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MATHEMATICS

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PEARSON EDEXCEL INTERNATIONAL A LEVEL

FURTHER PURE MATHEMATICS 3

STUDENT BOOK

PEARSON EDEXCEL INTERNATIONAL A LEVEL

FURTHER PURE MATHEMATICS 3

Student Book

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Endorsement Statement
In order to ensure that this resource offers high-quality support for the associated Pearson qualification, it has been through a review process by the awarding body. This process confirms that this resource fully covers the teaching and learning content of the specification or part of a specification at which it is aimed. It also confirms that it demonstrates an appropriate balance between the development of subject skills, knowledge and understanding, in addition to preparation for assessment.

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ABOUT THIS BOOK

The following three themes have been fully integrated throughout the Pearson Edexcel International Advanced Level in Mathematics series, so they can be applied alongside your learning.

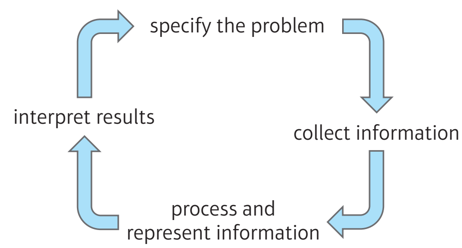
1. Mathematical argument, language and proof

- Rigorous and consistent approach throughout
- Notation boxes explain key mathematical language and symbols

2. Mathematical problem-solving

- Hundreds of problem-solving questions, fully integrated into the main exercises
- Problem-solving boxes provide tips and strategies
- Challenge questions provide extra stretch

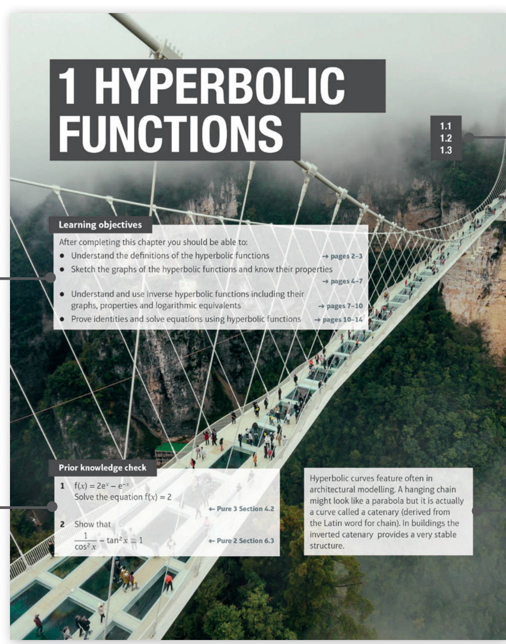
The Mathematical Problem-Solving Cycle



3. Transferable skills

- Transferable skills are embedded throughout this book, in the exercises and in some examples
- These skills are signposted to show students which skills they are using and developing

Finding your way around the book



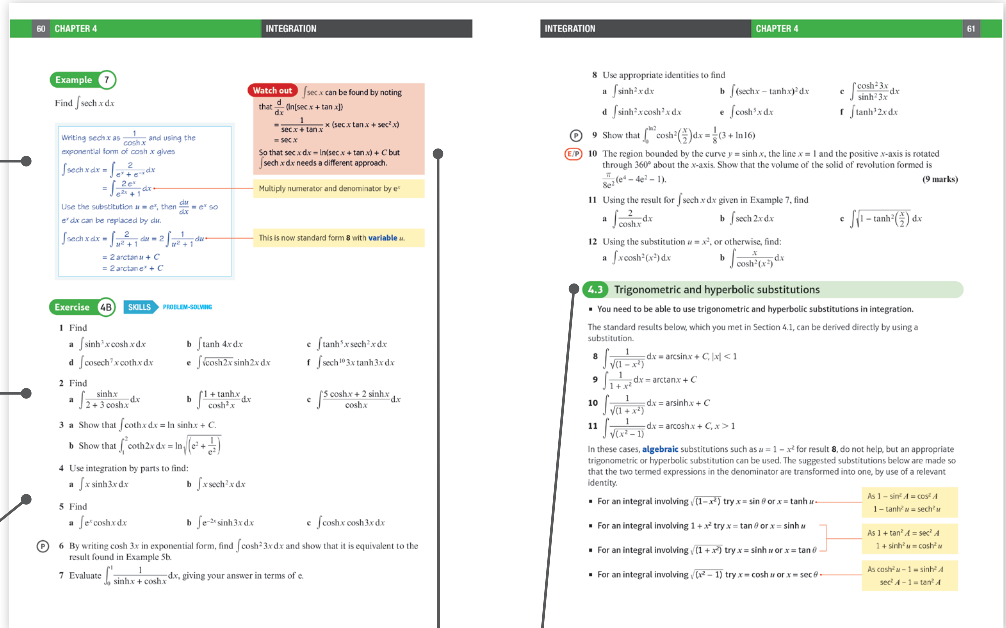
Each chapter is mapped to the specification content for easy reference

Each chapter starts with a list of Learning objectives

The *Prior knowledge check* helps make sure you are ready to start the chapter

The real world applications of the maths you are about to learn are highlighted at the start of the chapter

Glossary terms will be identified by bold blue text on their first appearance



Step-by-step worked examples focus on the key types of questions you'll need to tackle

Exercise questions are carefully graded so they increase in difficulty and gradually bring you up to exam standard

Exercises are packed with exam-style questions to ensure you are ready for the exams

Problem-solving boxes provide hints, tips and strategies, and *Watch out* boxes highlight areas where students often lose marks in their exams

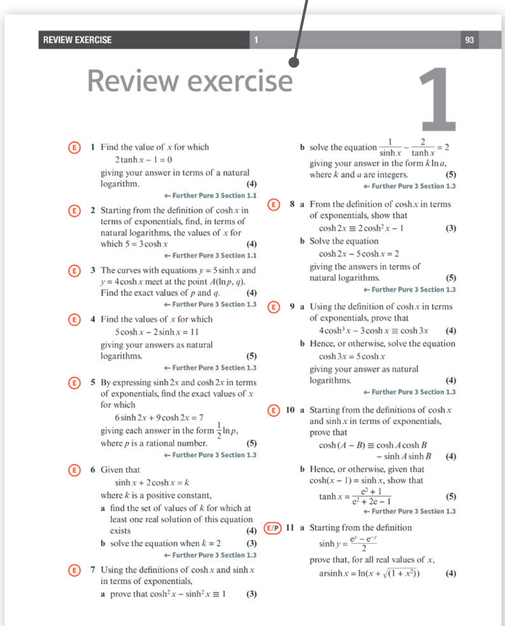
Each section begins with an explanation and key learning points

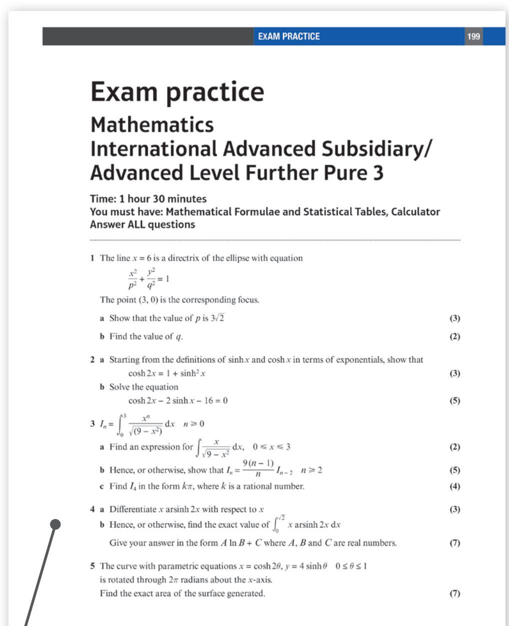
Exam-style questions are flagged with (E)

Problem-solving questions are flagged with (P)

Each chapter ends with a *Chapter review* and a *Summary of key points*

After every few chapters, a *Review exercise* helps you consolidate your learning with lots of exam-style questions





A full practice paper at the back of the book helps you prepare for the real thing

QUALIFICATION AND ASSESSMENT OVERVIEW

Qualification and content overview

Further Pure Mathematics 3 (FP3) is an **optional*** unit in the following qualifications:

International Advanced Subsidiary in Further Mathematics

International Advanced Level in Further Mathematics

*It is compulsory to study **either** FP2 **or** FP3 for the International Advanced Level in Further Mathematics.

