta(1-\cos^2\theta) + 15\cos^2\theta(1-\cos^2\theta)^2 - (1-\cos^2\theta)^3$$

Apply $\sin^2\theta \equiv 1 - \cos^2\theta$, $\sin^4\theta \equiv (\sin^2\theta)^2$ and $\sin^6\theta \equiv (\sin^2\theta)^3$

$$= \cos^6\theta - 15\cos^4\theta(1-\cos^2\theta) + 15\cos^2\theta(1 - 2\cos^2\theta + \cos^4\theta) - (1 - 3\cos^2\theta + 3\cos^4\theta - \cos^6\theta)$$

Multiply out the brackets.

Apply a cubic binomial expansion.

$$= \cos^6\theta - 15\cos^4\theta + 15\cos^6\theta + 15\cos^2\theta - 30\cos^4\theta + 15\cos^6\theta - 1 + 3\cos^2\theta - 3\cos^4\theta + \cos^6\theta$$

Expand the brackets.

$$= 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$$

Simplify.

Therefore,

$$\cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$$

You can also find trigonometric identities for $\sin^n\theta$ and $\cos^n\theta$ where n is a positive integer.

If $z = \cos\theta + i\sin\theta$, then

$$\frac{1}{z} = z^{-1} = (\cos\theta + i\sin\theta)^{-1}$$

$$= (\cos(-\theta) + i\sin(-\theta))$$ — Apply de Moivre's theorem.

$$= \cos\theta - i\sin\theta$$ — Use $\cos\theta = \cos(-\theta)$ and $-\sin\theta = \sin(-\theta)$.

It follows that

$$z + \frac{1}{z} = \cos\theta + i\sin\theta + \cos\theta - i\sin\theta = 2\cos\theta$$

$$z - \frac{1}{z} = \cos\theta + i\sin\theta - (\cos\theta - i\sin\theta) = 2i\sin\theta$$

Also,

$z^n = (\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta$ — By de Moivre's theorem.

$\frac{1}{z^n} = z^{-n} = (\cos\theta + \mathrm{i}\sin\theta)^{-n}$

$= (\cos(-n\theta) + \mathrm{i}\sin(-n\theta))$ — Apply de Moivre's theorem.

$= \cos n\theta - \mathrm{i}\sin n\theta$ — Use $\cos\theta = \cos(-\theta)$ and $\sin(-\theta) = -\sin\theta$.

It follows that

$$z^n + \frac{1}{z^n} = \cos n\theta + \mathrm{i}\sin n\theta + \cos n\theta - \mathrm{i}\sin n\theta = 2\cos n\theta$$

$$z^n - \frac{1}{z^n} = \cos n\theta + \mathrm{i}\sin n\theta - (\cos n\theta - \mathrm{i}\sin n\theta) = 2\mathrm{i}\sin n\theta$$

It is important that you remember and are able to apply these results:

- $z + \frac{1}{z} = 2\cos\theta$
- $z - \frac{1}{z} = 2\mathrm{i}\sin\theta$
- $z^n + \frac{1}{z^n} = 2\cos n\theta$
- $z^n - \frac{1}{z^n} = 2\mathrm{i}\sin n\theta$

Notation In exponential form, these results are equivalent to:

$\cos n\theta = \frac{1}{2}(\mathrm{e}^{\mathrm{i}n\theta} + \mathrm{e}^{-\mathrm{i}n\theta})$ $\quad$ $\sin n\theta = \frac{1}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}n\theta} - \mathrm{e}^{-\mathrm{i}n\theta})$

Example 11

Express $\cos^5\theta$ in the form $a\cos 5\theta + b\cos 3\theta + c\cos\theta$, where a, b and c are constants.

Let $z = \cos\theta + \mathrm{i}\sin\theta$

$\left(z + \frac{1}{z}\right)^5 = (2\cos\theta)^5 = 32\cos^5\theta$ — Use $z + \frac{1}{z} = 2\cos\theta$.

$= z^5 + {}^5C_1 z^4\left(\frac{1}{z}\right) + {}^5C_2 z^3\left(\frac{1}{z}\right)^2 + {}^5C_3 z^2\left(\frac{1}{z}\right)^3 + {}^5C_4 z\left(\frac{1}{z}\right)^4 + \left(\frac{1}{z}\right)^5$ — Apply the binomial expansion to $\left(z + \frac{1}{z}\right)^5$.

