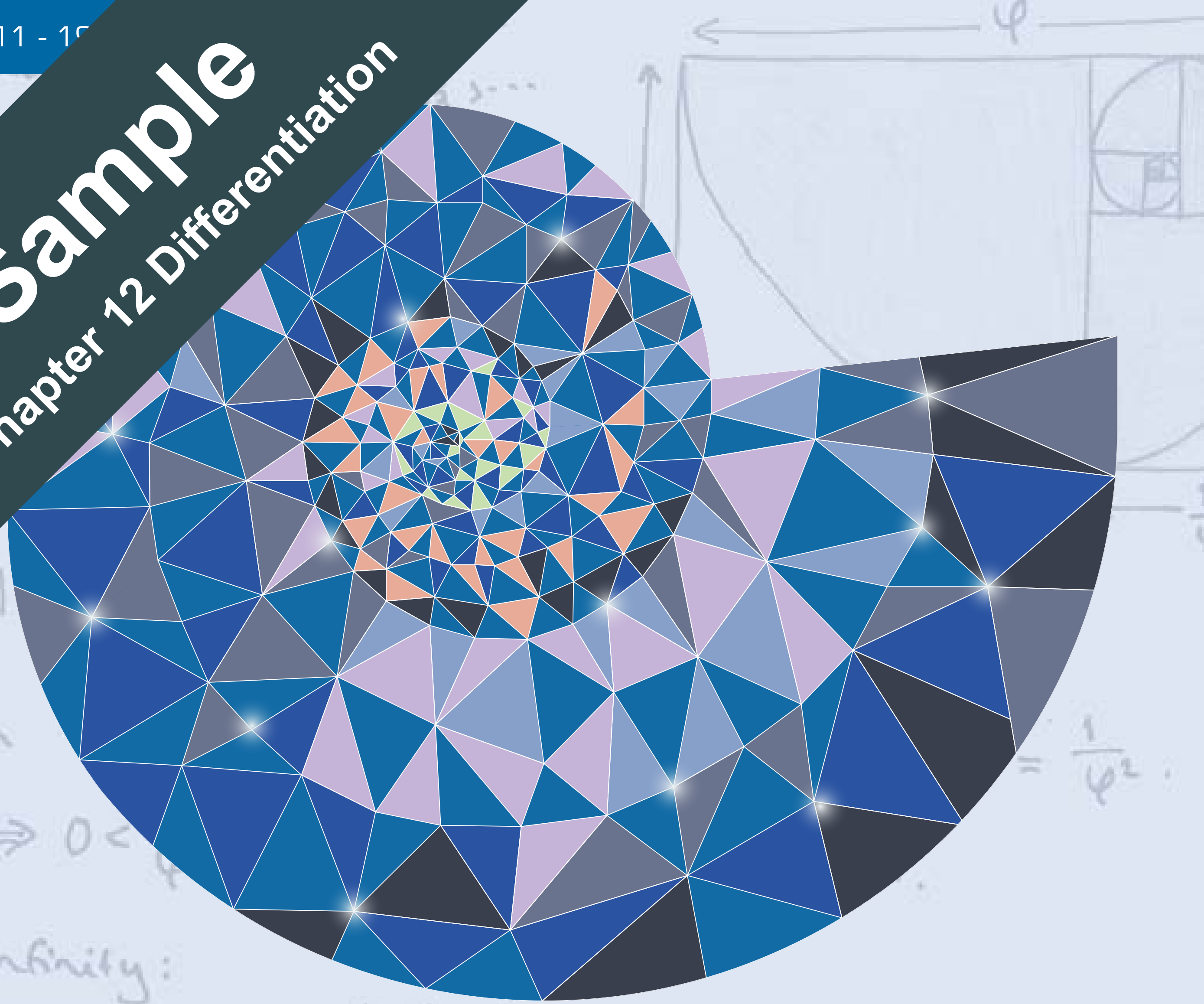
Sample

Chapter 12 Differentiation



Edexcel AS and A level Mathematics

Pure Mathematics

NEW FOR 2017

Year 1/AS

Pearson

Differentiation

12

Objectives

After completing this chapter you should be able to:

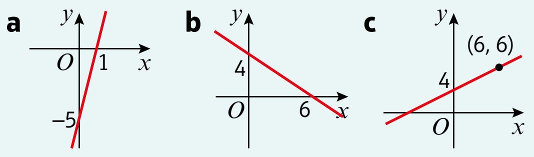
- Find the derivative, $f'(x)$ or $\frac{dy}{dx}$, of a simple function → pages 259–268
- Use the derivative to solve problems involving gradients, tangents and normals → pages 268–270
- Identify increasing and decreasing functions → pages 270–271
- Find the second order derivative, $f''(x)$ or $\frac{d^2y}{dx^2}$, of a simple function → pages 271–272
- Find stationary points of functions and determine their nature → pages 273–276
- Sketch the gradient function of a given function → pages 277–278
- Model real-life situations with differentiation → pages 279–281

Differentiation is part of **calculus**, one of the most powerful tools in mathematics. You will use differentiation in mechanics to model **rates of change**, such as **speed** and **acceleration**.

→ Exercise 12K Q5

Prior knowledge check

1 Find the gradients of these lines.



← Section 5.1

2 Write each of these expressions in the form x^n where n is a positive or negative real number.

a $x^3 \times x^7$ **b** $\sqrt[3]{x^2}$ **c** $\frac{x^2 \times x^3}{x^6}$

d $\sqrt{\frac{x^2}{\sqrt{x}}}$ ← Sections 1.1, 1.4

3 Find the equation of the straight line that passes through:

a (0, –2) and (6, 1) **b** (3, 7) and (9, 4)

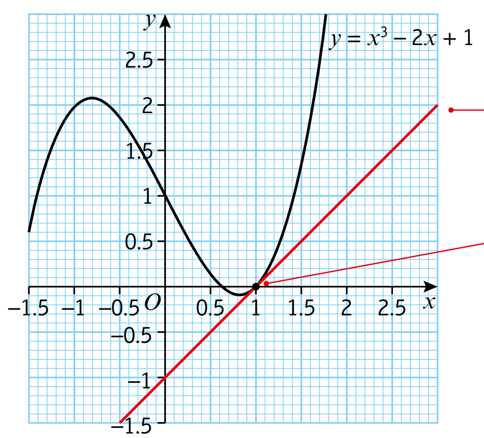
c (10, 5) and (–2, 8) ← Section 5.2

4 Find the equation of the perpendicular to the line $y = 2x - 5$ at the point (2, 1). ← Section 5.3

12.1 Gradients of curves

The gradient of a curve is **constantly changing**. You can use a tangent to find the gradient of a curve at any point on the curve. The tangent to a curve at a point A is the straight line that just touches the curve at A.

- **The gradient of a curve at a given point is defined as the gradient of the tangent to the curve at that point.**



The tangent to the curve at (1, 0) has gradient 1, so the gradient of the curve at the point (1, 0) is equal to 1.

The tangent **just touches** the curve at (1, 0). It does not cut the curve at this point, although it may cut the curve at another point.

Example 1

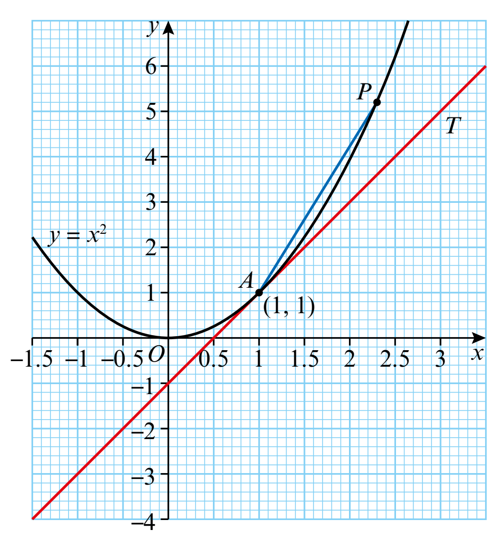
The diagram shows the curve with equation $y = x^2$.
The tangent, T, to the curve at the point $A(1, 1)$ is shown.
Point A is joined to point P by the chord AP.

a Calculate the gradient of the tangent, T.

b Calculate the gradient of the chord AP when P has coordinates:

i (2, 4)
ii (1.5, 2.25)
iii (1.1, 1.21)
iv (1.01, 1.0201)
v $(1 + h, (1 + h)^2)$

c Comment on the relationship between your answers to parts **a** and **b**.



a Gradient of tangent $= \frac{y_2 - y_1}{x_2 - x_1}$
$= \frac{3 - 1}{2 - 1}$
$= 2$

Use the formula for the gradient of a straight line between points (x_1, y_1) and (x_2, y_2). ← Section 5.1

The points used are (1, 1) and (2, 3).

b **i** Gradient of chord joining (1, 1) to (2, 4)
$= \frac{4 - 1}{2 - 1}$
$= 3$

Online Explore the gradient of the chord AP using GeoGebra.

ii Gradient of the chord joining (1, 1) to (1.5, 2.25)
$= \frac{2.25 - 1}{1.5 - 1}$
$= \frac{1.25}{0.5}$
$= 2.5$

This time (x_1, y_1) is (1, 1) and (x_2, y_2) is (1.5, 2.25).

iii Gradient of the chord joining (1, 1) to (1.1, 1.21)
$= \frac{1.21 - 1}{1.1 - 1}$
$= \frac{0.21}{0.1}$
$= 2.1$

iv Gradient of the chord joining (1, 1) to (1.01, 1.0201)
$= \frac{1.0201 - 1}{1.01 - 1}$
$= \frac{0.0201}{0.01}$
$= 2.01$

This point is closer to (1, 1) than (1.1, 1.21) is.

This gradient is closer to 2.

v Gradient of the chord joining (1, 1) to $(1 + h, (1 + h)^2)$
$= \frac{(1 + h)^2 - 1}{(1 + h) - 1}$
$= \frac{1 + 2h + h^2 - 1}{1 + h - 1}$
$= \frac{2h + h^2}{h}$
$= 2 + h$

h is a constant.
$(1 + h)^2 = (1 + h)(1 + h) = 1 + 2h + h^2$

This becomes $\frac{h(2 + h)}{h}$

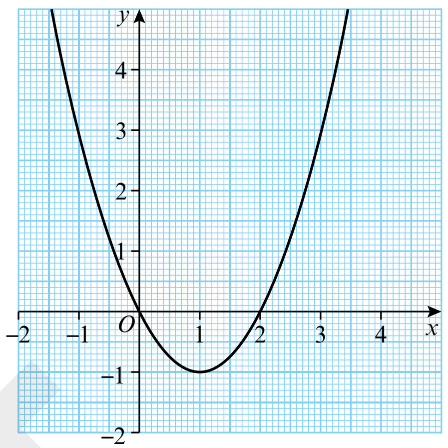
You can use this formula to confirm the answers to questions **i** to **iv**. For example, when $h = 0.5$, $(1 + h, (1 + h)^2) = (1.5, 2.25)$ and the gradient of the chord is $2 + 0.5 = 2.5$.

c As P gets closer to A, the gradient of the chord AP gets closer to the gradient of the tangent at A.

As h gets closer to zero, $2 + h$ gets closer to 2, so the gradient of the chord gets closer to the gradient of the tangent.

Exercise 12A

1 The diagram shows the curve with equation $y = x^2 - 2x$.



a Copy and complete this table showing estimates for the gradient of the curve.