Assessment overview

The following table gives an overview of the assessment for this unit.

We recommend that you study this information closely to help ensure that you are fully prepared for this course and know exactly what to expect in the assessment.

Unit	Percentage	Mark	Time	Availability
FP3: Further Pure Mathematics 3 Paper code WFM03/01	$33\frac{1}{3}$ % of IAS $16\frac{2}{3}$ % of IAL	75	1 hour 30 mins	January and June First assessment June 2020

IAS: International Advanced Subsidiary, IAL: International Advanced A Level.

Assessment objectives and weightings

		Minimum weighting in IAS and IAL
AO1	Recall, select and use their knowledge of mathematical facts, concepts and techniques in a variety of contexts.	30%
AO2	Construct rigorous mathematical arguments and proofs through use of precise statements, logical deduction and inference and by the manipulation of mathematical expressions, including the construction of extended arguments for handling substantial problems presented in unstructured form.	30%
AO3	Recall, select and use their knowledge of standard mathematical models to represent situations in the real world; recognise and understand given representations involving standard models; present and interpret results from such models in terms of the original situation, including discussion of the assumptions made and refinement of such models.	10%
AO4	Comprehend translations of common realistic contexts into mathematics; use the results of calculations to make predictions, or comment on the context; and, where appropriate, read critically and comprehend longer mathematical arguments or examples of applications.	5%
AO5	Use contemporary calculator technology and other permitted resources (such as formulae booklets or statistical tables) accurately and efficiently; understand when not to use such technology, and its limitations. Give answers to appropriate accuracy.	5%

Relationship of assessment objectives to units

FP3	Assessment objective				
	AO1	AO2	AO3	AO4	AO5
Marks out of 75	25–30	25–30	0–5	7–12	5–10
%	$33\frac{1}{3}$–40	$33\frac{1}{3}$–40	$6\frac{2}{3}$	$9\frac{1}{3}$–13	$6\frac{2}{3}$–$13\frac{1}{3}$

Calculators

Students may use a calculator in assessments for these qualifications. Centres are responsible for making sure that calculators used by their students meet the requirements given in the table below.

Students are expected to have available a calculator with at least the following keys: +, –, ×, ÷, π, x^2, $\sqrt{x}$, $\frac{1}{x}$, x^y, ln x, e^x, $x!$, sine, cosine and tangent and their inverses in degrees and decimals of a degree, and in radians; memory.

Prohibitions

Calculators with any of the following facilities are prohibited in all examinations:

- databanks
- retrieval of text or formulae
- built-in symbolic algebra manipulations
- symbolic differentiation and/or integration
- language translators
- communication with other machines or the internet

Extra online content

Whenever you see an *Online* box, it means that there is extra online content available to support you.

SolutionBank

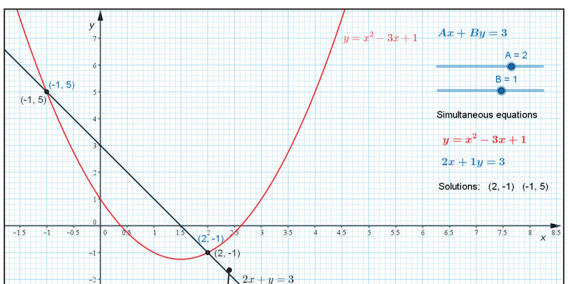
SolutionBank provides worked solutions for questions in the book. Download the solutions as a PDF or quickly find the solution you need online.

Use of technology

Explore topics in more detail, visualise problems and consolidate your understanding. Use pre-made GeoGebra activities or Casio resources for a graphic calculator.

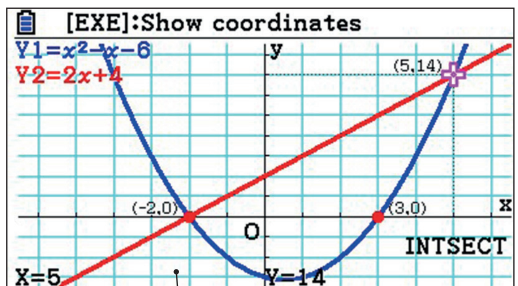
Online Find the point of intersection graphically using technology.

GeoGebra-powered interactives



Interact with the maths you are learning using GeoGebra's easy-to-use tools

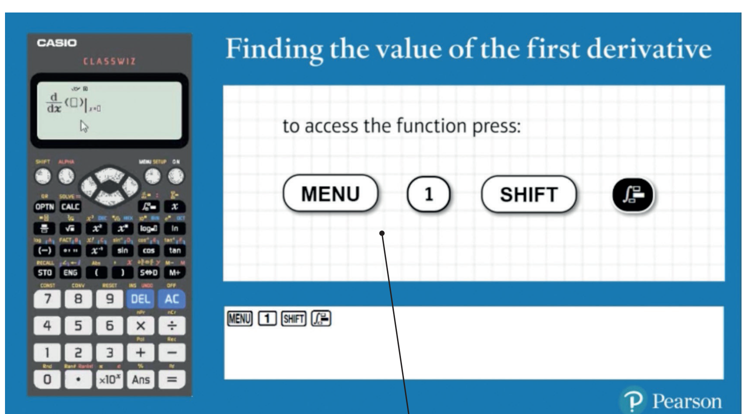
Graphic calculator interactives



Explore the maths you are learning and gain confidence in using a graphic calculator