$= z^5 + 5z^4\left(\frac{1}{z}\right) + 10z^3\left(\frac{1}{z^2}\right) + 10z^2\left(\frac{1}{z^3}\right) + 5z\left(\frac{1}{z^4}\right) + \left(\frac{1}{z^5}\right)$

$= z^5 + 5z^3 + 10z + \frac{10}{z} + \frac{5}{z^3} + \frac{1}{z^5}$ — Simplify.

$= \left(z^5 + \frac{1}{z^5}\right) + 5\left(z^3 + \frac{1}{z^3}\right) + 10\left(z + \frac{1}{z}\right)$ — Group z^n and $\frac{1}{z^n}$ terms.

$= 2\cos 5\theta + 5(2\cos 3\theta) + 10(2\cos\theta)$ — Use $z^n + \frac{1}{z^n} = 2\cos n\theta$

So, $32\cos^5\theta = 2\cos 5\theta + 10\cos 3\theta + 20\cos\theta$

$\Rightarrow \quad \cos^5\theta = \frac{1}{16}\cos 5\theta + \frac{5}{16}\cos 3\theta + \frac{5}{8}\cos\theta$ — This is in the required form with $a = \frac{1}{16}$, $b = \frac{5}{16}$ and $c = \frac{5}{8}$

Example 12

a Express $\sin^4\theta$ in the form $d\cos 4\theta + e\cos 2\theta + f$, where d, e and f are constants.

b Hence find the exact value of $\int_0^{\frac{\pi}{2}} \sin^4\theta \, d\theta$.

a Let $z = \cos\theta + i\sin\theta$

$$\left(z - \frac{1}{z}\right)^4 = (2i\sin\theta)^4 = 16i^4\sin^4\theta = 16\sin^4\theta$$

Use $z - \frac{1}{z} = 2i\sin\theta$, noting that $i^4 = 1$

$$= z^4 + {}^4C_1 z^3\left(-\frac{1}{z}\right) + {}^4C_2 z^2\left(-\frac{1}{z}\right)^2 + {}^4C_3 z^1\left(-\frac{1}{z}\right)^3 + \left(-\frac{1}{z}\right)^4$$

Apply the binomial expansion to $\left(z - \frac{1}{z}\right)^4$

$$= z^4 + 4z^3\left(-\frac{1}{z}\right) + 6z^2\left(\frac{1}{z^2}\right) + 4z\left(-\frac{1}{z^3}\right) + \left(\frac{1}{z^4}\right)$$

$$= z^4 - 4z^2 + 6 - \frac{4}{z^2} + \frac{1}{z^4}$$

Simplify.

$$= \left(z^4 + \frac{1}{z^4}\right) - 4\left(z^2 + \frac{1}{z^2}\right) + 6$$

Group z^n and $\frac{1}{z^n}$ terms.

$$= 2\cos 4\theta - 4(2\cos 2\theta) + 6$$

Use $z^n + \frac{1}{z^n} = 2\cos n\theta$

So, $16\sin^4\theta = 2\cos 4\theta - 8\cos 2\theta + 6$

$$\Rightarrow \quad \sin^4\theta = \frac{1}{8}\cos 4\theta - \frac{1}{2}\cos 2\theta + \frac{3}{8}$$

This is in the required form with $d = \frac{1}{8}$, $e = -\frac{1}{2}$ and $f = \frac{3}{8}$

b
$$\int_0^{\frac{\pi}{2}} \sin^4\theta \, d\theta = \int_0^{\frac{\pi}{2}} \left(\frac{1}{8}\cos 4\theta - \frac{1}{2}\cos 2\theta + \frac{3}{8}\right) d\theta$$

Use the answer from part **a**.

$$= \left[\frac{1}{32}\sin 4\theta - \frac{1}{4}\sin 2\theta + \frac{3}{8}\theta\right]_0^{\frac{\pi}{2}}$$

$\cos k\theta$ integrates to $\frac{1}{k}\sin k\theta$.

$$= \left(\frac{1}{32}\sin 2\pi - \frac{1}{4}\sin\pi + \frac{3}{8}\left(\frac{\pi}{2}\right)\right) - 0$$

$$= 0 - 0 + \frac{3\pi}{16}$$

$$= \frac{3\pi}{16}$$

Exercise 3D

Use de Moivre's theorem to prove the trigonometric identities:

(P) **1** **a** $\sin 3\theta \equiv 3\sin\theta - 4\sin^3\theta$

b $\sin 5\theta \equiv 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$

c $\cos 7\theta \equiv 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$

d $\cos^4\theta \equiv \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$

e $\sin^5\theta \equiv \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$

E/P **2** **a** Use de Moivre's theorem to show that

$$\cos 5\theta \equiv 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$$ **(5 marks)**

b Hence, given also that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, find all the solutions of $\cos 5\theta + 5\cos 3\theta = 0$ in the interval $0 \leqslant \theta < \pi$. Give your answers to 3 decimal places. **(6 marks)**

E/P **3** **a** Show that $32\cos^6\theta \equiv \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10$. **(6 marks)**

b Hence find $\int_0^{\frac{\pi}{6}} \cos^6\theta \, d\theta$ in the form $a\pi + b\sqrt{3}$ where a and b are rational constants to be found. **(3 marks)**

E/P **4** **a** Show that $32\cos^2\theta\sin^4\theta \equiv \cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2$. **(6 marks)**

b Hence find the exact value of $\int_0^{\frac{\pi}{3}} \cos^2\theta\sin^4\theta \, d\theta$. **(3 marks)**

P **5** By using de Moivre's theorem, or otherwise, compute the integrals.

a $\int_0^{\frac{\pi}{2}} \sin^6\theta \, d\theta$ **b** $\int_0^{\frac{\pi}{4}} \sin^2\theta\cos^4\theta \, d\theta$ **c** $\int_0^{\frac{\pi}{6}} \sin^3\theta\cos^5\theta \, d\theta$

E/P **6** **a** Use de Moivre's theorem to show that

$$\cos 6\theta \equiv 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$$ **(5 marks)**

b Hence find the six **distinct** solutions of the equation

$$32x^6 - 48x^4 + 18x^2 - \tfrac{3}{2} = 0$$

giving your answers to 3 decimal places where necessary. **(5 marks)**

Problem-solving

Use the substitution $x = \cos\theta$ to reduce the equation to the form $\cos 6\theta = k$. Find as many values of θ as you need to find six distinct values of x.

E/P **7** **a** Use de Moivre's theorem to show that $\sin 4\theta \equiv 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$ **(4 marks)**

b Hence, or otherwise, show that $\tan 4\theta \equiv \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$ **(4 marks)**

c Use your answer to part **b** to find, to 2 decimal places, the four solutions of the equation $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$ **(5 marks)**

3.5 nth roots of a complex number

You can use de Moivre's theorem to solve an equation of the form $z^n = w$, where $z, w \in \mathbb{C}$.
This is equivalent to finding the nth roots of w.

Just as a real number, x, has two square roots, $\sqrt{x}$ and $-\sqrt{x}$, any complex number has n distinct nth roots.

- **If z and w are non-zero complex numbers and n is a positive integer, then the equation $z^n = w$ has n distinct solutions.**

You can find the solutions to $z^n = w$ using de Moivre's theorem, and by considering the fact that the **argument of a complex number** is not unique.

Notation $\cos(\theta + 2k\pi) = \cos\theta$ and $\sin(\theta + 2k\pi) = \sin\theta$ for integer values of k.

- **For any complex number $z = r(\cos\theta + \mathrm{i}\sin\theta)$, you can write $z = r(\cos(\theta + 2k\pi) + \mathrm{i}\sin(\theta + 2k\pi))$, where k is any integer.**

Example 13

a Solve the equation $z^3 = 1$

b Represent your solutions to part **a** on an Argand diagram.

c Show that the three cube roots of 1 can be written as 1, ω and ω^2 where $1 + \omega + \omega^2 = 0$

a $z^3 = 1$

$z^3 = \cos 0 + \mathrm{i}\sin 0$ — Start by writing 1 in modulus–argument form.

$(r(\cos\theta + \mathrm{i}\sin\theta))^3 = \cos(0 + 2k\pi) + \mathrm{i}\sin(0 + 2k\pi),\ k \in \mathbb{Z}$ — Write z in modulus–argument form, and write the general form of the argument on the right-hand side by adding integer multiples of 2π.

$r^3(\cos 3\theta + \mathrm{i}\sin 3\theta) = \cos(0 + 2k\pi) + \mathrm{i}\sin(0 + 2k\pi),\ k \in \mathbb{Z}$ — Apply de Moivre's theorem to the left-hand side of the equation.