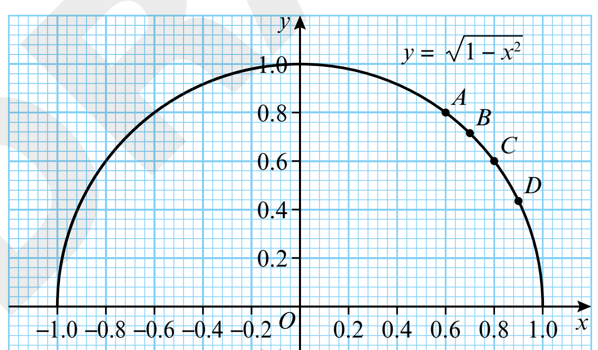
x-coordinate	–1	0	1	2	3
Estimate for gradient of curve					

b Write a hypothesis about the gradient of the curve at the point where $x = p$.

c Test your hypothesis by estimating the gradient of the graph at the point (1.5, –0.75).

Hint Place a ruler on the graph to approximate each tangent.

2 The diagram shows the curve with equation $y = \sqrt{1 - x^2}$.
The point A has coordinates (0.6, 0.8).
The points B, C and D lie on the curve with x-coordinates 0.7, 0.8 and 0.9 respectively.



a Verify that point A lies on the curve.

b Use a ruler to estimate the gradient of the curve at point A.

c Find the gradient of the line segments:

i AD

ii AC

iii AB

Hint Use algebra for part **c**.

d Comment on the relationship between your answers to parts **b** and **c**.

3 F is the point with coordinates (3, 9) on the curve with equation $y = x^2$.

a Find the gradients of the chords joining the point F to the points with coordinates:

i (4, 16) **ii** (3.5, 12.25) **iii** (3.1, 9.61)

iv (3.01, 9.0601) **v** $(3 + h, (3 + h)^2)$

b What do you deduce about the gradient of the tangent at the point (3, 9)?

4 G is the point with coordinates (4, 16) on the curve with equation $y = x^2$.

a Find the gradients of the chords joining the point G to the points with coordinates:

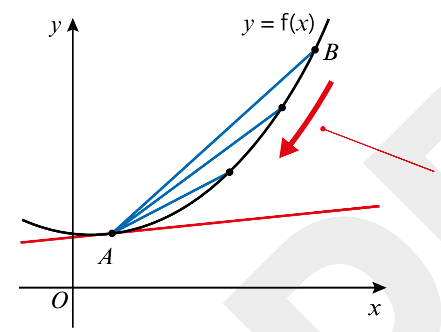
i (5, 25) **ii** (4.5, 20.25) **iii** (4.1, 16.81)

iv (4.01, 16.0801) **v** $(4 + h, (4 + h)^2)$

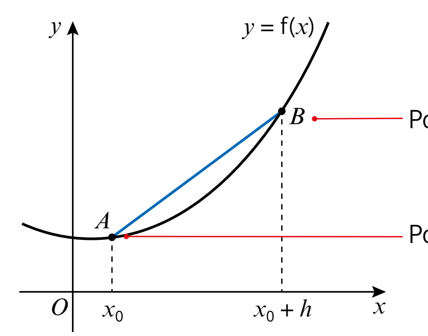
b What do you deduce about the gradient of the tangent at the point (4, 16)?

12.2 Finding the derivative

You can use algebra to find the exact gradient of a curve at a given point. This diagram shows two points, A and B, that lie on the curve with equation $y = f(x)$.



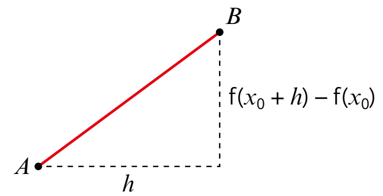
You can formalise this approach by letting the x-coordinate of A be x_0 and the x-coordinate of B be $x_0 + h$. Consider what happens to the gradient of AB as h gets smaller.



Notation h represents a **small change** in the value of x. You can also use δx to represent this small change. It is pronounced 'delta x'.

The vertical distance from A to B is $f(x_0 + h) - f(x_0)$.
The horizontal distance is $x_0 + h - x_0 = h$.

So the gradient of AB is $\dfrac{f(x_0 + h) - f(x_0)}{h}$



As h gets smaller, the gradient of AB gets closer to the gradient of the tangent to the curve at A. This means that the gradient of the **curve** at A is the **limit** of this expression as the value of h tends to 0.

You can use this to define the **gradient function**.

- **The gradient function, or derivative, of the curve $y = f(x)$ is written as $f'(x)$ or $\dfrac{dy}{dx}$.**

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The gradient function can be used to find the gradient of the curve for any value of x.

Notation $\lim_{h \to 0}$ means 'the limit as h tends to 0'. You can't evaluate the expression when $h = 0$, but as h gets smaller the expression gets closer to a fixed (or **limiting**) value.

Using this rule to find the derivative is called **differentiating from first principles**.

Example 2

The point A with coordinates (4, 16) lies on the curve with equation $y = x^2$.
At point A the curve has gradient g.

a Show that $g = \lim_{h \to 0} (8 + h)$.

b Deduce the value of g.

a $g = \lim_{h \to 0} \dfrac{f(4 + h) - f(4)}{h}$ — Use the definition of the derivative with $x = 4$.

$= \lim_{h \to 0} \dfrac{(4 + h)^2 - 4^2}{h}$ — The function is $f(x) = x^2$. Remember to square everything inside the brackets. ← Section 2.3

$= \lim_{h \to 0} \dfrac{16 + 8h + h^2 - 16}{h}$

$= \lim_{h \to 0} \dfrac{8h + h^2}{h}$ — The 16 and the −16 cancel, and you can cancel h in the fraction.

$= \lim_{h \to 0} (8 + h)$

b $g = 8$ — As $h \to 0$ the limiting value is 8, so the gradient at point A is 8.

Example 3

Prove, from first principles, that the derivative of x^3 is $3x^2$.

$f(x) = x^3$

$f'(x) = \lim_{h \to 0} \dfrac{f(x + h) - f(x)}{h}$

$= \lim_{h \to 0} \dfrac{(x + h)^3 - (x)^3}{h}$

$= \lim_{h \to 0} \dfrac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$ — $(x + h)^3 = (x + h)(x + h)^2 = (x + h)(x^2 + 2hx + h^2)$ which expands to give $x^3 + 3x^2h + 3xh^2 + h^3$

$= \lim_{h \to 0} \dfrac{3x^2h + 3xh^2 + h^3}{h}$

$= \lim_{h \to 0} \dfrac{h(3x^2 + 3xh + h^2)}{h}$ — Factorise the numerator.

$= \lim_{h \to 0} (3x^2 + 3xh + h^2)$

As $h \to 0$, $3xh \to 0$ and $h^2 \to 0$. — Any terms containing h, h^2, h^3, etc will have a limiting value of 0 as $h \to 0$.

So $f'(x) = 3x^2$

'From first principles' means that you have to use the definition of the derivative. You are starting your proof with a known definition, so this is an example of a proof by deduction.

Exercise 12B

1 For the function $f(x) = x^2$, use the definition of the derivative to show that:

a $f'(2) = 4$ **b** $f'(-3) = -6$ **c** $f'(0) = 0$ **d** $f'(50) = 100$

2 $f(x) = x^2$

a Show that $f'(x) = \lim_{h \to 0} (2x + h)$. **b** Hence deduce that $f'(x) = 2x$.

3 The point A with coordinates (−2, −8) lies on the curve with equation $y = x^3$.
At point A the curve has gradient g.

a Show that $g = \lim_{h \to 0} (12 - 6h + h^2)$. **b** Deduce the value of g.

(P) **4** The point A with coordinates (−1, 4) lies on the curve with equation $y = x^3 - 5x$.
The point B also lies on the curve and has x-coordinate $(-1 + h)$.

a Show that the gradient of the line segment AB is given by $h^2 - 3h - 2$.

b Deduce the gradient of the curve at point A.

Problem-solving Draw a sketch showing points A and B and the chord between them.

(E/P) **5** Prove, from first principles, that the derivative of $6x$ is 6. **(3 marks)**

(E/P) **6** Prove, from first principles, that the derivative of $4x^2$ is $8x$. **(4 marks)**

(E/P) **7** $f(x) = ax^2$, where a is a constant. Prove, from first principles, that $f'(x) = 2ax$. **(4 marks)**

Challenge

$f(x) = \frac{1}{x}$

a Given that $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$, show that $f'(x) = \lim_{h \to 0} \frac{-1}{x^2 + xh}$

b Deduce that $f'(x) = -\frac{1}{x^2}$

12.3 Differentiating x^n

You can use the definition of the derivative to find an expression for the derivative of x^n where n is any number. This is called **differentiation**.

- **For all real values of n, and for a constant a:**
  - **If $f(x) = x^n$ then $f'(x) = nx^{n-1}$**

    If $y = x^n$ then $\frac{dy}{dx} = nx^{n-1}$
  - **If $f(x) = ax^n$ then $f'(x) = anx^{n-1}$**

    If $y = ax^n$ then $\frac{dy}{dx} = anx^{n-1}$

Notation $f'(x)$ and $\frac{dy}{dx}$ both represent the derivative. You usually use $\frac{dy}{dx}$ when an expression is given in the form $y = \ldots$

Example 4

Find the derivative, $f'(x)$, when $f(x)$ equals:

a x^6 **b** $x^{\frac{1}{2}}$ **c** x^{-2} **d** $x^2 \times x^3$ **e** $\frac{x}{x^5}$

a $f(x) = x^6$

So $f'(x) = 6x^5$

Multiply by the power, then subtract 1 from the power:

$6 \times x^{6-1} = 6x^5$

b $f(x) = x^{\frac{1}{2}}$

So $f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$

$= \frac{1}{2\sqrt{x}}$

The new power is $\frac{1}{2} - 1 = -\frac{1}{2}$

$x^{-\frac{1}{2}} = \frac{1}{\sqrt{x}}$

← Section 1.4

c $f(x) = x^{-2}$

So $f'(x) = -2x^{-3}$

$= -\frac{2}{x^3}$

You can leave your answer in this form or write it as a fraction.

d $f(x) = x^2 \times x^3$

$= x^5$

So $f'(x) = 5x^4$

You need to write the function in the form x^n before you can use the rule.

$x^2 \times x^3 = x^{2+3}$

$= x^5$

e $f(x) = x \div x^5$

$= x^{-4}$

So $f'(x) = -4x^{-5}$

$= -\frac{4}{x^5}$

Use the laws of indices to simplify the fraction:

$x^1 \div x^5 = x^{1-5} = x^{-4}$

Example 5

Find $\frac{dy}{dx}$ when y equals:

a $7x^3$ **b** $-4x^{\frac{1}{2}}$ **c** $3x^{-2}$ **d** $\frac{8x^7}{3x}$ **e** $\sqrt{36x^3}$

a $\frac{dy}{dx} = 7 \times 3x^{3-1} = 21x^2$

Use the rule for differentiating ax^n with $a = 7$ and $n = 3$. Multiply by 3 then subtract 1 from the power.

b $\frac{dy}{dx} = -4 \times \frac{1}{2}x^{-\frac{1}{2}} = -2x^{-\frac{1}{2}} = -\frac{2}{\sqrt{x}}$

This is the same as differentiating $x^{\frac{1}{2}}$ then multiplying the result by −4.

c $\frac{dy}{dx} = 3 \times -2x^{-3} = -6x^{-3} = -\frac{6}{x^3}$

d $y = \frac{8}{3}x^6$

Write the expression in the form ax^n. Remember a can be any number, including fractions.