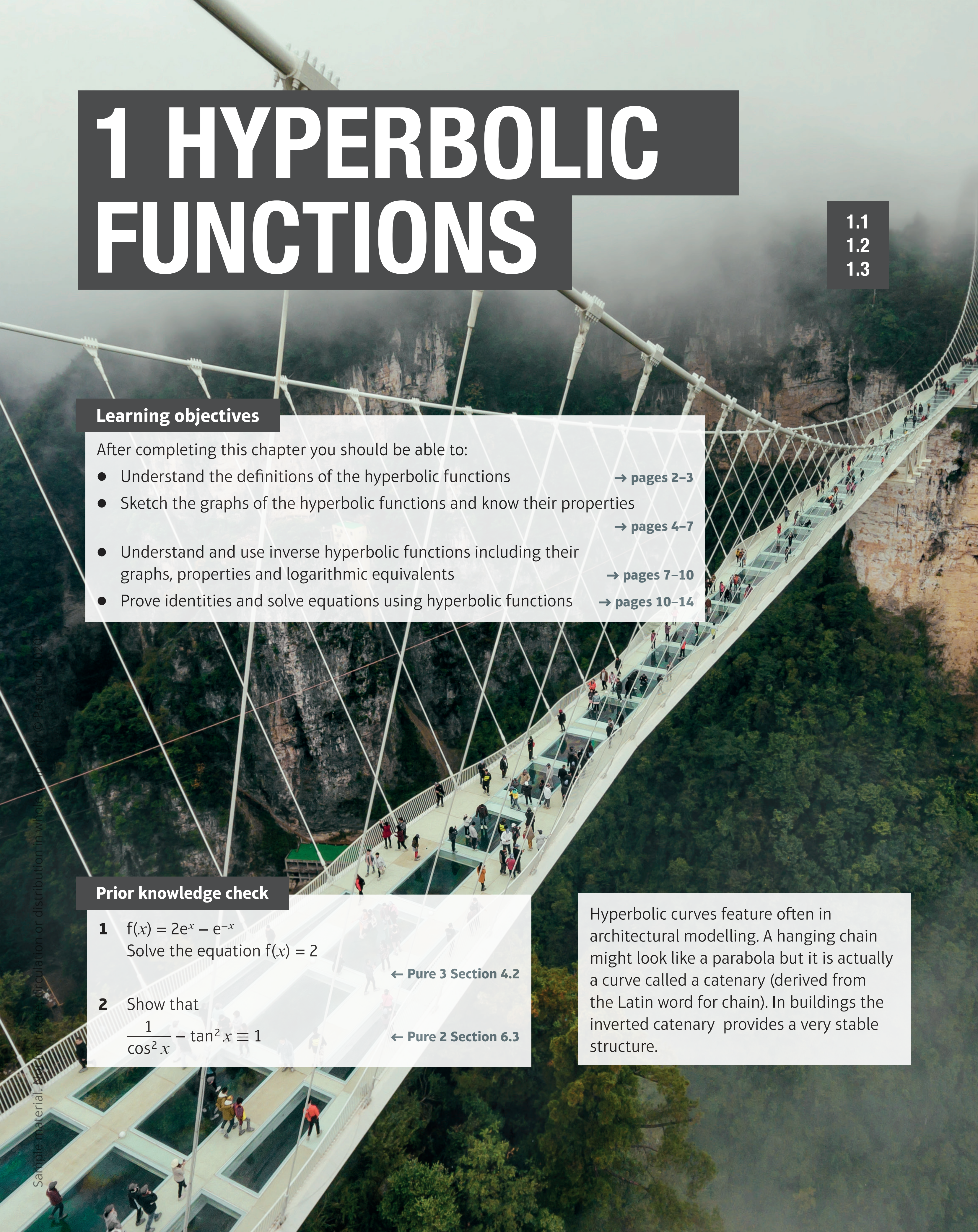
Calculator tutorials

Our helpful video tutorials will guide you through how to use your calculator in the exams. They cover both Casio's scientific and colour graphic calculators.



Step-by-step guide with audio instructions on exactly which buttons to press and what should appear on your calculator's screen

Online Work out each coefficient quickly using the ^nC_r and power functions on your calculator.



1 HYPERBOLIC FUNCTIONS

1.1
1.2
1.3

Learning objectives

After completing this chapter you should be able to:

- Understand the definitions of the hyperbolic functions → pages 2–3
- Sketch the graphs of the hyperbolic functions and know their properties → pages 4–7
- Understand and use inverse hyperbolic functions including their graphs, properties and logarithmic equivalents → pages 7–10
- Prove identities and solve equations using hyperbolic functions → pages 10–14

Prior knowledge check

1 $f(x) = 2e^{x} - e^{-x}$
Solve the equation $f(x) = 2$ ← Pure 3 Section 4.2

2 Show that
$$\frac{1}{\cos^2 x} - \tan^2 x \equiv 1$$
← Pure 2 Section 6.3

Hyperbolic curves feature often in architectural modelling. A hanging chain might look like a parabola but it is actually a curve called a catenary (derived from the Latin word for chain). In buildings the inverted catenary provides a very stable structure.

1.1 Introduction to hyperbolic functions

Hyperbolic functions have several properties in common with trigonometric **functions**, but they are defined in terms of **exponential** functions.

Notation x belongs to the set of real numbers, using the correct mathematical notation.

- Hyperbolic **sine** (or sinh) is defined as **sinh** $x \equiv \frac{e^x - e^{-x}}{2}$

Notation Often pronounced 'shine'.

- Hyperbolic **cosine** (or cosh) is defined as **cosh** $x \equiv \frac{e^x + e^{-x}}{2}$

Notation Often pronounced 'cosh'.

- Hyperbolic **tangent** (or tanh) is defined as **tanh** $x \equiv \frac{\sinh x}{\cosh x}$

Notation Often pronounced 'tanch' or 'than'.

You can use the definitions of sinh x and cosh x to write tanh x in exponential form.

$$\tanh x \equiv \frac{\sinh x}{\cosh x} \equiv \frac{e^x - e^{-x}}{2} \times \frac{2}{e^x + e^{-x}} \equiv \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Multiplying the numerator and denominator of the final expression through by e^x gives:

- $\tanh x \equiv \frac{e^{2x} - 1}{e^{2x} + 1}$

There are also hyperbolic functions corresponding to (i.e. connected to) the **reciprocal** trigonometric functions:

- Hyperbolic **cosecant** (or cosech) is defined as **cosech** $x \equiv \frac{1}{\sinh x} \equiv \frac{2}{e^x - e^{-x}}$

Notation Often pronounced 'cosech' or 'cosheck'.

- Hyperbolic **secant** (or sech) is defined as **sech** $x \equiv \frac{1}{\cosh x} \equiv \frac{2}{e^x + e^{-x}}$

Notation Often pronounced 'sheck' or 'setch'.

- Hyperbolic **cotangent** (or coth) is defined as **coth** $x \equiv \frac{1}{\tanh x} \equiv \frac{e^{2x} + 1}{e^{2x} - 1}$

Notation Often pronounced 'coth'.

Example 1 SKILLS ANALYSIS

Find, to 2 decimal places, the values of

a sinh 3 **b** cosh 1 **c** tanh 0.8

a $\sinh 3 = \frac{e^3 - e^{-3}}{2} = 10.02$ (2 d.p.)

b $\cosh 1 = \frac{e^1 + e^{-1}}{2} = 1.54$ (2 d.p.)

c $\tanh 0.8 = \frac{e^{1.6} - 1}{e^{1.6} + 1} = 0.66$ (2 d.p.)

Example 2

Find the exact value of tanh (ln 4).

$$\tanh(\ln 4) = \frac{e^{2\ln 4} - 1}{e^{2\ln 4} + 1} = \frac{e^{\ln 4^2} - 1}{e^{\ln 4^2} + 1} = \frac{e^{\ln 16} - 1}{e^{\ln 16} + 1}$$

$$= \frac{16 - 1}{16 + 1} = \frac{15}{17}$$

Use $e^{\ln k} = k$.

Example 3

Use the definition of sinh x to find, to 2 decimal places, the value of x for which sinh $x = 5$

$$\frac{e^x - e^{-x}}{2} = 5 \Rightarrow e^x - e^{-x} = 10$$

$$e^{2x} - 1 = 10e^x$$

Multiply both sides by e^x.

$$e^{2x} - 10e^x - 1 = 0$$

The substitution $u = e^x$ turns this into the quadratic equation $u^2 - 10u - 1 = 0$

$$e^x = 5 \pm \sqrt{26}$$

$$\Rightarrow e^x = 5 + \sqrt{26}$$

e^x cannot be negative.

So $x = \ln(5 + \sqrt{26}) = 2.31$ (2 d.p.)