So $r = 1$ — Compare the modulus on both sides to get $r = 1$.

$3\theta = 2k\pi$ — Compare the arguments on both sides.

$k = 0 \Rightarrow \theta = 0$, so $z_1 = \cos 0 + \mathrm{i}\sin 0 = 1$

$k = 1 \Rightarrow \theta = \frac{2\pi}{3}$

so $z_2 = \cos\left(\frac{2\pi}{3}\right) + \mathrm{i}\sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2}$

$k = -1 \Rightarrow \theta = -\frac{2\pi}{3}$

so $z_3 = \cos\left(-\frac{2\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{2\pi}{3}\right) = -\frac{1}{2} - \mathrm{i}\frac{\sqrt{3}}{2}$

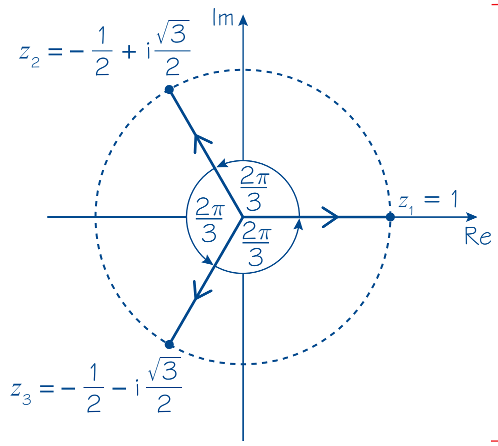
Therefore,

$z = 1,\ z = -\frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2}$ or $z = -\frac{1}{2} - \mathrm{i}\frac{\sqrt{3}}{2}$ — These are the cube roots of unity.

Problem-solving

Choose values of k to find the three distinct roots. By choosing values on either side of $k = 0$ you can find three different arguments in the interval $[-\pi, \pi]$.

b



Plot the points $z_1 = 1$, $z_2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ and $z_3 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$ on an Argand diagram:

The points z_1, z_2 and z_3 lie on a circle of radius 1 unit.

The angles between each of the **vectors** z_1, z_2 and z_3 are $\frac{2\pi}{3}$, as shown on the Argand diagram.

c Let $\omega = z_2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2} = e^{\frac{2\pi i}{3}}$

Then, $\omega^2 = \left(e^{\frac{2\pi i}{3}}\right)^2 = e^{\frac{4\pi i}{3}}$

$= e^{-\frac{2\pi i}{3}} = -\frac{1}{2} - i\frac{\sqrt{3}}{2} = z_3$

Notice that $\omega^* = \omega^2$.

$1 + \omega + \omega^2 =$

$1 + \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) + \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = 0$

Notation It can be proved that the sum of the nth roots of unity is zero, for any positive integer $n \geqslant 2$.

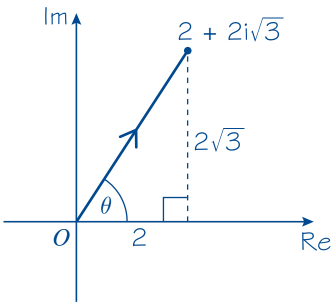
- In general, the solutions to $z^n = 1$ are $z = \cos\left(\frac{2\pi k}{n}\right) + i\sin\left(\frac{2\pi k}{n}\right) = e^{\frac{2\pi i k}{n}}$ for $k = 1, 2, \ldots, n$ and are known as the nth roots of unity.

  If n is a positive integer, then there is an nth root of unity $\omega = e^{\frac{2\pi i}{n}}$ such that:

  - the nth roots of unity are $1, \omega, \omega^2, \ldots, \omega^{n-1}$
  - $1, \omega, \omega^2, \ldots, \omega^{n-1}$ form the **vertices** of a regular n-gon
  - $1 + \omega + \omega^2 + \ldots + \omega^{n-1} = 0$

Example 14

Solve the equation $z^4 = 2 + 2\mathrm{i}\sqrt{3}$



modulus $= \sqrt{2^2 + (2\sqrt{3})^2} = \sqrt{4 + 12} = 4$

argument $= \arctan\left(\frac{2\sqrt{3}}{2}\right) = \frac{\pi}{3}$

So $z^4 = 4\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right)$

To solve an equation of the form $z^n = w$, start by writing w in modulus–argument form.