$\frac{dy}{dx} = 6 \times \frac{8}{3}x^5 = 16x^5$

e $y = \sqrt{36} \times \sqrt{x^3} = 6 \times (x^3)^{\frac{1}{2}} = 6x^{\frac{3}{2}}$

$\frac{dy}{dx} = 6 \times \frac{3}{2}x^{\frac{1}{2}} = 9x^{\frac{1}{2}} = 9\sqrt{x}$

$\frac{3}{2} - 1 = \frac{1}{2}$

Simplify the number part as much as possible.

Exercise 12C

Hint Make sure that the functions are in the form x^n before you differentiate.

1 Find $f'(x)$ given that $f(x)$ equals:

a x^7 **b** x^8 **c** x^4 **d** $x^{\frac{1}{3}}$ **e** $x^{\frac{1}{4}}$ **f** $\sqrt[3]{x}$

g x^{-3} **h** x^{-4} **i** $\frac{1}{x^2}$ **j** $\frac{1}{x^5}$ **k** $\frac{1}{\sqrt{x}}$ **l** $\frac{1}{\sqrt[3]{x}}$

m $x^3 \times x^6$ **n** $x^2 \times x^3$ **o** $x \times x^2$ **p** $\frac{x^2}{x^4}$ **q** $\frac{x^3}{x^2}$ **r** $\frac{x^6}{x^3}$

2 Find $\frac{dy}{dx}$ given that y equals:

a $3x^2$ **b** $6x^9$ **c** $\frac{1}{2}x^4$ **d** $20x^{\frac{1}{4}}$ **e** $6x^{\frac{5}{4}}$

f $10x^{-1}$ **g** $\frac{4x^6}{2x^3}$ **h** $\frac{x}{8x^5}$ **i** $-\frac{2}{\sqrt{x}}$ **j** $\sqrt{\frac{5x^4 \times 10x}{2x^2}}$

3 Find the gradient of the curve with equation $y = 3\sqrt{x}$ at the point where:

a $x = 4$ **b** $x = 9$

c $x = \frac{1}{4}$ **d** $x = \frac{9}{16}$

E/P 4 Given that $2y^2 - x^3 = 0$ and $y > 0$, find $\frac{dy}{dx}$ **(2 marks)**

Problem-solving

Try rearranging unfamiliar equations into a form you recognise.

12.4 Differentiating quadratics

You can differentiate a function with more than one term by differentiating the terms **one-at-a-time**. The highest power of x in a **quadratic function** is x^2, so the highest power of x in its derivative will be x.

- **For the quadratic curve with equation $y = ax^2 + bx + c$, the derivative is given by**

$$\frac{dy}{dx} = 2ax + b$$

Links The derivative is a **straight line** with gradient $2a$. It crosses the x-axis once, at the point where the quadratic curve has zero gradient. This is the **turning point** of the quadratic curve. ← Section 5.1

You can find this expression for $\frac{dy}{dx}$ by differentiating each of the terms one-at-a-time:

ax^2 → Differentiate → $2ax^1 = 2ax$
The quadratic term tells you the slope of the gradient function.

$bx = bx^1$ → Differentiate → $1bx^0 = b$
An x term differentiates to give a constant.

c → Differentiate → 0
Constant terms disappear when you differentiate.

Example 6

Find $\frac{dy}{dx}$ given that y equals:

a $x^2 + 3x$ **b** $8x - 7$ **c** $4x^2 - 3x + 5$

a $y = x^2 + 3x$
So $\frac{dy}{dx} = 2x + 3$

Differentiate the terms one-at-a-time.

b $y = 8x - 7$
So $\frac{dy}{dx} = 8$

The constant term disappears when you differentiate. The line $y = -7$ would have **zero gradient**.

c $y = 4x^2 - 3x + 5$

$4x^2 - 3x + 5$ is a quadratic expression with $a = 4$, $b = -3$ and $c = 5$.

So $\frac{dy}{dx} = 8x - 3$

The derivative is $2ax + b = 2 \times 4x - 3 = 8x - 3$.

Example 7

Let $f(x) = 4x^2 - 8x + 3$.

a Find the gradient of $y = f(x)$ at the point $\left(\frac{1}{2}, 0\right)$.

b Find the coordinates of the point on the graph of $y = f(x)$ where the gradient is 8.

c Find the gradient of $y = f(x)$ at the points where the curve meets the line $y = 4x - 5$.

a As $y = 4x^2 - 8x + 3$
$\frac{dy}{dx} = f'(x) = 8x - 8 + 0$
So $f'\left(\frac{1}{2}\right) = -4$

Differentiate to find the gradient function. Then substitute the x-coordinate value to obtain the gradient.

b $\frac{dy}{dx} = f'(x) = 8x - 8 = 8$

Put the gradient function equal to 8. Then solve the equation you have obtained to give the value of x.

So $x = 2$

Substitute this value of x into f(x) to give the value of y and interpret your answer in words.

So $y = f(2) = 3$
The point where the gradient is 8 is (2, 3).

c $4x^2 - 8x + 3 = 4x - 5$

To find the points of intersection, set the equation of the curve equal to the equation of the line. Solve the resulting quadratic equation to find the x-coordinates of the points of intersection. ← Section 4.4

$4x^2 - 12x + 8 = 0$
$x^2 - 3x + 2 = 0$
$(x - 2)(x - 1) = 0$
So $x = 1$ or $x = 2$
At $x = 1$, the gradient is 0.
At $x = 2$, the gradient is 8, as in part **b**.

Substitute the values of x into $f'(x) = 8x - 8$ to give the gradients at the specified points.

Online Use your calculator to check solutions to quadratic equations quickly.

Exercise 12D

1 Find $\frac{dy}{dx}$ when y equals:

a $2x^2 - 6x + 3$ **b** $\frac{1}{2}x^2 + 12x$ **c** $4x^2 - 6$

d $8x^2 + 7x + 12$ **e** $5 + 4x - 5x^2$

2 Find the gradient of the curve with equation:

a $y = 3x^2$ at the point (2, 12) **b** $y = x^2 + 4x$ at the point (1, 5)

c $y = 2x^2 - x - 1$ at the point (2, 5) **d** $y = \frac{1}{2}x^2 + \frac{3}{2}x$ at the point (1, 2)

e $y = 3 - x^2$ at the point (1, 2) **f** $y = 4 - 2x^2$ at the point (−1, 2)

3 Find the y-coordinate and the value of the gradient at the point P with x-coordinate 1 on the curve with equation $y = 3 + 2x - x^2$.

4 Find the coordinates of the point on the curve with equation $y = x^2 + 5x - 4$ where the gradient is 3.

(P) **5** Find the gradients of the curve $y = x^2 - 5x + 10$ at the points A and B where the curve meets the line $y = 4$.

(P) **6** Find the gradients of the curve $y = 2x^2$ at the points C and D where the curve meets the line $y = x + 3$.

(P) **7** $f(x) = x^2 - 2x - 8$

a Sketch the graph of $y = f(x)$.

b On the same set of axes, sketch the graph of $y = f'(x)$.

c Explain why the x-coordinate of the turning point of $y = f(x)$ is the same as the x-coordinate of the point where the graph of $y = f'(x)$ crosses the x-axis.

12.5 Differentiating functions with two or more terms

You can use the rule for differentiating ax^n to differentiate functions with two or more terms. You need to be able to rearrange **each term** into the form ax^n, where a is a constant and n is a real number. Then you can differentiate the terms **one-at-a-time**.

- **If $y = f(x) \pm g(x)$, then $\frac{dy}{dx} = f'(x) \pm g'(x)$.**

Example 8

Find $\frac{dy}{dx}$ given that y equals:

a $4x^3 + 2x$ **b** $x^3 + x^2 - x^{\frac{1}{2}}$ **c** $\frac{1}{3}x^{\frac{1}{2}} + 4x^2$

a $y = 4x^3 + 2x$

So $\frac{dy}{dx} = 12x^2 + 2$

Differentiate the terms one-at-a-time.

b $y = x^3 + x^2 - x^{\frac{1}{2}}$

So $\frac{dy}{dx} = 3x^2 + 2x - \frac{1}{2}x^{-\frac{1}{2}}$

Be careful with the third term. You multiply the term by $\frac{1}{2}$ and then reduce the power by 1 to get $-\frac{1}{2}$

c $y = \frac{1}{3}x^{\frac{1}{2}} + 4x^2$

So $\frac{dy}{dx} = \frac{1}{3} \times \frac{1}{2}x^{-\frac{1}{2}} + 8x$

$= \frac{1}{6}x^{-\frac{1}{2}} + 8x$

Check that each term is in the form ax^n before differentiating.

Example 9

Differentiate:

a $\frac{1}{4\sqrt{x}}$ **b** $x^3(3x + 1)$ **c** $\frac{x - 2}{x^2}$

a Let $y = \frac{1}{4\sqrt{x}}$

$= \frac{1}{4}x^{-\frac{1}{2}}$

Therefore $\frac{dy}{dx} = -\frac{1}{8}x^{-\frac{3}{2}}$

Use the laws of indices to write the expression in the form ax^n.
$\frac{1}{4\sqrt{x}} = \frac{1}{4} \times \frac{1}{\sqrt{x}} = \frac{1}{4} \times \frac{1}{x^{\frac{1}{2}}} = \frac{1}{4}x^{-\frac{1}{2}}$

b Let $y = x^3(3x + 1)$

$= 3x^4 + x^3$

Multiply out the brackets to give a polynomial function.

Therefore $\frac{dy}{dx} = 12x^3 + 3x^2$

$= 3x^2(4x + 1)$

Differentiate each term.

c Let $y = \frac{x - 2}{x^2}$

$= \frac{1}{x} - \frac{2}{x^2}$

Express the single fraction as two separate fractions, and simplify: $\frac{x}{x^2} = \frac{1}{x}$

$= x^{-1} - 2x^{-2}$

Therefore $\frac{dy}{dx} = -x^{-2} + 4x^{-3}$

Write each term in the form ax^n then differentiate.

$= -\frac{1}{x^2} + \frac{4}{x^3}$

$= \frac{4 - x}{x^3}$

You can write the answer as a single fraction with denominator x^3.

Exercise 12E

1 Differentiate:

a $x^4 + x^{-1}$ **b** $2x^5 + 3x^{-2}$ **c** $6x^{\frac{3}{2}} + 2x^{-\frac{1}{2}} + 4$

2 Find the gradient of the curve with equation $y = f(x)$ at the point A where:

a $f(x) = x^3 - 3x + 2$ and A is at $(-1, 4)$ **b** $f(x) = 3x^2 + 2x^{-1}$ and A is at $(2, 13)$

3 Find the point or points on the curve with equation $y = f(x)$, where the gradient is zero:

a $f(x) = x^2 - 5x$ **b** $f(x) = x^3 - 9x^2 + 24x - 20$

c $f(x) = x^{\frac{3}{2}} - 6x + 1$ **d** $f(x) = x^{-1} + 4x$

4 Differentiate:

a $2\sqrt{x}$ **b** $\frac{3}{x^2}$ **c** $\frac{1}{3x^3}$ **d** $\frac{1}{3}x^3(x - 2)$

e $\frac{2}{x^3} + \sqrt{x}$ **f** $\sqrt[3]{x} + \frac{1}{2x}$ **g** $\frac{2x + 3}{x}$ **h** $\frac{3x^2 - 6}{x}$

i $\frac{2x^3 + 3x}{\sqrt{x}}$ **j** $x(x^2 - x + 2)$ **k** $3x^2(x^2 + 2x)$ **l** $(3x - 2)\left(4x + \frac{1}{x}\right)$

5 Find the gradient of the curve with equation $y = f(x)$ at the point A where:

a $f(x) = x(x + 1)$ and A is at $(0, 0)$ **b** $f(x) = \frac{2x - 6}{x^2}$ and A is at $(3, 0)$

c $f(x) = \frac{1}{\sqrt{x}}$ and A is at $\left(\frac{1}{4}, 2\right)$ **d** $f(x) = 3x - \frac{4}{x^2}$ and A is at $(2, 5)$

E/P 6 $f(x) = \frac{12}{p\sqrt{x}} + x$, where p is a real constant and $x > 0$.