Exercise 1A

SKILLS ANALYSIS

1 Use your calculator to find, to 2 decimal places, the value of:

a sinh 4 **b** $\cosh\left(\frac{1}{2}\right)$ **c** tanh (–2) **d** sech 5

2 Write, in terms of e:

a sinh 1 **b** cosh 4 **c** tanh 0.5 **d** sech (–1)

3 Find the exact values of:

a sinh (ln 2) **b** cosh (ln 3) **c** tanh (ln 2) **d** cosech (ln π)

In questions **4** to **8**, use the definitions of the hyperbolic functions (in terms of exponentials) to find each answer, then check your answers using an inverse hyperbolic function on your calculator.

4 Find, to 2 decimal places, the values of x for which cosh $x = 2$

5 Find, to 2 decimal places, the values of x for which sinh $x = 1$

6 Find, to 2 decimal places, the values of x for which tanh $x = -\frac{1}{2}$

7 Find, to 2 decimal places, the values of x for which coth $x = 10$

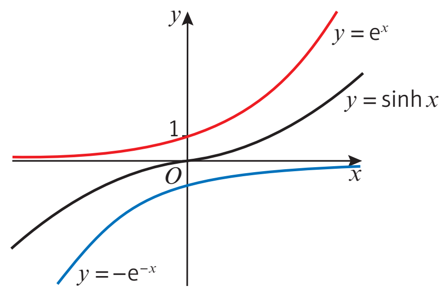
8 Find, to 2 decimal places, the values of x for which sech $x = \frac{1}{8}$

1.2 Sketching graphs of hyperbolic functions

You can sketch the graphs of the hyperbolic functions by considering the graphs of $y = e^x$ and $y = e^{-x}$

$$\sinh x = \frac{e^x - e^{-x}}{2} = \frac{e^x + (-e^{-x})}{2}$$

so the graph of $y = \sinh x$ is the 'average' of the graphs of $y = e^x$ and $y = -e^{-x}$

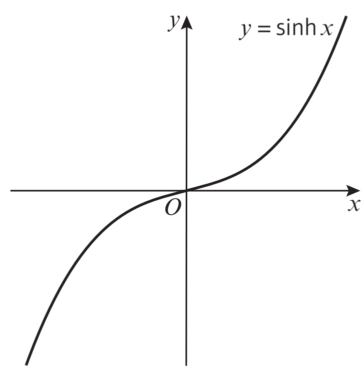


For the graph of $y = \sinh x$

- when x is large and positive, e^{-x} is small, so $\sinh x \approx \frac{1}{2}e^x$
- when x is large and negative, e^x is small, so $\sinh x \approx -\frac{1}{2}e^{-x}$

■ **For any value a, sinh $(-a) = -$sinh a**

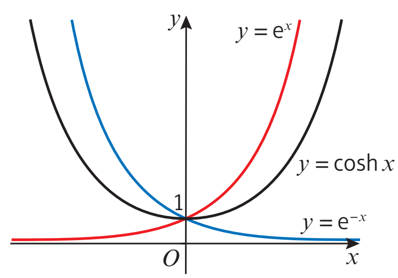
Notation $f(x) = \sinh x$ is an **odd** function since $f(-x) = -f(x)$



Consider the graphs of $y = e^x$ and $y = e^{-x}$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

so the graph of $y = \cosh x$ is the 'average' of the graphs of $y = e^x$ and $y = e^{-x}$

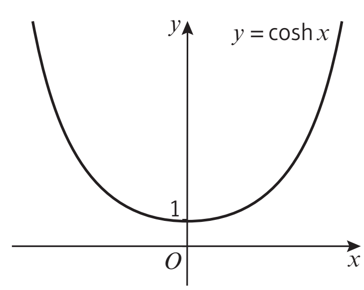


For the graph of $y = \cosh x$

- when x is large and positive, e^{-x} is small, so $\cosh x \approx \frac{1}{2}e^x$
- when x is large and negative, e^x is small, so $\cosh x \approx \frac{1}{2}e^{-x}$

■ **For any value a, cosh $(-a) =$ cosh a**

Notation $f(x) = \cosh x$ is an **even** function because $f(-x) = f(x)$



Example 4 SKILLS CRITICAL THINKING

Sketch the graph of $y = \tanh x$

$\tanh x = \dfrac{\sinh x}{\cosh x}$

When $x = 0$, $\tanh x = \dfrac{0}{1} = 0$

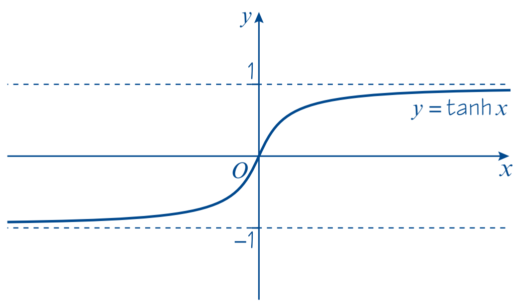
When x is large and positive, $\sinh x \approx \dfrac{1}{2}e^x$ and $\cosh x \approx \dfrac{1}{2}e^x$, so $\tanh x \approx 1$.

When x is large and negative, $\sinh x \approx -\dfrac{1}{2}e^{-x}$ and $\cosh x \approx \dfrac{1}{2}e^{-x}$, so $\tanh x \approx -1$.

As $x \to \infty$, $\tanh x \to 1$ and as $x \to -\infty$, $\tanh x \to -1$

For $f(x) = \tanh x$, $x \in \mathbb{R}$, the range of f is $-1 < f(x) < 1$

$y = -1$ and $y = 1$ are asymptotes to the curve.



Online Explore graphs of hyberbolic functions using GeoGebra.

Consider the graphs of $y = \sinh x$ and $y = \cosh x$ to work out the behaviour of $y = \tanh x$ as $x \to \infty$ and $x \to -\infty$

You should always include any **asymptotes** on a sketch graph.

Example 5 SKILLS CRITICAL THINKING

Sketch the graph of $y = \operatorname{sech} x$

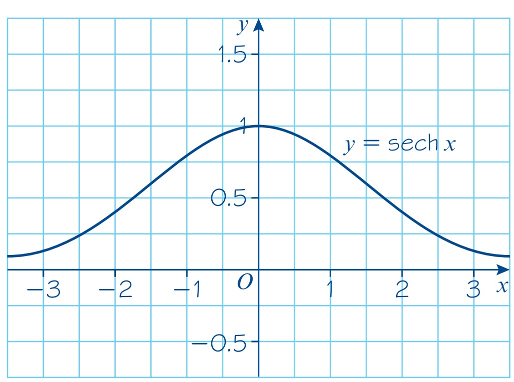
Using $\operatorname{sech} x = \dfrac{1}{\cosh x}$

When $x = 0$, $\operatorname{sech} x = \dfrac{1}{1} = 1$

As $x \to \infty$, $\cosh x \to \infty$, so $\operatorname{sech} x \to 0$

As $x \to -\infty$, $\cosh x \to \infty$, so $\operatorname{sech} x \to 0$

The x-axis is an asymptote to the curve.



Check this sketch using the graphic function on your calculator.

Example 6 SKILLS CRITICAL THINKING

Sketch the graph of $y = \text{cosech}\, x$, $x \neq 0$

Using $\text{cosech}\, x = \dfrac{1}{\sinh x}$

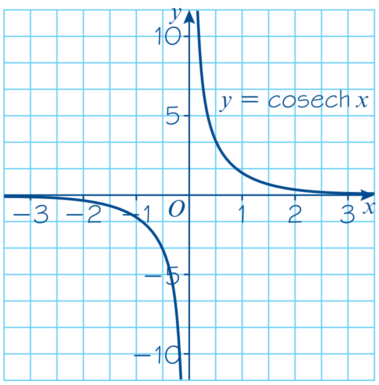
For positive x, as $x \to 0$, $\text{cosech}\, x \to \infty$

For negative x, as $x \to 0$, $\text{cosech}\, x \to -\infty$

As $x \to \infty$, $\sinh x \to \infty$, so $\text{cosech}\, x \to 0$

As $x \to -\infty$, $\sinh x \to -\infty$, so $\text{cosech}\, x \to 0$

The x- and y-axes are asymptotes to the curve.