$(r(\cos\theta + \mathrm{i}\sin\theta))^4$

$= 4\left(\cos\left(\frac{\pi}{3} + 2k\pi\right) + \mathrm{i}\sin\left(\frac{\pi}{3} + 2k\pi\right)\right), k \in \mathbb{Z}$

Now let $z = r(\cos\theta + \mathrm{i}\sin\theta)$, and write the general form of the argument on the RHS by adding integer multiples of 2π.

$r^4(\cos 4\theta + \mathrm{i}\sin 4\theta)$

$= 4\left(\cos\left(\frac{\pi}{3} + 2k\pi\right) + \mathrm{i}\sin\left(\frac{\pi}{3} + 2k\pi\right)\right), k \in \mathbb{Z}$

Apply de Moivre's theorem to the LHS.

So $r^4 = 4 \Rightarrow r = \sqrt[4]{4} = \sqrt{2}$

Compare the modulus on both sides to get $r = \sqrt{2}$.

$4\theta = \frac{\pi}{3} + 2k\pi$

Compare the arguments on both sides.

$k = 0 \Rightarrow \theta = \frac{\pi}{12}$, so $z_1 = \sqrt{2}\left(\cos\frac{\pi}{12} + \mathrm{i}\sin\frac{\pi}{12}\right)$

$k = 1 \Rightarrow \theta = \frac{7\pi}{12}$, so $z_2 = \sqrt{2}\left(\cos\frac{7\pi}{12} + \mathrm{i}\sin\frac{7\pi}{12}\right)$

When $k = 1$, $4\theta = \frac{\pi}{3} + 2\pi$

$\Rightarrow \theta = \frac{\pi}{12} + \frac{2\pi}{4} = \frac{7\pi}{12}$

$k = -1 \Rightarrow \theta = -\frac{5\pi}{12}$, so $z_3 = \sqrt{2}\left(\cos\left(-\frac{5\pi}{12}\right) + \mathrm{i}\sin\left(-\frac{5\pi}{12}\right)\right)$

$k = -2 \Rightarrow \theta = -\frac{11\pi}{12}$, so $z_4 = \sqrt{2}\left(\cos\left(-\frac{11\pi}{12}\right) + \mathrm{i}\sin\left(-\frac{11\pi}{12}\right)\right)$

Watch out Make sure you choose n **consecutive** values of k to get n distinct roots. If an argument is not in the interval $[-\pi, \pi]$ you can add or subtract a multiple of 2π.

or $z = \sqrt{2}e^{\frac{\pi\mathrm{i}}{12}}$, $z = \sqrt{2}e^{\frac{7\pi\mathrm{i}}{12}}$, $z = \sqrt{2}e^{-\frac{5\pi\mathrm{i}}{12}}$ or $z = \sqrt{2}e^{-\frac{11\pi\mathrm{i}}{12}}$

These are the solutions in the form $re^{\mathrm{i}\theta}$.

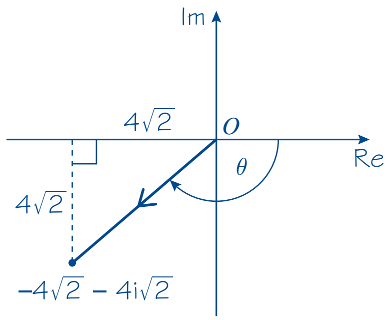
You can also use the exponential form of a complete number when solving equations.

Example 15

Solve the equation $z^3 + 4\sqrt{2} + 4i\sqrt{2} = 0$

$z^3 + 4\sqrt{2} + 4i\sqrt{2} = 0$

$z^3 = -4\sqrt{2} - 4i\sqrt{2}$



$\text{modulus} = \sqrt{(-4\sqrt{2})^2 + (-4\sqrt{2})^2} = \sqrt{32 + 32} = \sqrt{64} = 8$

$\text{argument} = -\pi + \arctan\left(\frac{4\sqrt{2}}{4\sqrt{2}}\right) = -\pi + \frac{\pi}{4} = -\frac{3\pi}{4}$

Find the modulus and argument of $-4\sqrt{2} - 4i\sqrt{2}$.