Given that $f'(2) = 3$, find p, giving your answer in the form $a\sqrt{2}$ where a is a rational number. **(4 marks)**

P 7 $f(x) = (2 - x)^9$

a Find the first 3 terms, in ascending powers of x, of the binomial expansion of $f(x)$, giving each term in its simplest form.

b If x is small, so that x^2 and higher powers can be ignored, show that $f'(x) \approx 9216x - 2304$.

Hint Use the binomial expansion with $a = 2$, $b = -x$ and $n = 9$. ← Section 8.3

12.6 Gradients, tangents and normals

You can use the derivative to find the equation of the tangent to a curve at a given point. On the curve with equation $y = f(x)$, the gradient of the tangent at a point A with x-coordinate a will be $f'(a)$.

- **The tangent to the curve $y = f(x)$ at the point with coordinates $(a, f(a))$ has equation**

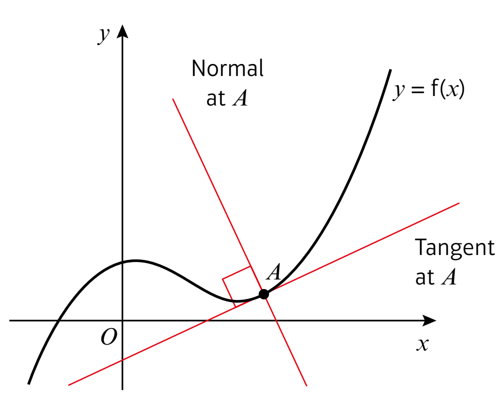
$$y - f(a) = f'(a)(x - a)$$

Links The equation of a straight line with gradient m that passes through the point (x_1, y_1) is $y - y_1 = m(x - x_1)$. ← Section 5.2

The **normal** to a curve at point A is the straight line through A which is perpendicular to the tangent to the curve at A. The gradient of the normal will be $-\frac{1}{f'(a)}$

- **The normal to the curve $y = f(x)$ at the point with coordinates $(a, f(a))$ has equation**

$$y - f(a) = -\frac{1}{f'(a)}(x - a)$$



Example 10

Find the equation of the tangent to the curve $y = x^3 - 3x^2 + 2x - 1$ at the point $(3, 5)$.

$y = x^3 - 3x^2 + 2x - 1$

$\frac{dy}{dx} = 3x^2 - 6x + 2$

First differentiate to determine the gradient function.

When $x = 3$, the gradient is 11.

Then substitute for x to calculate the value of the gradient of the curve and of the tangent when $x = 3$.

So the equation of the tangent at $(3, 5)$ is

$y - 5 = 11(x - 3)$

$y = 11x - 28$

You can now use the line equation and simplify.

Example 11

Find the equation of the normal to the curve with equation $y = 8 - 3\sqrt{x}$ at the point where $x = 4$.

$y = 8 - 3\sqrt{x}$

$= 8 - 3x^{\frac{1}{2}}$

$\frac{dy}{dx} = -\frac{3}{2}x^{-\frac{1}{2}}$

Write each term in the form ax^n and differentiate to obtain the gradient function, which you can use to find the gradient at any point.

When $x = 4$, $y = 2$ and gradient of curve and of tangent $= -\frac{3}{4}$

Find the y-coordinate when $x = 4$ by substituting into the equation of the curve and calculating $8 - 3\sqrt{4} = 8 - 6 = 2$.

Find the gradient of the curve, by calculating $\frac{dy}{dx} = -\frac{3}{2}(4)^{-\frac{1}{2}} = -\frac{3}{2} \times \frac{1}{2} = -\frac{3}{4}$

So gradient of normal is $\frac{4}{3}$.

Gradient of normal $= -\frac{1}{\text{gradient of curve}}$

$= -\frac{1}{\left(-\frac{3}{4}\right)} = \frac{4}{3}$

Equation of normal is

$y - 2 = \frac{4}{3}(x - 4)$

$3y - 6 = 4x - 16$

$3y - 4x + 10 = 0$

Simplify by multiplying both sides by 3 and collecting terms.

Online Explore the tangent and normal to the curve using GeoGebra.

Exercise 12F

1 Find the equation of the tangent to the curve:

a $y = x^2 - 7x + 10$ at the point $(2, 0)$ **b** $y = x + \frac{1}{x}$ at the point $\left(2, 2\frac{1}{2}\right)$

c $y = 4\sqrt{x}$ at the point $(9, 12)$ **d** $y = \frac{2x - 1}{x}$ at the point $(1, 1)$

e $y = 2x^3 + 6x + 10$ at the point $(-1, 2)$ **f** $y = x^2 - \frac{7}{x^2}$ at the point $(1, -6)$

2 Find the equation of the normal to the curve:

a $y = x^2 - 5x$ at the point $(6, 6)$ **b** $y = x^2 - \frac{8}{\sqrt{x}}$ at the point $(4, 12)$

P 3 Find the coordinates of the point where the tangent to the curve $y = x^2 + 1$ at the point $(2, 5)$ meets the normal to the same curve at the point $(1, 2)$.

(P) 4 Find the equations of the normals to the curve $y = x + x^3$ at the points (0, 0) and (1, 2), and find the coordinates of the point where these normals meet.

(P) 5 For $f(x) = 12 - 4x + 2x^2$, find the equations of the tangent and the normal at the point where $x = -1$ on the curve with equation $y = f(x)$.

(E/P) 6 The point P with x-coordinate $\frac{1}{2}$ lies on the curve with equation $y = 2x^2$.
The normal to the curve at P intersects the curve at points P and Q.
Find the coordinates of Q. **(6 marks)**

Problem-solving

Draw a sketch showing the curve, the point P and the normal. This will help you check that your answer makes sense.

Challenge

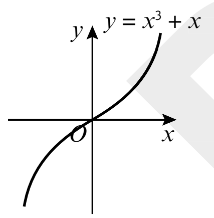
The line L is a tangent to the curve with equation $y = 4x^2 + 1$. L cuts the y-axis at (0, −8) and has a positive gradient. Find the equation of L in the form $y = mx + c$.

Hint Use the discriminant to find the value of m when the line just touches the curve. ← Section 2.5

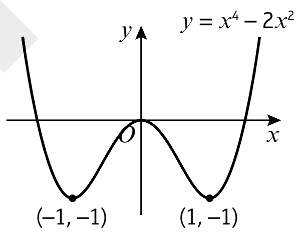
12.7 Increasing and decreasing functions

You can use the derivative to determine whether a function is **increasing** or **decreasing** on a given interval.

- **The function $f(x)$ is increasing on the interval $[a, b]$ if $f'(x) \geqslant 0$ for all values of x such that $a < x < b$.**
- **The function $f(x)$ is decreasing on the interval $[a, b]$ if $f'(x) \leqslant 0$ for all values of x such that $a < x < b$.**



The function $f(x) = x^3 + x$ is increasing for all real values of x.



The function $f(x) = x^4 - 2x^2$ is increasing on the interval [−1, 0] and decreasing on the interval [0, 1].

Notation The **interval** $[a, b]$ is the set of all real numbers, x, that satisfy $a \leqslant x \leqslant b$.

Example 12

Show that the function $f(x) = x^3 + 24x + 3$ is increasing for all real values of x.

$f(x) = x^3 + 24x + 3$
$f'(x) = 3x^2 + 24$
$x^2 \geqslant 0$ for all real values of x
So $3x^2 + 24 \geqslant 0$ for all real values of x.
So $f(x)$ is increasing for all real values of x.

First differentiate to obtain the gradient function.

State that the condition for an increasing function is met. In fact $f'(x) \geqslant 24$ for all real values of x.

Example 13

Find the interval on which the function $f(x) = x^3 + 3x^2 - 9x$ is decreasing.

$f(x) = x^3 + 3x^2 - 9x$
$f'(x) = 3x^2 + 6x - 9$
If $f'(x) \leqslant 0$ then $3x^2 + 6x - 9 \leqslant 0$
So $3(x^2 + 2x - 3) \leqslant 0$
$3(x + 3)(x - 1) \leqslant 0$
So $-3 \leqslant x \leqslant 1$
So $f(x)$ is decreasing on the interval [−3, 1].

Find $f'(x)$ and put this expression $\leqslant 0$.

Solve the inequality by considering the three regions $x \leqslant -3$, $-3 \leqslant x \leqslant 1$ and $x \geqslant 1$, or by sketching the curve with equation $y = 3(x + 3)(x - 1)$ ← Section 3.5

Write the answer clearly.

Online Explore increasing and decreasing functions using GeoGebra.

Exercise 12G

1 Find the values of x for which $f(x)$ is an increasing function, given that $f(x)$ equals:
a $3x^2 + 8x + 2$ **b** $4x - 3x^2$ **c** $5 - 8x - 2x^2$ **d** $2x^3 - 15x^2 + 36x$
e $3 + 3x - 3x^2 + x^3$ **f** $5x^3 + 12x$ **g** $x^4 + 2x^2$ **h** $x^4 - 8x^3$

2 Find the values of x for which $f(x)$ is a decreasing function, given that $f(x)$ equals:
a $x^2 - 9x$ **b** $5x - x^2$ **c** $4 - 2x - x^2$ **d** $2x^3 - 3x^2 - 12x$
e $1 - 27x + x^3$ **f** $x + \frac{25}{x}$ **g** $x^{\frac{1}{2}} + 9x^{-\frac{1}{2}}$ **h** $x^2(x + 3)$

(E/P) 3 Show that the function $f(x) = 4 - x(2x^2 + 3)$ is decreasing for all $x \in \mathbb{R}$. **(3 marks)**

(E/P) 4 **a** Given that the function $f(x) = x^2 + px$ is increasing on the interval [−1, 1], find one possible value for p. **(2 marks)**
b State with justification whether this is the only possible value for p. **(1 mark)**

12.8 Second order derivatives

You can find the rate of change of the gradient function by differentiating a function twice.

$y = 5x^3$ → Differentiate → $\frac{dy}{dx} = 15x^2$ → Differentiate → $\frac{d^2y}{dx^2} = 30x$

This is the gradient function. It describes the rate of change of the function with respect to x.

This is the **rate of change of the gradient function**. It is called the second order derivative. It can also be written as $f''(x)$.

- **Differentiating a function $y = f(x)$ twice gives you the second order derivative, $f''(x)$ or $\frac{d^2y}{dx^2}$**

Notation The derivative is also called the **first order derivative** or **first derivative**. The **second order derivative** is sometimes just called the **second derivative**.

Example 14

Given that $y = 3x^5 + \dfrac{4}{x^2}$ find:

a $\dfrac{dy}{dx}$ **b** $\dfrac{d^2y}{dx^2}$

a $y = 3x^5 + \dfrac{4}{x^2}$

$= 3x^5 + 4x^{-2}$ — Express the fraction as a negative power of x.