Check this sketch using the graphic function on your calculator.

Example 7 SKILLS CRITICAL THINKING

Sketch the graph of $y = \coth x$, $x \neq 0$.

Using $\coth x = \dfrac{1}{\tanh x}$

For positive x, as $x \to 0$, $\coth x \to \infty$

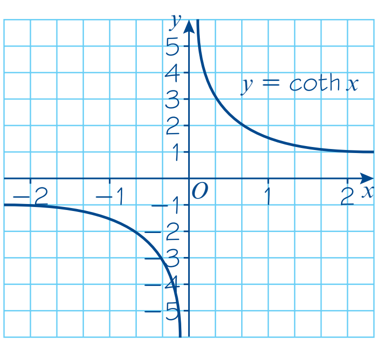
For negative x, as $x \to 0$, $\coth x \to -\infty$

As $x \to \infty$, $\tanh x \to 1$, so $\coth x \to 1$

As $x \to -\infty$, $\tanh x \to -1$, so $\coth x \to -1$

The y-axis is an asymptote to the curve.

$y = -1$ and $y = 1$ are asymptotes to the curve.



Check this sketch using the graphic function on your calculator.

Exercise 1B

1 On the same diagram, sketch the graphs of $y = \cosh 2x$ and $y = 2\cosh x$

2 a On the same diagram, sketch the graphs of $y = \text{sech}\, x$ and $y = \sinh x$

b Show that, at the point of intersection of the graphs, $x = \frac{1}{2}\ln(2 = \sqrt{5})$

3 Find the range of each hyberbolic function.

a $f(x) = \sinh x,\ x \in \mathbb{R}$

b $f(x) = \cosh x,\ x \in \mathbb{R}$

c $f(x) = \tanh x,\ x \in \mathbb{R}$

d $f(x) = \text{sech}\, x,\ x \in \mathbb{R}$

e $f(x) = \text{cosech}\, x,\ x \in \mathbb{R},\ x \neq 0$

f $f(x) = \coth x,\ x \in \mathbb{R},\ x \neq 0$

4 a Sketch the graph of $y = 1 + \coth x,\ x \in \mathbb{R},\ x \neq 0$

b Write down the equations of the asymptotes to this curve.

5 a Sketch the graph of $y = 3\tanh x,\ x \in \mathbb{R}$

b Write down the equations of the asymptotes to this curve.

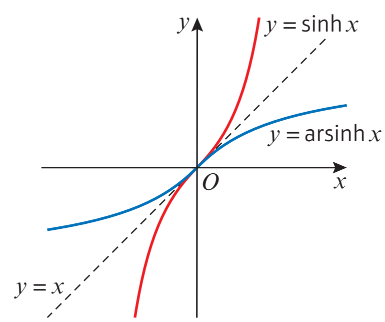
Challenge Sketch the graph of $y = \sinh x + \cosh x$

1.3 Inverse hyperbolic functions

You can define and use the inverses of the hyperbolic functions.

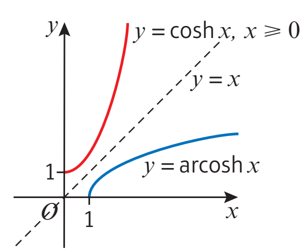
If $f(x) = \sinh x$, the inverse function f^{-1} is called $\text{arsinh}\, x$.

The graph of $y = \text{arsinh}\, x$ is the reflection of the graph of $y = \sinh x$ in the line $y = x$.



The inverse of a function is defined only if the function is one-to-one, so for $\cosh x$ the **domain** must be restricted in order to define an inverse.

For $f(x) = \cosh x,\ x \geqslant 0,\ f^{-1}(x) = \text{arcosh}\, x,\ x \geqslant 1$



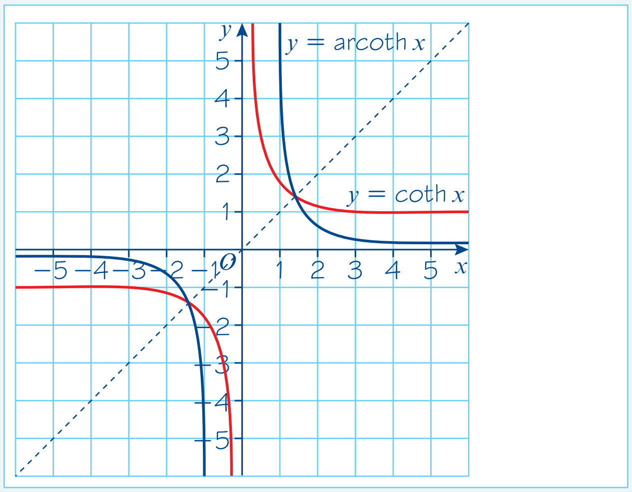
- The following table shows the inverse hyperbolic functions, with domains restricted where necessary.

Notation **arsinh**, **arcosh** and **artanh** are sometimes written as $\sinh^{-1}$, $\cosh^{-1}$ and $\tanh^{-1}$

Hyperbolic function	Inverse hyperbolic function
$y = \sinh x$	$y = \text{arsinh}\, x$
$y = \cosh x,\ x \geqslant 0$	$y = \text{arcosh}\, x,\ x \geqslant 1$
$y = \tanh x$	$y = \text{artanh}\, x,\ \lvert x\rvert < 1$
$y = \text{sech}\, x,\ x \geqslant 0$	$y = \text{arsech}\, x,\ 1 < x \leqslant 1$
$y = \text{cosech}\, x,\ x \neq 0$	$y = \text{arcosech}\, x,\ x \neq 0$
$y = \coth x,\ x \neq 0$	$y = \text{arcoth}\, x,\ \lvert x\rvert > 1$

Example 8 SKILLS REASONING/CRITICAL THINKING

Sketch the graph of $y = \text{arcoth}\, x,\ |x| > 1$



Reflect the graph of $y = \coth x$ in the line $y = x$

You can express the inverse hyperbolic functions in terms of **natural logarithms**.

Example 9 SKILLS ANALYSIS

Show that $\text{arsinh}\, x = \ln\left(x + \sqrt{x^2 + 1}\right)$

Let $y = \text{arsinh}\, x$

$x = \sinh y$

$x = \dfrac{e^y - e^{-y}}{2}$ — Use the definition of sinh.

$e^y - e^{-y} = 2x$

$e^{2y} - 1 = 2xe^y$ — Multiply by e^y

$e^{2y} - 2xe^y - 1 = 0$

$(e^y - x)^2 - x^2 - 1 = 0$

$e^y = x \pm \sqrt{x^2 + 1}$

$e^y = x - \sqrt{x^2 + 1}$ can be ignored since $\sqrt{x^2 + 1} > x$, and would give a negative value of e^y, which is not possible.

So $e^y = x + \sqrt{x^2 + 1}$

$y = \ln\left(x + \sqrt{x^2 + 1}\right) \Rightarrow \text{arsinh}\, x = \ln\left(x + \sqrt{x^2 + 1}\right)$

Problem-solving

$e^{2y} - 2x e^y - 1 = 0$ is a quadratic in e^y
You can write it as $(e^y)^2 - 2x e^y - 1 = 0$ and then complete the square.