$(re^{i\theta})^3 = 8e^{\left(-\frac{3\pi}{4} + 2k\pi\right)i}$

$r^3e^{3i\theta} = 8e^{\left(-\frac{3\pi}{4} + 2k\pi\right)i}$

Write $z = re^{i\theta}$ and use $(re^{i\theta})n = r^ne^{in\theta}$. Remember to write the general form of the argument on the right-hand side by adding integer multiples of 2π.

So $r^3 = 8 \Rightarrow r = \sqrt[3]{8} = 2$

Compare the modulus on both sides to get $r = 2$.

$3\theta = -\frac{3\pi}{4} + 2k\pi$

Compare the arguments on both sides.

$k = 0 \Rightarrow \theta = -\frac{\pi}{4}$, so $z_1 = 2e^{\frac{-\pi i}{4}}$

$k = 1 \Rightarrow \theta = \frac{5\pi}{12}$, so $z_2 = 2e^{\frac{5\pi i}{12}}$

$k = -1 \Rightarrow \theta = -\frac{11\pi}{12}$, so $z_3 = 2e^{\frac{-11\pi i}{12}}$

or $z = 2\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$, $z = 2\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)$

or $z = 2\left(\cos\left(-\frac{11\pi}{12}\right) + i\sin\left(-\frac{11\pi}{12}\right)\right)$.

Choose values of k to find three distinct roots. Either choose values that produce arguments in the interval $-\pi < \theta \leqslant \pi$, or add or subtract multiples of 2π as necessary.

Exercise 3E

1 Solve the equations, expressing your answers for z in the form $x + iy$, where $x, y \in \mathbb{R}$.

a $z^4 - 1 = 0$ **b** $z^3 - i = 0$ **c** $z^3 = 27$

d $z^4 + 64 = 0$ **e** $z^4 + 4 = 0$ **f** $z^3 + 8i = 0$

2 Solve the equations, expressing the roots in the form $r(\cos\theta + i\sin\theta)$, where $-\pi < \theta \leqslant \pi$.

a $z^7 = 1$ **b** $z^4 + 16i = 0$ **c** $z^5 + 32 = 0$

d $z^3 = 2 + 2i$ **e** $z^4 + 2i\sqrt{3} = 2$ **f** $z^3 + 32\sqrt{3} + 32i = 0$

3 Solve the equations, expressing the roots in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. Give θ to 2 decimal places.

a $z^4 = 3 + 4i$ **b** $z^3 = \sqrt{11} - 4i$ **c** $z^4 = -\sqrt{7} + 3i$

(P) **4** **a** Find the three roots of the equation $(z + 1)^3 = -1$
Give your answers in the form $x + iy$, where $x, y \in \mathbb{R}$.

b Plot the points representing these three roots on an Argand diagram.

c Given that these three points lie on a circle, find its centre and radius.

(P) **5** **a** Find the five roots of the equation $z^5 - 1 = 0$
Give your answers in the form $r(\cos\theta + i\sin\theta)$, where $-\pi < \theta \leqslant \pi$.

b Hence or otherwise, show that

$$\cos\left(\frac{2\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right) = -\frac{1}{2}$$

Problem-solving

Use the fact that the sum of the five roots of unity is zero.

(E) **6** **a** Find the modulus and argument of $-2 - 2i\sqrt{3}$ **(2 marks)**

b Hence find all the solutions of the equation $z^4 + 2 + 2i\sqrt{3} = 0$
Give your answers in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$ and illustrate the roots on an Argand diagram. **(4 marks)**

(E) **7** Find the four distinct roots of the equation $z^4 = 2(1 - i\sqrt{3})$ in exponential form, and show these roots on an Argand diagram. **(7 marks)**

E/P **8** $z = \sqrt{6} + i\sqrt{2}$

a Find the modulus and argument of z. **(2 marks)**

b Find the values of w such that $w^3 = z^4$, giving your answers in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. **(4 marks)**

P **9** **a** Solve the equation

$$1 + z + z^2 + z^3 + z^4 + z^5 + z^6 + z^7 = 0$$

b Hence deduce that $(z^2 + 1)$ and $(z^4 + 1)$ are factors of

$$1 + z + z^2 + z^3 + z^4 + z^5 + z^6 + z^7.$$

Problem-solving

$1 + z + z^2 + z^3 + \ldots + z^7$ is the sum of a geometric series.