So $\dfrac{dy}{dx} = 15x^4 - 8x^{-3}$ — Differentiate once to get the first order derivative.

$= 15x^4 - \dfrac{8}{x^3}$

b $\dfrac{d^2y}{dx^2} = 60x^3 + 24x^{-4}$ — Differentiate a second time to get the second order derivative.

$= 60x^3 + \dfrac{24}{x^4}$

Example 15

Given that $f(x) = 3\sqrt{x} + \dfrac{1}{2\sqrt{x}}$, find:

a $f'(x)$ **b** $f''(x)$

a $f(x) = 3\sqrt{x} + \dfrac{1}{2\sqrt{x}}$

$= 3x^{\frac{1}{2}} + \frac{1}{2}x^{-\frac{1}{2}}$

$f'(x) = \frac{3}{2}x^{-\frac{1}{2}} - \frac{1}{4}x^{-\frac{3}{2}}$ — Don't rewrite your expression for $f'(x)$ as a fraction. It will be easier to differentiate again if you leave it in this form.

b $f''(x) = -\frac{3}{4}x^{-\frac{3}{2}} + \frac{3}{8}x^{-\frac{5}{2}}$ — The coefficient for the second term is $\left(-\frac{3}{2}\right) \times \left(-\frac{1}{4}\right) = +\frac{3}{8}$. The new power is $-\frac{3}{2} - 1 = -\frac{5}{2}$

Exercise 12H

1 Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ when y equals:

a $12x^2 + 3x + 8$ **b** $15x + 6 + \dfrac{3}{x}$ **c** $9\sqrt{x} - \dfrac{3}{x^2}$ **d** $(5x + 4)(3x - 2)$ **e** $\dfrac{3x + 8}{x^2}$

2 The displacement of a particle in metres at time t seconds is modelled by the function

$$f(t) = \frac{t^2 + 2}{\sqrt{t}}$$

The acceleration of the particle in m s^{-2} is the second derivative of this function. Find an expression for the acceleration of the particle at time t seconds.

Links The velocity of the particle will be $f'(t)$ and its acceleration will be $f''(t)$.
→ Statistics and Mechanics Year 2, Section 6.2

(P) 3 Given that $y = (2x - 3)^3$, find the value of x when $\dfrac{d^2y}{dx^2} = 0$.

(P) 4 $f(x) = px^3 - 3px^2 + x^2 - 4$
When $x = 2$, $f''(x) = -1$. Find the value of p.

Problem-solving When you differentiate with respect to x, you treat any other letters as constants.

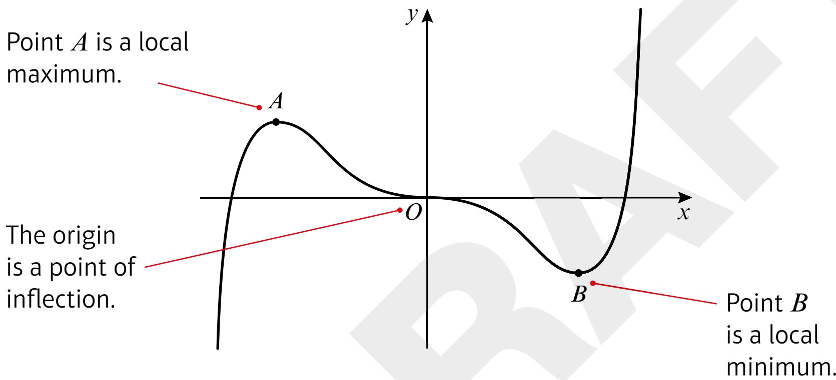
12.9 Stationary points

A **stationary point** on a curve is any point where the curve has **gradient zero**. You can determine whether a stationary point is a **local maximum**, a **local minimum** or a **point of inflection** by looking at the gradient of the curve on either side.

■ **Any point on the curve $y = f(x)$ where $f'(x) = 0$ is called a stationary point. For a small positive value h:**

Type of stationary point	$f'(x - h)$	$f'(x)$	$f'(x + h)$
Local maximum	Positive	0	Negative
Local minimum	Negative	0	Positive
Point of inflection	Negative	0	Negative
	Positive	0	Positive

Notation The plural of maximum is **maxima** and the plural of minimum is **minima**.



Notation Point A is called a **local** maximum because it is not the largest value the function can take. It is just the largest value in that immediate vicinity.

Example 16

a Find the coordinates of the stationary point on the curve with equation $y = x^4 - 32x$.

b By considering points on either side of the stationary point, determine whether it is a local maximum, a local minimum or a point of inflection.

a $y = x^4 - 32x$

$\dfrac{dy}{dx} = 4x^3 - 32$

Let $\dfrac{dy}{dx} = 0$ — Differentiate and let $\dfrac{dy}{dx} = 0$.

Then $4x^3 - 32 = 0$

$4x^3 = 32$

$x^3 = 8$

$x = 2$ — Solve the equation to find the value of x.

So $y = 2^4 - 32 \times 2$

$= 16 - 64$

$= -48$ — Substitute the value of x into the original equation to find the value of y.

So $(2, -48)$ is a stationary point.

b Now consider the gradient on either side of $(2, -48)$.

Value of x	$x = 1.9$	$x = 2$	$x = 2.1$
Gradient	-4.56 which is $-$ve	0	5.04 which is +ve
Shape of curve			

From the shape of the curve, the point $(2, -48)$ is a local minimum point.

Make a table where you consider a value of x slightly less than 2 and a value of x slightly greater than 2.

Calculate the gradient for each of these values of x close to the stationary point.

Deduce the shape of the curve.

Online Explore the solution using GeoGebra.

In some cases you can use the **second derivative**, $f''(x)$, to determine the nature of a stationary point.

- **If a function $f(x)$ has a stationary point when $x = a$, then:**
  - **if $f''(a) > 0$, the point is a local minimum**
  - **if $f''(a) < 0$, the point is a local maximum**

  If $f''(a) = 0$, the point could be a local minimum, a local maximum or a point of inflection. You will need to look at points on either side to determine its nature.

Hint $f''(x)$ tells you the **rate of change** of the gradient function. When $f'(x) = 0$ and $f''(x) > 0$ the gradient is **increasing** from a negative value to a positive value, so the stationary point is a **minimum**.

Example 17

a Find the coordinates of the stationary points on the curve with equation
$$y = 2x^3 - 15x^2 + 24x + 6$$

b Find $\frac{d^2y}{dx^2}$ and use it to determine the nature of the stationary points.

a $y = 2x^3 - 15x^2 + 24x + 6$

$\frac{dy}{dx} = 6x^2 - 30x + 24$

Putting $6x^2 - 30x + 24 = 0$

$6(x - 4)(x - 1) = 0$

So $x = 4$ or $x = 1$

When $x = 1$,

$y = 2 - 15 + 24 + 6 = 17$

When $x = 4$,

$y = 2 \times 64 - 15 \times 16 + 24 \times 4 + 6$

$= -10$

So the stationary points are at $(1, 17)$ and $(4, -10)$.

Differentiate and put the derivative equal to zero.

Solve the equation to obtain the values of x for the stationary points.

Substitute $x = 1$ and $x = 4$ into the original equation of the curve to obtain the values of y which correspond to these values.

b $\frac{d^2y}{dx^2} = 12x - 30$

When $x = 1$, $\frac{d^2y}{dx^2} = -18$ which is < 0

So $(1, 17)$ is a local maximum point.

When $x = 4$, $\frac{d^2y}{dx^2} = 18$ which is > 0

So $(4, -10)$ is a local minimum point.

Differentiate again to obtain the second derivative.

Substitute $x = 1$ and $x = 4$ into the second derivative expression. If the second derivative is negative then the point is a local maximum point. If it is positive then the point is a local minimum point.

Example 18

a The curve with equation $y = \frac{1}{x} + 27x^3$ has stationary points at $x = \pm a$. Find the value of a.

b Sketch the graph of $y = \frac{1}{x} + 27x^3$.

a $y = x^{-1} + 27x^3$

$\frac{dy}{dx} = -x^{-2} + 81x^2 = -\frac{1}{x^2} + 81x^2$

When $\frac{dy}{dx} = 0$:

$-\frac{1}{x^2} + 81x^2 = 0$

$81x^2 = \frac{1}{x^2}$

$81x^4 = 1$

$x^4 = \frac{1}{81}$

$x = \pm\frac{1}{3}$

So $a = \frac{1}{3}$

Write $\frac{1}{x}$ as x^{-1} to differentiate.

Set $\frac{dy}{dx} = 0$ to determine the x-coordinates of the stationary points.

You need to consider the positive and negative roots: $\left(-\frac{1}{3}\right)^4 = \left(-\frac{1}{3}\right) \times \left(-\frac{1}{3}\right) \times \left(-\frac{1}{3}\right) \times \left(-\frac{1}{3}\right) = \frac{1}{81}$

b $\frac{d^2y}{dx^2} = 2x^{-3} + 162x = \frac{2}{x^3} + 162x$

When $x = -\frac{1}{3}$, $y = \frac{1}{\left(-\frac{1}{3}\right)} + 27\left(-\frac{1}{3}\right)^3 = -4$

and $\frac{d^2y}{dx^2} = \frac{2}{\left(-\frac{1}{3}\right)^3} + 162\left(-\frac{1}{3}\right) = -108$

which is negative.

So the curve has a local maximum at $\left(-\frac{1}{3}, -4\right)$.

When $x = \frac{1}{3}$,

$y = \frac{1}{\left(\frac{1}{3}\right)} + 27\left(\frac{1}{3}\right)^3 = 4$

and

$\frac{d^2y}{dx^2} = \frac{2}{\left(\frac{1}{3}\right)^3} + 162\left(\frac{1}{3}\right) = 108$

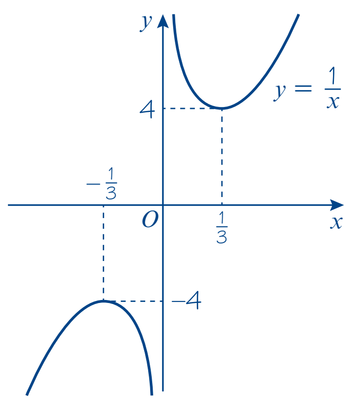
which is positive.

To sketch the curve, you need to find the coordinates of the stationary points and determine their natures. Differentiate your expression for $\frac{dy}{dx}$ to find $\frac{d^2y}{dx^2}$

Substitute $x = -\frac{1}{3}$ and $x = \frac{1}{3}$ into the equation of the curve to find the y-coordinates of the stationary points.

Online Check your solution using your calculator.

So the curve has a local minimum at $\left(\frac{1}{3}, 4\right)$.
The curve has an asymptote at $x = 0$.
As $x \to \infty$, $y \to \infty$.
As $x \to -\infty$, $y \to -\infty$.



$\frac{1}{x} \to \pm\infty$ as $x \to 0$ so $x = 0$ is an asymptote of the curve.

Mark the coordinates of the stationary points on your sketch, and label the curve with its equation.

You could check $\frac{dy}{dx}$ at specific points to help with your sketch:

- When $x = \frac{1}{4}$, $\frac{dy}{dx} = -10.9375$ which is negative.
- When $x = 1$, $\frac{dy}{dx} = 80$ which is positive.

Exercise 12I

Hint For each part of questions **1** and **2**:
- Find $f'(x)$.
- Set $f'(x) = 0$ and solve to find the value of x at the stationary point.
- Find the corresponding value of $f(x)$.