Example 10

Show that $\text{arcosh}\,x = \ln(x + \sqrt{x^2 - 1}),\ x \geqslant 1$

Let $y = \text{arcosh}\,x$

$x = \cosh y$

$x = \dfrac{e^y + e^{-y}}{2}$ — Use the definition of cosh.

$e^y + e^{-y} = 2x$

$e^{2y} + 1 = 2xe^y$ — Multiply by e^y

$e^{2y} - 2xe^y + 1 = 0$ — Form (i.e. create) and solve a quadratic in e^y

$(e^y - x)^2 - x^2 + 1 = 0$

So $e^y = x \pm \sqrt{x^2 - 1}$

$y = \ln(x \pm \sqrt{x^2 - 1})$ — Note that both $x + \sqrt{x^2 - 1}$ and $x - \sqrt{x^2 - 1}$ are positive.

$\Rightarrow \text{arcosh}\,x = \ln(x + \sqrt{x^2 - 1})$ — arcosh x is always non-negative. For all values of $x > 1$, $x - \sqrt{x^2 - 1} < 1$ so the value of $\ln(x - \sqrt{x^2 - 1})$ is negative.

You can use a similar method to express artanh x in terms of natural logarithms.

The following formulae are provided in the formula booklet and can be used directly unless you are asked to prove them.

- $\text{arsinh}\,x = \ln(x + \sqrt{x^2 + 1})$
- $\text{arcosh}\,x = \ln(x + \sqrt{x^2 - 1}),\ x \geqslant 1$
- $\text{artanh}\,x = \frac{1}{2}\ln\left(\frac{1 + x}{1 - x}\right),\ |x| < 1$

Example 11

Express as natural logarithms

a arsinh 1 **b** arcosh 2 **c** artanh $\frac{1}{3}$

a $\text{arsinh}\,1 = \ln(1 + \sqrt{1^2 + 1}) = \ln(1 + \sqrt{2})$

b $\text{arcosh}\,2 = \ln(2 + \sqrt{2^2 - 1}) = \ln(2 + \sqrt{3})$

c $\text{artanh}\,\frac{1}{3} = \frac{1}{2}\ln\left(\dfrac{1 + \frac{1}{3}}{1 - \frac{1}{3}}\right) = \frac{1}{2}\ln 2 = \ln\sqrt{2}$ — Use $a\ln x = \ln x^a$

Exercise 1C

SKILLS ANALYSIS

1 Sketch the graph of $y = \text{artanh}\,x,\ |x| < 1$

(P) 2 Sketch the graph of $y = \text{arcosech}\,x,\ 0 < x < 1$

(E/P) 3 Prove that $\text{artanh}\,x = \frac{1}{2}\ln\left(\frac{1 + x}{1 - x}\right),\ |x| < 1$ **(5 marks)**

4 Express as natural logarithms

a arsinh 2 **b** arcosh 3 **c** $\text{artanh}\,\frac{1}{2}$

5 Express as natural logarithms

a $\text{arsinh}\,\sqrt{2}$ **b** $\text{arcosh}\,\sqrt{5}$ **c** artanh 0.1

6 Express as natural logarithms

a arsinh (−3) **b** $\text{arcosh}\,\frac{3}{2}$ **c** $\text{artanh}\,\frac{1}{\sqrt{3}}$

E/P **7** Given that $\text{artanh}\,x + \text{artanh}\,y = \ln\sqrt{3}$, prove that $y = \dfrac{2x-1}{x-2}$ **(6 marks)**

1.4 Identities and equations

You can find and use identities for the hyperbolic functions that are similar to the trigonometric identities. You can solve **equations** involving hyperbolic functions.

Example 12

SKILLS REASONING/ARGUMENTATION

Prove that $\cosh^2 A - \sinh^2 A \equiv 1$

$$\text{LHS} \equiv \cosh^2 A - \sinh^2 A \equiv \left(\frac{e^A + e^{-A}}{2}\right)^2 - \left(\frac{e^A - e^{-A}}{2}\right)^2$$

Use the definitions of cosh and sinh.

$$\equiv \left(\frac{e^{2A} + 2 + e^{-2A}}{4}\right) - \left(\frac{e^{2A} - 2 + e^{-2A}}{4}\right)$$

$e^A \times e^{-A} = \frac{e^A}{e^A} = 1$

$$\equiv \frac{4}{4} \equiv 1 \equiv \text{RHS}$$

- $\cosh^2 A - \sinh^2 A \equiv 1$ (this formula is given to you in the formula booklet)

Example 13

Prove that $\sinh(A + B) \equiv \sinh A \cosh B + \cosh A \sinh B$

$$\text{RHS} \equiv \sinh A \cosh B + \cosh A \sinh B$$

$$\equiv \left(\frac{e^A - e^{-A}}{2}\right)\left(\frac{e^B + e^{-B}}{2}\right) + \left(\frac{e^A + e^{-A}}{2}\right)\left(\frac{e^B - e^{-B}}{2}\right)$$

$$\equiv \left(\frac{e^{A+B} + e^{A-B} - e^{-A+B} - e^{-A-B}}{4}\right) + \left(\frac{e^{A+B} - e^{A-B} + e^{-A+B} - e^{-A-B}}{4}\right)$$

$$\equiv \frac{2e^{A+B} - 2e^{-A-B}}{4} \equiv \frac{e^{A+B} - e^{-(A+B)}}{2} \equiv \sinh(A + B) \equiv \text{LHS}$$

You can prove other sinh and cosh addition formulae similarly, giving:

- $\sinh(A \pm B) \equiv \sinh A \cosh B \pm \cosh A \sinh B$
- $\cosh(A \pm B) \equiv \cosh A \cosh B \pm \sinh A \sinh B$

Example 14

Prove that $\cosh 2A \equiv 1 + 2\sinh^2 A$

$$\text{RHS} \equiv 1 + 2\sinh^2 A$$
$$\equiv 1 + 2\left(\frac{e^A - e^{-A}}{2}\right)\left(\frac{e^A - e^{-A}}{2}\right)$$
$$\equiv 1 + 2\left(\frac{e^{2A} - 2 + e^{-2A}}{4}\right) \equiv 1 - 1 + \left(\frac{e^{2A} + e^{-2A}}{2}\right)$$
$$\equiv \cosh 2A \equiv \text{LHS}$$

Use the definition of sinh.