Challenge

a Find the six roots of the equation $z^6 = 1$ in the form $e^{i\theta}$, where $-\pi < \theta \leqslant \pi$.

b Hence show that the solutions to $(z + 1)^6 = z^6$ are

$z = -\frac{1}{2} + \frac{1}{2}i\cot\left(\frac{k\pi}{6}\right)$, $k = 1, 2, 3, 4, 5$.

Chapter review 3

P **1** **a** Use $e^{i\theta} = \cos\theta + i\sin\theta$ to show that $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$

b Hence prove that $\cos A\cos B \equiv \dfrac{\cos(A + B) + \cos(A - B)}{2}$

E/P **2** Given that $z = r(\cos\theta + i\sin\theta)$, $r \in \mathbb{R}$, prove by induction that $z^n = r^n(\cos n\theta + i\sin n\theta)$, $n \in \mathbb{Z}^+$. **(5 marks)**

3 Express $\dfrac{(\cos 3x + i\sin 3x)^2}{\cos x - i\sin x}$ in the form $\cos nx + i\sin nx$ where n is an integer to be determined.

4 Use de Moivre's theorem to evaluate:

a $(-1 + i)^8$ **b** $\dfrac{1}{\left(\frac{1}{2} - \frac{1}{2}i\right)^{16}}$

E/P **5** **a** Given $z = \cos\theta + i\sin\theta$, use de Moivre's theorem to show that $z^n + \dfrac{1}{z^n} = 2\cos n\theta$. **(4 marks)**

b Express $\left(z^2 + \dfrac{1}{z^2}\right)^3$ in terms of $\cos 6\theta$ and $\cos 2\theta$. **(3 marks)**

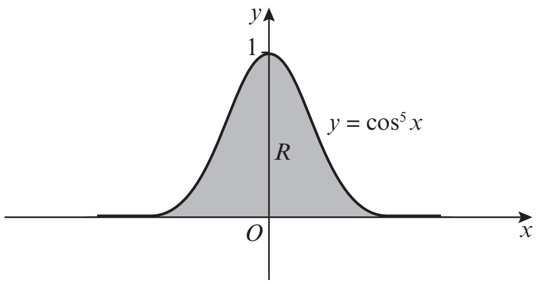
c Hence, or otherwise, find constants a and b such that $\cos^3 2\theta = a\cos 6\theta + b\cos 2\theta$. **(3 marks)**

d Hence, or otherwise, show that $\int_0^{\frac{\pi}{6}} \cos^3 2\theta \, d\theta = k\sqrt{3}$, where k is a rational constant. **(4 marks)**

E/P **6** **a** Show that

$\cos^5\theta \equiv \frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)$ **(5 marks)**

The diagram shows the curve with equation $y = \cos^5 x$, $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$. The finite region R is **bounded** by the curve and the x-axis.



b Calculate the exact area of R. **(6 marks)**

E/P **7** **a** Show that

$\sin^6\theta \equiv -\frac{1}{32}(\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10)$ **(5 marks)**

b Using the substitution $\alpha = \left(\frac{\pi}{2} - \theta\right)$, or otherwise, find a similar **identity** for $\cos^6\theta$. **(3 marks)**

c Given that $\int_0^a (\cos^6\theta + \sin^6\theta)\,d\theta = \frac{5\pi}{32}$, find the exact value of a. **(5 marks)**

E/P **8** Use de Moivre's theorem to show that

$\sin 6\theta \equiv \sin 2\theta(16\cos^4\theta - 16\cos^2\theta + 3)$ **(5 marks)**

E/P **9** **a** Use de Moivre's theorem to show that

$\cos 5\theta \equiv 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$ **(5 marks)**

b Hence find all solutions to the equation

$16x^5 - 20x^3 + 5x + 1 = 0$

giving your answers to 3 decimal places where necessary. **(5 marks)**

E/P **10** **a** Show that

$\sin^5\theta \equiv \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$ **(5 marks)**

b Hence solve the equation

$\sin 5\theta - 5\sin 3\theta + 9\sin\theta = 0$ for $0 \leqslant \theta < \pi$ **(4 marks)**

E/P **11** **a** Use de Moivre's theorem to show that $\cos 5\theta \equiv \cos\theta(16\cos^4\theta - 20\cos^2\theta + 5)$ **(5 marks)**

b By solving the equation $\cos 5\theta = 0$, deduce that $\cos^2\left(\frac{\pi}{10}\right) = \frac{5 + \sqrt{5}}{8}$ **(4 marks)**

c Hence, or otherwise, write down the exact values of $\cos^2\left(\frac{3\pi}{10}\right)$, $\cos^2\left(\frac{7\pi}{10}\right)$ and $\cos^2\left(\frac{9\pi}{10}\right)$. **(3 marks)**