1 Find the least value of the following functions:

a $f(x) = x^2 - 12x + 8$ **b** $f(x) = x^2 - 8x - 1$ **c** $f(x) = 5x^2 + 2x$

2 Find the greatest value of the following functions:

a $f(x) = 10 - 5x^2$ **b** $f(x) = 3 + 2x - x^2$ **c** $f(x) = (6 + x)(1 - x)$

3 Find the coordinates of the points where the gradient is zero on the curves with the given equations. Establish whether these points are local maximum points, local minimum points or points of inflection in each case.

a $y = 4x^2 + 6x$ **b** $y = 9 + x - x^2$ **c** $y = x^3 - x^2 - x + 1$

d $y = x(x^2 - 4x - 3)$ **e** $y = x + \frac{1}{x}$ **f** $y = x^2 + \frac{54}{x}$

g $y = x - 3\sqrt{x}$ **h** $y = x^{\frac{1}{2}}(x - 6)$ **i** $y = x^4 - 12x^2$

4 Sketch the curves with equations given in question **3** parts **a**, **b**, **c** and **d**, labelling any stationary points with their coordinates.

(P) **5** By considering the gradient on either side of the stationary point on the curve $y = x^3 - 3x^2 + 3x$, show that this point is a point of inflection.
Sketch the curve $y = x^3 - 3x^2 + 3x$.

(P) **6** Find the maximum value and hence the range of values for the function $f(x) = 27 - 2x^4$.

(P) **7** $f(x) = x^4 + 3x^3 - 5x^2 - 3x + 1$

a Find the coordinates of the stationary points of $f(x)$, and determine the nature of each.

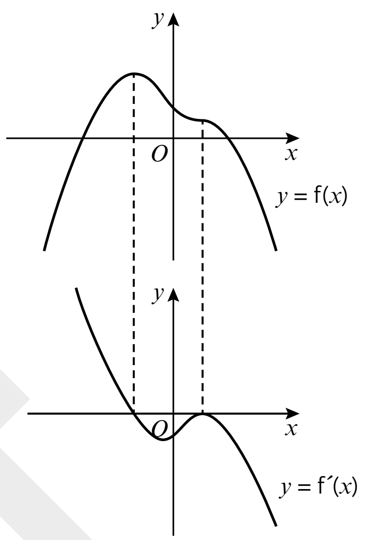
b Sketch the graph of $y = f(x)$.

Hint Use the **factor theorem** with small positive integer values of x to find one factor of $f'(x)$. ← Section 7.2

12.10 Sketching gradient functions

You can use the features of a given function to sketch the corresponding gradient function. This table shows you features of the graph of a function, $y = f(x)$, and the graph of its gradient function, $y = f'(x)$, at corresponding values of x.

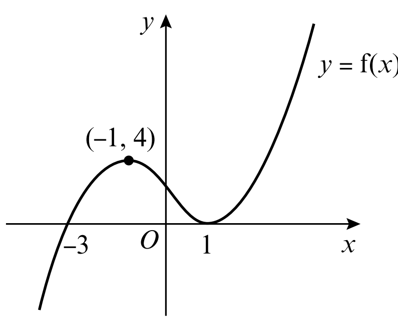
$y = f(x)$	$y = f'(x)$
Maximum or minimum	Cuts the x-axis
Point of inflection	Touches the x-axis
Positive gradient	Above the x-axis
Negative gradient	Below the x-axis
Vertical asymptote	Vertical asymptote
Horizontal asymptote	Horizontal asymptote at the x-axis

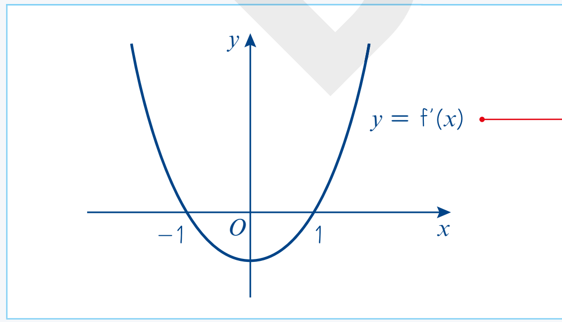


Example 19

The diagram shows the curve with equation $y = f(x)$.
The curve has stationary points at $(-1, 4)$ and $(1, 0)$, and cuts the x-axis at $(-3, 0)$.

Sketch the gradient function, $y = f'(x)$, showing the coordinates of any points where the curve cuts or meets the x-axis.





x	$y = f(x)$	$y = f'(x)$
$x < -1$	Positive gradient	Above x-axis
$x = -1$	Maximum	Cuts x-axis
$-1 < x < 1$	Negative gradient	Below x-axis
$x = 1$	Minimum	Cuts x-axis
$x > 1$	Positive gradient	Above x-axis

Watch out Ignore any points where the curve $y = f(x)$ cuts the x-axis. These will not tell you anything about the features of the graph of $y = f'(x)$.

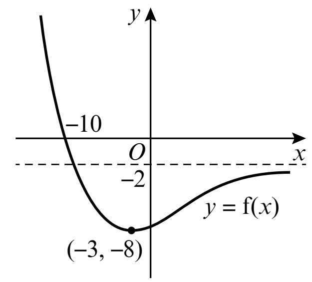
Online Use GeoGebra to explore the key features linking $y = f(x)$ and $y = f'(x)$.

Example 20

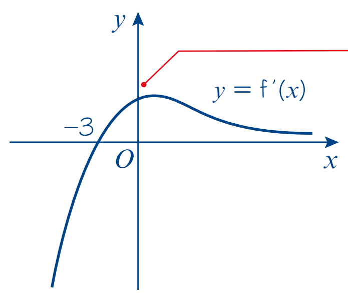
The diagram shows the curve with equation $y = f(x)$. The curve has an asymptote at $y = -2$ and a turning point at $(-3, -8)$. It cuts the x-axis at $(-10, 0)$.

a Sketch the graph of $y = f'(x)$.

b State the equation of the asymptote of $y = f'(x)$.



a



b $y = 0$

Draw your sketch on a separate set of axes. The graph of $y = f'(x)$ will have the same horizontal scale but will have a different vertical scale.

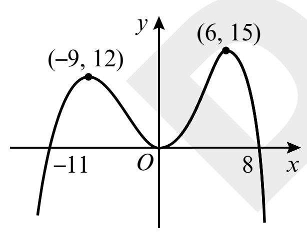
You don't have enough information to work out the coordinates of the y-intercept, or the local maximum, of the graph of the gradient function. The graph of $y = f(x)$ is a smooth curve so the graph of $y = f'(x)$ will also be a smooth curve.

If $y = f(x)$ has any **horizontal asymptotes** then the graph of $y = f'(x)$ will have an asymptote at the x-axis.

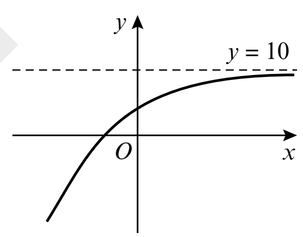
Exercise 12J

1 For each graph given, sketch the graph of the corresponding gradient function on a separate set of axes. Show the coordinates of any points where the curve cuts or meets the x-axis, and give the equations of any asymptotes.

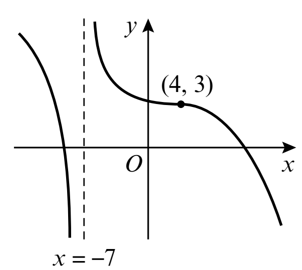
a



b



c



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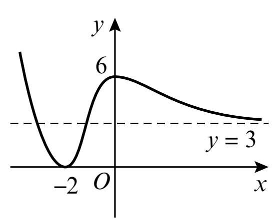
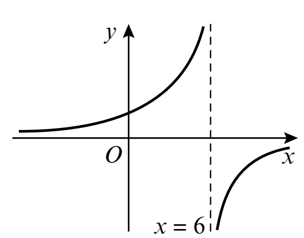
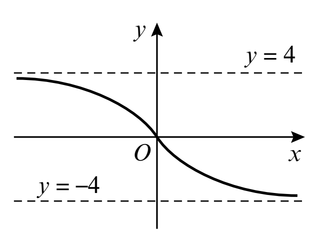


e



f



(P) 2 $f(x) = (x + 1)(x - 4)^2$

a Sketch the graph of $y = f(x)$.

b On a separate set of axes, sketch the graph of $y = f'(x)$.

c Show that $f'(x) = (x - 4)(3x - 2)$.

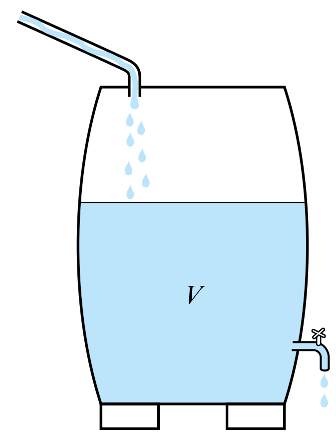
d Use the derivative to determine the exact coordinates of the points where the gradient function cuts the coordinate axes.

Hint This is an x^3 graph with a positive coefficient of x^3. ← Section 4.1

12.11 Modelling with differentiation

You can think of $\frac{dy}{dx}$ as $\frac{\text{small change in } y}{\text{small change in } x}$. It represents the **rate of change** of y with respect to x.

If you replace y and x with variables that represent real-life quantities, you can use the derivative to model lots of real-life situations involving rates of change.



The volume of water in this water butt is constantly changing over time. If V represents the volume of water in the water butt in litres, and t represents the time in seconds, then you could model V as a function of t. If $V = f(t)$ then $\frac{dV}{dt} = f'(t)$ would represent the **rate of change** of volume with respect to time. The units of $\frac{dV}{dt}$ would be litres per second.

Example 21

Given that the volume, V cm^3, of an expanding sphere is related to its radius, r cm, by the formula $V = \frac{4}{3}\pi r^3$, find the rate of change of volume with respect to radius at the instant when the radius is 5 cm.

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2$ — Differentiate V with respect to r. Remember that π is a constant.

When $r = 5$, $\frac{dV}{dr} = 4\pi \times 5^2$ — Substitute $r = 5$.

$= 314$ (3 s.f.)

So the rate of change is 314 cm^3 per cm. — Interpret the answer with units.

Example 22

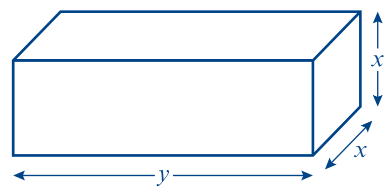
A large tank in the shape of a cuboid is to be made from 54 m^2 of sheet metal. The tank has a horizontal base and no top. The height of the tank is x metres. Two opposite vertical faces are squares.

a Show that the volume, V m^3, of the tank is given by $V = 18x - \frac{2}{3}x^3$

b Given that x can vary, use differentiation to find the maximum or minimum value of V.

c Justify that the value of V you have found is a maximum.

a Let the length of the tank be y metres.



Draw a sketch.

Problem-solving

You don't know the length of the tank. Write it as y metres to simplify your working.
You could also draw a sketch to help you find the correct expressions for the surface area and volume of the tank.

Total area, $A = 2x^2 + 3xy$

So $54 = 2x^2 + 3xy$

$y = \frac{54 - 2x^2}{3x}$

Rearrange to find y in terms of x.

But $V = x^2y$

So $V = x^2\left(\frac{54 - 2x^2}{3x}\right)$

Substitute the expression for y into the equation.