Given a trigonometric **identity**, it is generally possible to write down the corresponding hyperbolic identity using what is known as **Osborn's rule**:

- Replace cos by cosh: $\cos A \rightarrow \cosh A$
- Replace sin by sinh: $\sin A \rightarrow \sinh A$

However …

- replace any product (or **implied** product) of two sin terms by **minus** the product of two sinh terms:

e.g. $\sin A \sin B \rightarrow -\sinh A \sinh B$

$\tan^2 A \rightarrow -\tanh^2 A$

This is the implied product of two sin terms because $\tan^2 A \equiv \frac{\sin^2 A}{\cos^2 A}$

Example 15

SKILLS ANALYSIS

Write down the hyperbolic identity corresponding to

a $\cos 2A \equiv 2\cos^2 A - 1$ **b** $\tan(A - B) \equiv \frac{\tan A - \tan B}{1 + \tan A \tan B}$

a $\cosh 2A \equiv 2\cosh^2 A - 1$

b $\tanh(A - B) \equiv \frac{\tanh A - \tanh B}{1 - \tanh A \tanh B}$

Implied product of two sin terms because $\tan A \tan B \equiv \frac{\sin A \sin B}{\cos A \cos B}$

Example 16

Given that $\sinh x = \frac{3}{4}$, find the exact value of

a $\cosh x$ **b** $\tanh x$ **c** $\sinh 2x$

a Using $\cosh^2 x - \sinh^2 x \equiv 1$,

$\cosh^2 x - \frac{9}{16} = 1 \Rightarrow \cosh^2 x = \frac{25}{16}$

$\Rightarrow \cosh x = \frac{5}{4}$

$\cosh x \geqslant 1$, so $\cosh x = -\frac{5}{4}$ is not possible.

b Using $\tanh x \equiv \frac{\sinh x}{\cosh x}$

$\tanh x = \frac{3}{4} \div \frac{5}{4} = \frac{3}{5}$

c Using $\sinh 2x \equiv 2\sinh x \cosh x$

$\sinh 2x = 2 \times \frac{3}{4} \times \frac{5}{4} = \frac{15}{8}$

Example 17

Solve $6\sinh x - 2\cosh x = 7$ for real values of x.

$$6\left(\frac{e^x - e^{-x}}{2}\right) - 2\left(\frac{e^x + e^{-x}}{2}\right) = 7$$

There is no hyperbolic identity that will easily transform the equation into an equation in just one hyperbolic function, so use the basic definitions.

$$3e^x - 3e^{-x} - e^x - e^{-x} = 7$$
$$2e^x - 7 - 4e^{-x} = 0$$
$$2e^{2x} - 7e^x - 4 = 0$$
$$(2e^x + 1)(e^x - 4) = 0$$
$$e^x = -\frac{1}{2},\ e^x = 4$$
$$e^x = 4$$

There are no real values of x for which $e^x = -\frac{1}{2}$

$$x = \ln 4$$

Example 18

Solve $2\cosh^2 x - 5\sinh x = 5$, giving your answers as natural logarithms.

Using $\cosh^2 x - \sinh^2 x \equiv 1$,

Use this identity to transform the equation into an equation in just one hyperbolic function.

$$2(1 + \sinh^2 x) - 5\sinh x = 5$$
$$2\sinh^2 x - 5\sinh x - 3 = 0$$
$$(2\sinh x + 1)(\sinh x - 3) = 0$$

So $\sinh x = -\frac{1}{2}$ or $\sinh x = 3$

Then,

$$x = \text{arsinh}\left(-\frac{1}{2}\right) \quad \text{or } x = \text{arsinh}\, 3$$

$$\Rightarrow x = \ln\left(-\frac{1}{2} + \sqrt{\frac{1}{4} + 1}\right) \text{ or } x = \ln\left(3 + \sqrt{9 + 1}\right)$$

Use $\text{arsinh}\, x = \ln\left(x + \sqrt{x^2 + 1}\right)$.

$$\Rightarrow x = \ln\left(-\frac{1}{2} + \frac{\sqrt{5}}{2}\right) \quad \text{or } x = \ln\left(3 + \sqrt{10}\right)$$

Example 19

Solve $\cosh 2x - 5\cosh x + 4 = 0$, giving your answers as natural logarithms where appropriate.

Using $\cosh 2x \equiv 2\cosh^2 x - 1$,

Use this identity to transform the equation into an equation in just one hyperbolic function.

$$2\cosh^2 x - 1 - 5\cosh x + 4 = 0$$
$$2\cosh^2 x - 5\cosh x + 3 = 0$$
$$(2\cosh x - 3)(\cosh x - 1) = 0$$

So $\cosh x = \frac{3}{2}$ or $\cosh x = 1$

$$\Rightarrow x = \ln\left(\frac{3}{2} \pm \sqrt{\frac{9}{4} - 1}\right) \text{ or } x = 0$$

You can use $\text{arcosh}\, x = \ln\left(x + \sqrt{x^2 - 1}\right)$, but remember that both $\ln\left(x + \sqrt{x^2 - 1}\right)$ and $\ln\left(x - \sqrt{x^2 - 1}\right)$ are possible.

For any value of k greater than 1, $\cosh x = k$ will give two values of x, one positive and one negative.

$$\Rightarrow x = \ln\left(\frac{3}{2} \pm \frac{\sqrt{5}}{2}\right) \text{ or } x = 0$$

Exercise 1D SKILLS REASONING/ARGUMENTATION

1 Prove the following identities, using the definitions of $\sinh x$ and $\cosh x$.

a $\sinh 2A \equiv 2 \sinh A \cosh A$

b $\cosh (A - B) \equiv \cosh A \cosh B - \sinh A \sinh B$

c $\cosh 3A \equiv 4 \cosh^3 A - 3 \cosh A$

d $\sinh A - \sinh B \equiv 2 \sinh \left(\frac{A - B}{2}\right) \cosh \left(\frac{A + B}{2}\right)$

2 Use Osborn's rule to write down the hyperbolic identities corresponding to the following trigonometric identities.

a $\sin (A - B) \equiv \sin A \cos B - \cos A \sin B$

b $\sin 3A \equiv 3 \sin A - 4 \sin^3 A$

c $\cos A + \cos B \equiv 2 \cos \left(\frac{A + B}{2}\right) \cos \left(\frac{A - B}{2}\right)$

d $\cos 2A \equiv \frac{1 - \tan^2 A}{1 + \tan^2 A}$

e $\cos 2A \equiv \cos^4 A - \sin^4 A$

3 Given that $\cosh x = 2$, find the exact values of

a $\sinh x$

b $\tanh x$

c $\cosh 2x$

4 Given that $\sinh x = -1$, find the exact values of

a $\cosh x$

b $\sinh 2x$

c $\tanh 2x$

5 Solve the following equations, giving your answers as natural logarithms.

a $3 \sinh x + 4 \cosh x = 4$

b $7 \sinh x - 5 \cosh x = 1$

c $30 \cosh x = 15 + 26 \sinh x$

d $13 \sinh x - 7 \cosh x + 1 = 0$

e $\cosh 2x - 5 \sinh x = 13$

f $3 \sinh^2 x - 13 \cosh x + 7 = 0$

g $\sinh 2x - 7 \sinh x = 0$

h $4 \cosh x + 13e^{-x} = 11$

i $2 \tanh x = \cosh x$

E **6** **a** Starting from the definitions of $\sinh x$ and $\cosh x$ in terms of exponentials, prove that

$$\cosh 2x \equiv 2 \cosh^2 x - 1$$ **(3 marks)**

b Solve the equation

$$\cosh 2x - 3 \cosh x = 8$$

giving your answers as exact logarithms. **(5 marks)**

E **7** Solve the equation

$$2 \sinh^2 x - 5 \cosh x = 5$$

giving your answer in terms of natural logarithms in simplest form. **(6 marks)**

(P) **8** Joshua is asked to prove the following identity:

$$\frac{1+\tanh^2 x}{1-\tanh^2 x} \equiv 2\cosh^2 x - 1$$

His answer is below.

$$\frac{1+\tanh^2 x}{1-\tanh^2 x} \equiv \frac{\text{sech}^2 x}{2-\text{sech}^2 x}$$ (using $\text{sech}^2 x \equiv 1 + \tanh^2 x$: same identity as the trig one)

$$\equiv \frac{\text{sech}^2 x}{2} - 1$$ (splitting the fraction up and cancelling)

$$\equiv \frac{2}{\text{sech}^2 x} - 1$$ (taking the reciprocal of both terms)

$$\equiv 2\cosh^2 x - 1$$

Joshua has made three errors. Explain the errors and provide a correct proof.