E/P **12** **a** Use de Moivre's theorem to find an expression for $\tan 3\theta$ in terms of $\tan \theta$. **(4 marks)**

b Deduce that $\cot 3\theta = \dfrac{\cot^3\theta - 3\cot\theta}{3\cot^2\theta - 1}$ **(2 marks)**

E **13** **a** Express $4 - 4\text{i}$ in the form $r(\cos\theta + \text{i}\sin\theta)$, where $r > 0$, $-\pi < \theta \leqslant \pi$, where r and θ are exact values. **(2 marks)**

b Hence, or otherwise, solve the equation $z^5 = 4 - 4\text{i}$, leaving your answers in the form $z = R\text{e}^{\text{i}k\pi}$, where R is the modulus of z and k is a **rational number** such that $-1 \leqslant k \leqslant 1$. **(4 marks)**

c Show on an Argand diagram the points representing the roots. **(2 marks)**

E/P **14** **a** Find the cube roots of $2 - 2\text{i}$ in the form $r\text{e}^{\text{i}\theta}$ where $r > 0$ and $-\pi < \theta \leqslant \pi$. **(5 marks)**

These cube roots are represented by points A, B and C in the Argand diagram, with A in the fourth quadrant and ABC going anticlockwise. The **midpoint** of AB is M, and M represents the complex number w.

b Draw an Argand diagram, showing the points A, B, C and M. **(2 marks)**

c Find the modulus and argument of w. **(2 marks)**

d Find w^6 in the form $a + b\text{i}$. **(3 marks)**

Challenge

Show that the points on an Argand diagram that represent the roots of $\left(\dfrac{z+1}{z}\right)^6 = 1$ lie on a straight line.

Summary of key points

1 You can use **Euler's relation**, $\text{e}^{\text{i}\theta} = \cos\theta + \text{i}\sin\theta$, to write a complex number z in exponential form:

$$z = r\text{e}^{\text{i}\theta}$$

where $r = |z|$ and $\theta = \arg z$.

2 For any two complex numbers $z_1 = r_1\text{e}^{\text{i}\theta_1}$ and $z_2 = r_2\text{e}^{\text{i}\theta_2}$

- $z_1 z_2 = r_1 r_2 \text{e}^{\text{i}(\theta_1 + \theta_2)}$
- $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\text{e}^{\text{i}(\theta_1 - \theta_2)}$

3 **De Moivre's theorem:**
For any integer n, $(r(\cos\theta + \text{i}\sin\theta))^n = r^n(\cos n\theta + \text{i}\sin n\theta)$

4
- $z + \dfrac{1}{z} = 2\cos\theta$
- $z^n + \dfrac{1}{z^n} = 2\cos n\theta$
- $z - \dfrac{1}{z} = 2\text{i}\sin\theta$
- $z^n - \dfrac{1}{z^n} = 2\text{i}\sin n\theta$

5 If z and w are non-zero complex numbers and n is a positive integer, then the equation $z^n = w$ has n distinct solutions.

6 For any complex number $z = r(\cos\theta + \text{i}\sin\theta)$, you can write

$$z = r(\cos(\theta + 2k\pi) + \text{i}\sin(\theta + 2k\pi))$$

where k is any integer.

7 In general, the solutions to $z^n = 1$ are $z = \cos\left(\frac{2\pi k}{n}\right) + \text{i}\sin\left(\frac{2\pi k}{n}\right) = \text{e}^{\frac{2\pi \text{i}k}{n}}$ for $k = 1, 2, \ldots, n$ and are known as the nth roots of unity.

If n is a positive integer, then there is an nth root of unity $\omega = \text{e}^{\frac{2\pi\text{i}}{n}}$ such that:

- The nth roots of unity are $1, \omega, \omega^2, \ldots, \omega^{n-1}$
- $1, \omega, \omega^2, \ldots, \omega^{n-1}$ form the vertices of a regular n-gon
- $1 + \omega + \omega^2 + \ldots + \omega^{n-1} = 0$

8 The nth roots of any complex number s lie on the vertices of a regular n-gon with its centre at the origin.