$= \frac{x}{3}(54 - 2x^2)$

So $V = 18x - \frac{2}{3}x^3$

Simplify.

b $\frac{dV}{dx} = 18 - 2x^2$

Put $\frac{dV}{dx} = 0$

Differentiate V with respect to x and put $\frac{dV}{dx} = 0$.

$0 = 18 - 2x^2$

So $x^2 = 9$

$x = -3$ or 3

Rearrange to find x.
x is a length so use the positive solution.

But x is a length so $x = 3$

When $x = 3$, $V = 18 \times 3 - \frac{2}{3} \times 3^3$

$= 54 - 18$

$= 36$

Substitute the value of x into the expression for V.

$V = 36$ is a maximum or minimum value of V.

c $\frac{d^2V}{dx^2} = -4x$

Find the second derivative of V.

When $x = 3$, $\frac{d^2V}{dx^2} = -4 \times 3 = -12$

This is negative, so $V = 36$ is the maximum value of V.

$\frac{d^2V}{dx^2} < 0$ so $V = 36$ is a maximum.

Exercise 12K

1 Find $\frac{d\theta}{dt}$ where $\theta = t^2 - 3t$.

2 Find $\frac{dA}{dr}$ where $A = 2\pi r$.

3 Given that $r = \frac{12}{t}$, find the value of $\frac{dr}{dt}$ when $t = 3$.

4 The surface area, A cm², of an expanding sphere of radius r cm is given by $A = 4\pi r^2$. Find the rate of change of the area with respect to the radius at the instant when the radius is 6 cm.

5 The displacement, s metres, of a car from a fixed point at time t seconds is given by $s = t^2 + 8t$. Find the rate of change of the displacement with respect to time at the instant when $t = 5$.

P **6** A rectangular garden is fenced on three sides, and the house forms the fourth side of the rectangle.

a Given that the total length of the fence is 80 m, show that the area, A, of the garden is given by the formula $A = y(80 - 2y)$, where y is the distance from the house to the end of the garden.

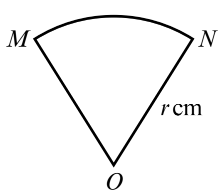
b Given that the area is a maximum for this length of fence, find the dimensions of the enclosed garden, and the area which is enclosed.

P **7** A closed cylinder has total surface area equal to 600π.

a Show that the volume, V cm³, of this cylinder is given by the formula $V = 300\pi r - \pi r^3$, where r cm is the radius of the cylinder.

b Find the maximum volume of such a cylinder.

P **8** A sector of a circle has area 100 cm².

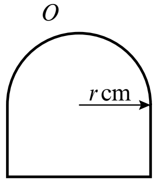


a Show that the perimeter of this sector is given by the formula

$$P = 2r + \frac{200}{r},\ r > \sqrt{\frac{100}{\pi}}$$

b Find the minimum value for the perimeter.

E/P **9** A shape consists of a rectangular base with a semicircular top, as shown.



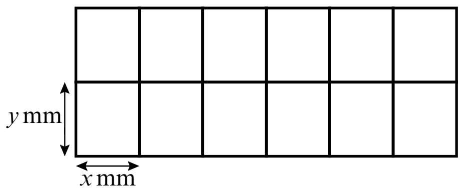
a Given that the perimeter of the shape is 40 cm, show that its area, A cm², is given by the formula

$$A = 40r - 2r^2 - \frac{\pi r^2}{2}$$

where r cm is the radius of the semicircle. **(2 marks)**

b Hence find the maximum value for the area of the shape. **(4 marks)**

E/P **10** The shape shown is a wire frame in the form of a large rectangle split by parallel lengths of wire into 12 smaller equal-sized rectangles.



a Given that the total length of wire used to complete the whole frame is 1512 mm, show that the area of the whole shape, A mm², is given by the formula

$$A = 1296x - \frac{108x^2}{7}$$

where x mm is the width of one of the smaller rectangles. **(4 marks)**

b Hence find the maximum area which can be enclosed in this way. **(4 marks)**

Mixed exercise 12

E/P **1** Prove, from first principles, that the derivative of $10x^2$ is $20x$. **(4 marks)**

P **2** The point A with coordinates (1, 4) lies on the curve with equation $y = x^3 + 3x$. The point B also lies on the curve and has x-coordinate $(1 + \delta x)$.

a Show that the gradient of the line segment AB is given by $(\delta x)^2 + 3\delta x + 6$.

b Deduce the gradient of the curve at point A.

3 A curve is given by the equation $y = 3x^2 + 3 + \frac{1}{x^2}$, where $x > 0$. At the points A, B and C on the curve, $x = 1$, 2 and 3 respectively. Find the gradient of the curve at A, B and C.

E **4** Calculate the x-coordinates of the points on the curve with equation $y = 7x^2 - x^3$ at which the gradient is equal to 16. **(4 marks)**

5 Find the x-coordinates of the two points on the curve with equation $y = x^3 - 11x + 1$ where the gradient is 1. Find the corresponding y-coordinates.

E **6** The function f is defined by $f(x) = x + \frac{9}{x}$, $x \in \mathbb{R}$, $x \neq 0$.

a Find $f'(x)$. **(2 marks)**

b Solve $f'(x) = 0$. **(2 marks)**

E **7** Given that

$$y = 3\sqrt{x} - \frac{4}{\sqrt{x}}, \quad x > 0,$$

find $\frac{dy}{dx}$ **(3 marks)**

E/P **8** A curve has equation $y = 12x^{\frac{1}{2}} - x^{\frac{3}{2}}$.

a Show that $\frac{dy}{dx} = \frac{3}{2}x^{-\frac{1}{2}}(4 - x)$. **(2 marks)**

b Find the coordinates of the point on the curve where the gradient is zero. **(2 marks)**

E **9** **a** Expand $(x^{\frac{3}{2}} - 1)(x^{-\frac{1}{2}} + 1)$. **(2 marks)**

b A curve has equation $y = (x^{\frac{3}{2}} - 1)(x^{-\frac{1}{2}} + 1)$, $x > 0$. Find $\frac{dy}{dx}$ **(2 marks)**

c Use your answer to part **b** to calculate the gradient of the curve at the point where $x = 4$. **(1 mark)**

E **10** Differentiate with respect to x:

$$2x^3 + \sqrt{x} + \frac{x^2 + 2x}{x^2}$$ **(3 marks)**

E/P **11** The curve with equation $y = ax^2 + bx + c$ passes through the point (1, 2). The gradient of the curve is zero at the point (2, 1). Find the values of a, b and c. **(5 marks)**

E/P **12** A curve C has equation $y = x^3 - 5x^2 + 5x + 2$.

a Find $\frac{dy}{dx}$ in terms of x. **(2 marks)**

b The points P and Q lie on C. The gradient of C at both P and Q is 2. The x-coordinate of P is 3.

i Find the x-coordinate of Q. **(3 marks)**

ii Find an equation for the tangent to C at P, giving your answer in the form $y = mx + c$, where m and c are constants. **(3 marks)**

iii If this tangent intersects the coordinate axes at the points R and S, find the length of RS, giving your answer as a surd. **(3 marks)**

13 A curve has equation $y = \frac{8}{x} - x + 3x^2$, $x > 0$. Find the equations of the tangent and the normal to the curve at the point where $x = 2$.

E/P **14** The normals to the curve $2y = 3x^3 - 7x^2 + 4x$, at the points $O(0, 0)$ and $A(1, 0)$, meet at the point N.

a Find the coordinates of N. **(7 marks)**

b Calculate the area of triangle OAN. **(3 marks)**

E/P **15** A curve C has equation $y = x^3 - 2x^2 - 4x - 1$ and cuts the y-axis at a point P. The line L is a tangent to the curve at P, and cuts the curve at the point Q. Show that the distance PQ is $2\sqrt{17}$. **(7 marks)**

E **16** Given that $y = x^{\frac{3}{2}} + \frac{48}{x}$, $x > 0$

a find the value of x and the value of y when $\frac{dy}{dx} = 0$. **(5 marks)**

b show that the value of y which you found in part **a** is a minimum. **(2 marks)**

17 A curve has equation $y = x^3 - 5x^2 + 7x - 14$. Determine, by calculation, the coordinates of the stationary points of the curve.

E/P **18** The function f, defined for $x \in \mathbb{R}$, $x > 0$, is such that:

$$f'(x) = x^2 - 2 + \frac{1}{x^2}$$

a Find the value of $f''(x)$ at $x = 4$. **(4 marks)**

b Prove that f is an increasing function. **(3 marks)**

E **19** A curve has equation $y = x^3 - 6x^2 + 9x$. Find the coordinates of its local maximum. **(4 marks)**

20 $f(x) = 3x^4 - 8x^3 - 6x^2 + 24x + 20$

a Find the coordinates of the stationary points of f(x), and determine the nature of each of them.

b Sketch the graph of $y = f(x)$.

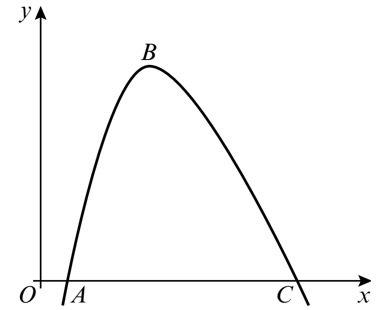
E **21** The diagram shows part of the curve with equation $y = f(x)$, where:

$$f(x) = 200 - \frac{250}{x} - x^2, \; x > 0$$

The curve cuts the x-axis at the points A and C. The point B is the maximum point of the curve.

a Find $f'(x)$. **(3 marks)**

b Use your answer to part **a** to calculate the coordinates of B. **(4 marks)**



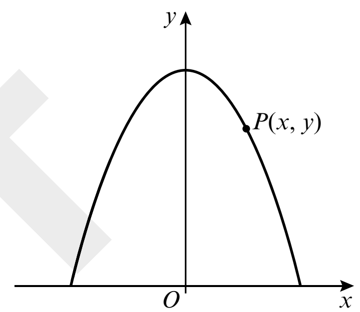
E/P **22** The diagram shows the part of the curve with equation $y = 5 - \frac{1}{2}x^2$ for which $y > 0$. The point $P(x, y)$ lies on the curve and O is the origin.

a Show that $OP^2 = \frac{1}{4}x^4 - 4x^2 + 25$. **(3 marks)**

Taking $f(x) = \frac{1}{4}x^4 - 4x^2 + 25$:

b Find the values of x for which $f'(x) = 0$. **(4 marks)**

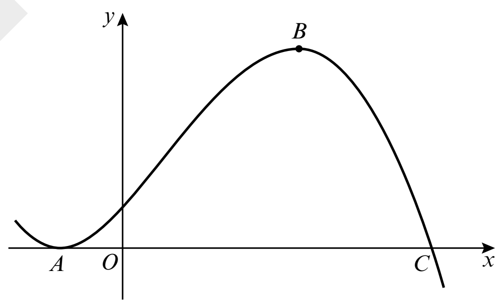
c Hence, or otherwise, find the minimum distance from O to the curve, showing that your answer is a minimum. **(4 marks)**



E **23** The diagram shows part of the curve with equation $y = 3 + 5x + x^2 - x^3$. The curve touches the x-axis at A and crosses the x-axis at C. The points A and B are stationary points on the curve.

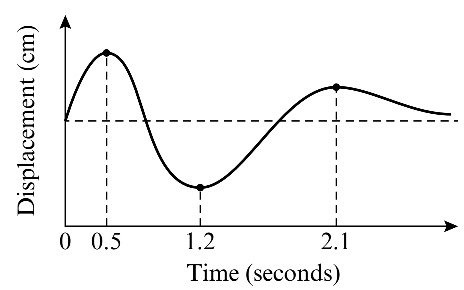
a Show that C has coordinates $(3, 0)$. **(1 mark)**

b Using calculus and showing all your working, find the coordinates of A and B. **(5 marks)**



P **24** The motion of a damped spring is modelled using this graph.