(E/P) **9** **a** Express $10\cosh x + 6\sinh x$ in the form $R\cosh(x + a)$ where $R > 0$. Give the value of a correct to 3 decimal places. **(4 marks)**

Hint Use the identity for $\cosh(A + B)$

b Write down the minimum value of $10\cosh x + 6\sinh x$ **(1 mark)**

c Use your answer to part **a** to solve the equation $10\cosh x + 6\sinh x = 11$
Give your answers to 3 decimal places. **(4 marks)**

Chapter review 1

SKILLS PROBLEM-SOLVING, REASONING/ARGUMENTATION

1 Find the exact value of **a** $\sinh(\ln 3)$ **b** $\cosh(\ln 5)$ **c** $\tanh\left(\ln\frac{1}{4}\right)$

2 Given that $\text{artanh}\, x - \text{artanh}\, y = \ln 5$, find y in terms of x.

(E/P) **3** Using the definitions of $\sinh x$ and $\cosh x$, prove that

$$\sinh(A - B) \equiv \sinh A\cosh B - \cosh A\sinh B$$ **(5 marks)**

(E/P) **4** Using definitions in terms of exponentials, prove that

$$\sinh x \equiv \frac{2\tanh\frac{1}{2}x}{1-\tanh^2\frac{1}{2}x}$$ **(5 marks)**

(E) **5** Solve, giving your answers as natural logarithms

$$9\cosh x - 5\sinh x = 15$$ **(6 marks)**

(E) **6** Solve, giving your answers as natural logarithms

$$23\sinh x - 17\cosh x + 7 = 0$$ **(6 marks)**

(E) **7** Solve, giving your answers as natural logarithms

$$3\cosh^2 x + 11\sinh x = 17$$ **(6 marks)**

E **8** **a** On the same diagram, sketch the graphs of $y = 6 + \sinh x$ and $y = \sinh 3x$ **(2 marks)**

b Using the identity $\sinh 3x \equiv 3\sinh x + 4\sinh^3 x$, show that the graphs **intersect** where $\sinh x = 1$ and hence find the exact **coordinates** of the point of **intersection**. **(5 marks)**

E/P **9** **a** Given that $13\cosh x + 5\sinh x = R\cosh(x + \alpha)$, $R > 0$, use the identity $\cosh(A + B) \equiv \cosh A\cosh B + \sinh A\sinh B$ to find the values of R and α, giving the value of α to 3 decimal places. **(4 marks)**

b Write down the minimum value of $13\cosh x + 5\sinh x$ **(1 mark)**

E/P **10** **a** Express $3\cosh x + 5\sinh x$ in the form $R\sinh(x + \alpha)$, where $R > 0$
Give α to 3 decimal places. **(4 marks)**

b Use the answer to part **a** to solve the equation $3\cosh x + 5\sinh x = 8$, giving your answer to 2 decimal places. **(3 marks)**

c Solve $3\cosh x + 5\sinh x = 8$ by using the definitions of $\cosh x$ and $\sinh x$. **(4 marks)**

Challenge

SKILLS CREATIVITY

Sketch the graph of $y = (\text{arsinh } x)^2$

Summary of key points

1 • Hyperbolic sine (or **sinh**) is defined as $\sinh x \equiv \frac{e^x - e^{-x}}{2}, x \in \mathbb{R}$

• Hyperbolic cosine (or **cosh**) is defined as $\cosh x \equiv \frac{e^x + e^{-x}}{2}, x \in \mathbb{R}$

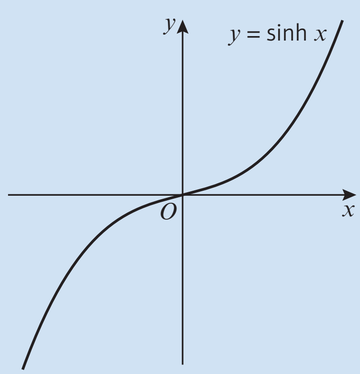
• Hyperbolic tangent (or **tanh**) is defined as $\tanh x \equiv \frac{\sinh x}{\cosh x} \equiv \frac{e^{2x} - 1}{e^{2x} + 1}, x \in \mathbb{R}$

• Hyperbolic cosecant (or **cosech**) is defined as $\text{cosech } x \equiv \frac{2}{e^x - e^{-x}}, x \in \mathbb{R}$

• Hyperbolic secant (or **sech**) is defined as $\text{sech } x \equiv \frac{2}{e^x + e^{-x}}, x \in \mathbb{R}$

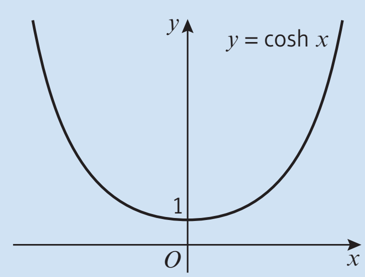
• Hyperbolic cotangent (or **coth**) is defined as $\coth x \equiv \frac{e^{2x} + 1}{e^{2x} - 1}, x \in \mathbb{R}$

2 • The graph of $y = \sinh x$:



For any value a, $\sinh(-a) = -\sinh a$.

• The graph of $y = \cosh x$:



For any value a, $\cosh(-a) = \cosh a$.

3 The table shows the inverse hyperbolic functions, with domains restricted where necessary.

Hyberbolic function	**Inverse hyperbolic function**
$y = \sinh x$	$y = \text{arsinh}\, x$
$y = \cosh x,\ x \geqslant 0$	$y = \text{arcosh}\, x,\ x \geqslant 1$
$y = \tanh x$	$y = \text{artanh}\, x,\ \lvert x \rvert < 1$
$y = \text{sech}\, x,\ x \geqslant 0$	$y = \text{arsech}\, x,\ 0 < x \leqslant 1$
$y = \text{cosech}\, x,\ x \neq 0$	$y = \text{arcosech}\, x,\ x \neq 0$
$y = \coth x,\ x \neq 0$	$y = \text{arcoth}\, x,\ \lvert x \rvert > 0$

4 The following formulae are given to you in the formula booklet in the examination.

- $\cosh^2 x - \sinh^2 x = 1$
- $\sinh 2x = 2 \sinh x \cosh x$
- $\cosh 2x = \cosh^2 x + \sinh^2 x$
- $\text{arcosh}\, x = \ln\{x + \sqrt{x^2 - 1}\} \quad x \geqslant 1$
- $\text{arsinh}\, x = \ln\{x + \sqrt{x^2 + 1}\}$
- $\text{artanh}\, x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \quad (\lvert x \rvert < 1)$

5 $\text{sech}^2 A = 1 - \tanh^2 A \qquad \text{cosech}^2 A = \coth^2 A - 1$

6
- $\sinh(A \pm B) \equiv \sinh A \cosh B \pm \cosh A \sinh B$
- $\cosh(A \pm B) \equiv \cosh A \cosh B \pm \sinh A \sinh B$

7 If $f(x) = \sinh x$, then the inverse function f^{-1} is called arsinh x (sometimes written as $\sinh^{-1} x$).

8 If $y = \text{arsinh}\, x$ then $x = \sinh y$

9 The graph of $y = \text{arsinh}\, x$ is the reflection of the graph $y = \sinh x$ in the line $y = x$.

10 The inverse of a function is defined only if the function is one-to-one, so for cosh x the domain must be restricted in order to define an inverse.

For $f(x) = \cosh x \quad x \geqslant 0, \quad f^{-1}(x) = \text{arcosh}\, x \quad (x \geqslant 1)$