On a separate graph, sketch the gradient function for this model. Choose suitable labels and units for each axis, and indicate the coordinates of any points where the gradient function crosses the horizontal axis.



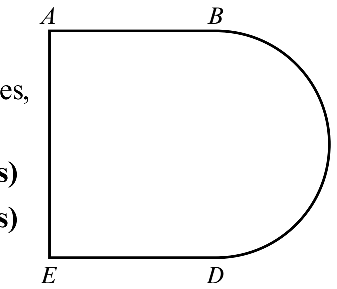
25 The volume, V cm^3, of a tin of radius r cm is given by the formula $V = \pi(40r - r^2 - r^3)$. Find the positive value of r for which $\frac{dV}{dr} = 0$, and find the value of V which corresponds to this value of r.

P **26** The total surface area, A cm^2, of a cylinder with a fixed volume of 1000 cm^3 is given by the formula $A = 2\pi x^2 + \frac{2000}{x}$, where x cm is the radius. Show that when the rate of change of the area with respect to the radius is zero, $x^3 = \frac{500}{\pi}$

E/P **27** A wire is bent into the plane shape $ABCDE$ as shown. Shape $ABDE$ is a rectangle and BCD is a semicircle with diameter BD. The area of the region enclosed by the wire is R m^2, $AE = x$ metres, and $AB = ED = y$ metres. The total length of the wire is 2 m.

a Find an expression for y in terms of x. **(3 marks)**

b Prove that $R = \frac{x}{8}(8 - 4x - \pi x)$. **(4 marks)**



Given that x can vary, using calculus and showing your working:

c find the maximum value of R. (You do not have to prove that the value you obtain is a maximum.) **(5 marks)**

E/P **28** A cylindrical biscuit tin has a close-fitting lid which overlaps the tin by 1 cm, as shown. The radii of the tin and the lid are both x cm. The tin and the lid are made from a thin sheet of metal of area 80π cm^2 and there is no wastage. The volume of the tin is V cm^3.

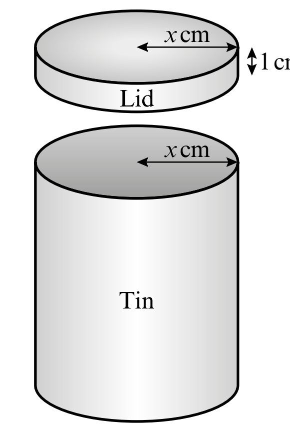
a Show that $V = \pi(40x - x^2 - x^3)$. **(5 marks)**

Given that x can vary:

b use differentiation to find the positive value of x for which V is stationary. **(3 marks)**

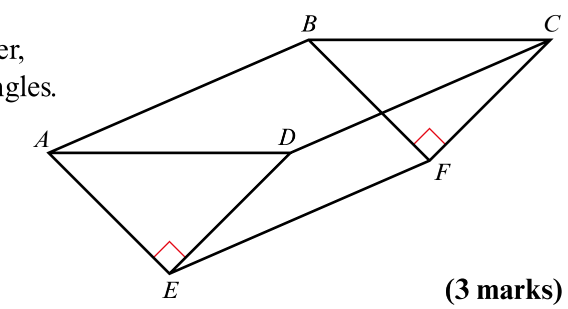
c Prove that this value of x gives a maximum value of V. **(2 marks)**

d Find this maximum value of V. **(1 mark)**



e Determine the percentage of the sheet metal used in the lid when V is a maximum. **(2 marks)**

E **29** The diagram shows an open tank for storing water, $ABCDEF$. The sides $ABFE$ and $CDEF$ are rectangles. The triangular ends ADE and BCF are isosceles, and $\angle AED = \angle BFC = 90°$. The ends ADE and BCF are vertical and EF is horizontal.



Given that $AD = x$ metres:

a show that the area of triangle ADE is $\frac{1}{4}x^2$ m^2 **(3 marks)**

Given also that the capacity of the container is 4000 m^3 and that the total area of the two triangular and two rectangular sides of the container is S m^2:

b show that $S = \frac{x^2}{2} + \frac{16\,000\sqrt{2}}{x}$ **(4 marks)**

Given that x can vary:

c use calculus to find the minimum value of S. **(6 marks)**

d justify that the value of S you have found is a minimum. **(2 marks)**

Challenge

a Find the first four terms in the binomial expansion of $(x + h)^7$, in ascending powers of h.

b Hence prove, from first principles, that the derivative of x^7 is $7x^6$.

Summary of key points

1 The **gradient** of a **curve** at a given point is defined as the gradient of the **tangent** to the curve at that point.

2 The **gradient function**, or **derivative**, of the curve $y = \mathrm{f}(x)$ is written as $\mathrm{f}'(x)$ or $\frac{\mathrm{d}y}{\mathrm{d}x}$

$$\mathrm{f}'(x) = \lim_{h \to 0} \frac{\mathrm{f}(x+h) - \mathrm{f}(x)}{h}$$

The gradient function can be used to find the gradient of the curve for any value of x.

3 For all real values of n, and for a constant a:

- If $\mathrm{f}(x) = x^n$ then $\mathrm{f}'(x) = nx^{n-1}$
- If $y = x^n$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = nx^{n-1}$
- If $\mathrm{f}(x) = ax^n$ then $\mathrm{f}'(x) = anx^{n-1}$
- If $y = ax^n$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = anx^{n-1}$

4 For the quadratic curve with equation $y = ax^2 + bx + c$, the derivative is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2ax + b$$

5 If $y = \mathrm{f}(x) \pm \mathrm{g}(x)$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}'(x) \pm \mathrm{g}'(x)$.

6 The tangent to the curve $y = \mathrm{f}(x)$ at the point with coordinates $(a, \mathrm{f}(a))$ has equation

$$y - \mathrm{f}(a) = \mathrm{f}'(a)(x - a)$$

7 The normal to the curve $y = \mathrm{f}(x)$ at the point with coordinates $(a, \mathrm{f}(a))$ has equation

$$y - \mathrm{f}(a) = -\frac{1}{\mathrm{f}'(a)}(x - a)$$

8
- The function $\mathrm{f}(x)$ is **increasing** on the interval $[a, b]$ if $\mathrm{f}'(x) \geqslant 0$ for all values of x such that $a < x < b$.
- The function $\mathrm{f}(x)$ is **decreasing** on the interval $[a, b]$ if $\mathrm{f}'(x) \leqslant 0$ for all values of x such that $a < x < b$.

9 Differentiating a function $y = \mathrm{f}(x)$ twice gives you the second order derivative, $\mathrm{f}''(x)$ or $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$

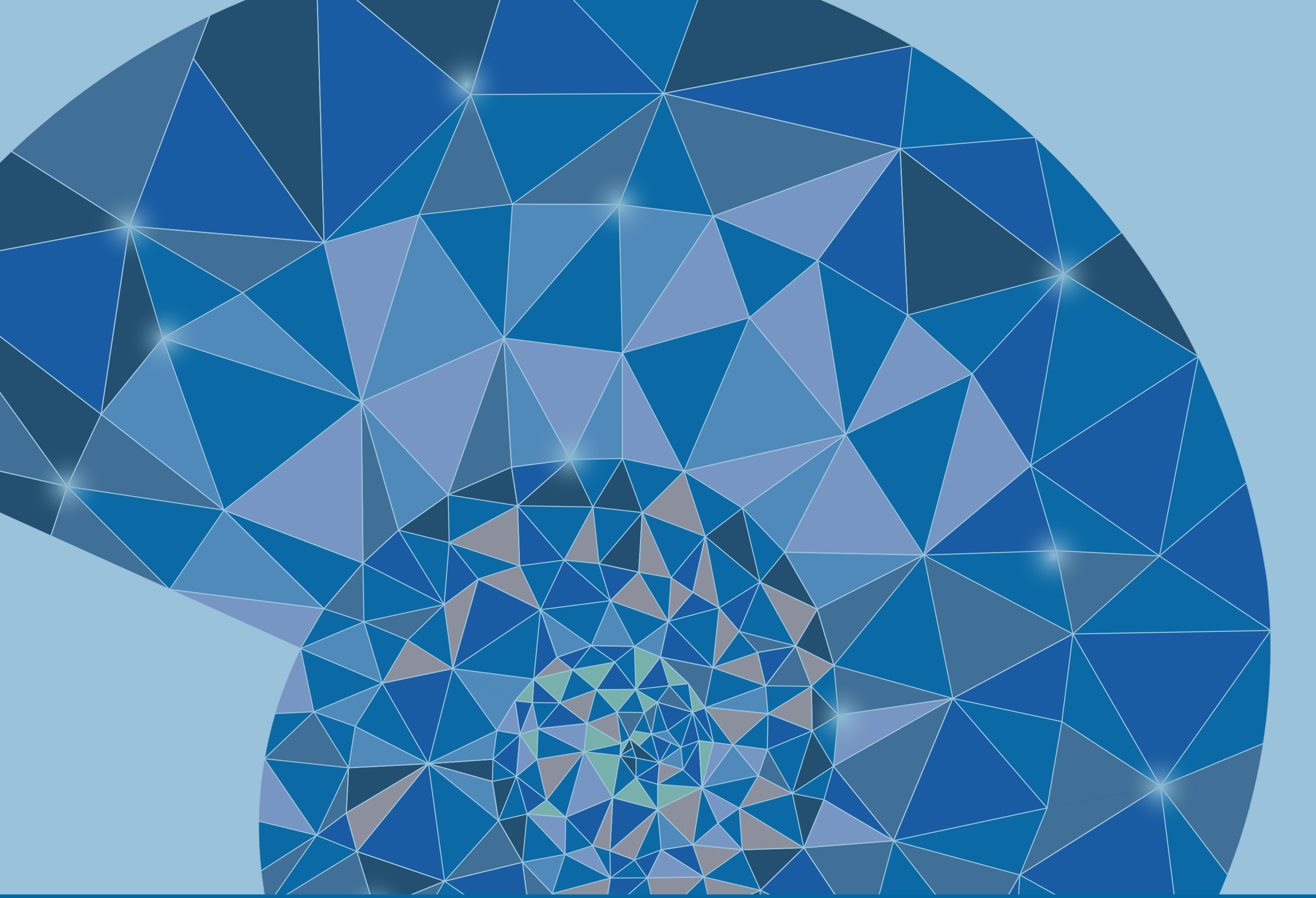
10 Any point on the curve $y = \mathrm{f}(x)$ where $\mathrm{f}'(x) = 0$ is called a **stationary point**. For a small positive value h:

Type of stationary point	$\mathrm{f}'(x-h)$	$\mathrm{f}'(x)$	$\mathrm{f}'(x+h)$
Local maximum	Positive	0	Negative
Local minimum	Negative	0	Positive
Point of inflection	Negative	0	Negative
	Positive	0	Positive

11 If a function $\mathrm{f}(x)$ has a stationary point when $x = a$, then:

- if $\mathrm{f}''(a) > 0$, the point is a local minimum
- if $\mathrm{f}''(a) < 0$, the point is a local maximum.

If $\mathrm{f}''(a) = 0$, the point could be a local minimum, a local maximum or a point of inflection. You will need to look at points on either side to determine its nature.



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