ENHANCING THE CONCEPTUAL UNDERSTANDING OF SEQUENCES AND SERIES WITH TECHNOLOGY

Jay L. Schiffman

Rowan University

201 Mullica Hill Road

Glassboro, NJ 08028-1701

schiffman@rowan.edu

Abstract: The TI-89 and Mathematica are employed to explore the formal definition of limit for sequences, investigate partial sums and obtain sums of series. We delve deeper into these topics to enhance our calculus course.

Our journey commences with the TI-89 enabling students to be exposed to rich activities supplementing a traditional second semester calculus course. We explore connections between the algebra employed in verifying the limit of a sequence with how the definition works by assigning epsilon a small prescribed value. Later activities include determining the minimal number of terms for the sum of the terms in the harmonic series to exceed a prescribed positive integer and explore an interesting connection between the harmonic series and the transcendental number e. Finally, one can consider variations of the harmonic series such as the convergent harmonic series of squares and the divergent harmonic series of primes proven by Euler. Technology is employed to obtain empirical evidence for the concepts and aids in broadening one's understanding.

We begin by furnishing a snippet of a rich conceptual activity which places the formal definition of a limit in context. We first show that $\lim\limits_{n \to +\infty} \frac{3 \cdot n - 1}{4 \cdot n + 5} = \frac{3}{4}$ via the formal definition of limit

(ε, N). One then conceptualizes this in **FIGURES 1-2,** noting that the TI-89 can solve rational inequalities. The solve command is accessed from the Algebra Menu. We solve the following inequalities and determine the minimal number of terms required for all terms of the sequence to lie within the respective epsilon neighborhoods of the limit $\frac{3}{4}: \varepsilon = 0.1 \text{ } and \text{ } \varepsilon = 0.01.$ We first furnish the formal proof which proceeds as follows: Let $\varepsilon > 0$ be given. We must produce a $N > 0,\ N(\varepsilon) \ni n > N \Rightarrow |\frac{3 \cdot n - 1}{4 \cdot n + 5} - \frac{3}{4}| < \varepsilon.$ Now

$$|\frac{3 \cdot n - 1}{4 \cdot n + 5} - \frac{3}{4}| = |\frac{(3 \cdot n - 1) \cdot (4) - (3) \cdot (4 \cdot n + 5)}{4 \cdot (4 \cdot n + 5)}| = |\frac{12 \cdot n - 4 - 12 \cdot n - 15}{4 \cdot (4 \cdot n + 5)}| =$$

$$|\frac{-19}{4 \cdot (4 \cdot n + 5)}| = \frac{|-19|}{|4 \cdot (4 \cdot n + 5)|} = \frac{|-19|}{|4| \cdot |4 \cdot n + 5|} = \frac{19}{4 \cdot (4 \cdot n + 5)}.$$

$$|\frac{3 \cdot n - 1}{4 \cdot n + 5} - \frac{3}{4}| < \varepsilon \Leftrightarrow \frac{19}{4 \cdot (4 \cdot n + 5)} < \varepsilon \Leftrightarrow \frac{4 \cdot (4 \cdot n + 5)}{19} > \frac{1}{\varepsilon} \Leftrightarrow 4 \cdot n + 5 > \frac{19}{4 \cdot \varepsilon} \Leftrightarrow$$

$$4 \cdot n > \frac{19}{4 \cdot \varepsilon} - 5 \Leftrightarrow n > \frac{1}{4} \cdot \left(\frac{19}{4 \cdot \varepsilon} - 5\right).$$

Choosing $N = \frac{1}{4} \cdot \left(\frac{19}{4 \cdot \varepsilon} - 5\right)$, we see that $|\frac{3 \cdot n - 1}{4 \cdot n + 5} - \frac{3}{4}| < \varepsilon \quad \forall n > N$ so that $\lim_{n \to +\infty} \frac{3 \cdot n - 1}{4 \cdot n + 5} = \frac{3}{4}$ and the proof is complete.

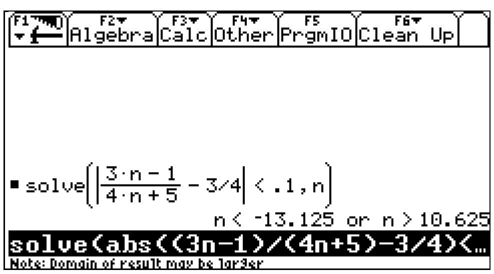


FIGURE 1: Number of Terms Needed

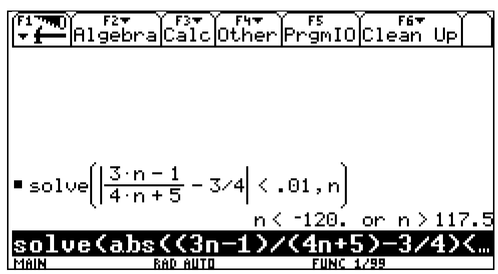


FIGURE 2: Number of Terms Needed

From the eleventh term onward **(FIGURE 1),** each of the terms of the sequence lies within 0.1 of the limit $\frac{3}{4}$. Similarly **(FIGURE 2),** each of the terms of the sequence from the one hundred eighteenth onward lies within 0.01 unit of the limit $\frac{3}{4}$. One can also see a capsule using a Table in sequence mode for the first sixteen terms in **FIGURES 3-6. FIGURE 6** verifies that from the eleventh term onward**,** each of the terms of the sequence lies within 0.1 of the limit $\frac{3}{4}$.

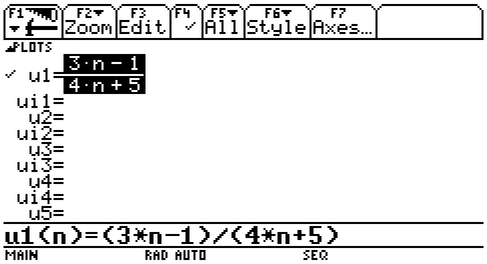


FIGURE 3: The Sequence Inputted

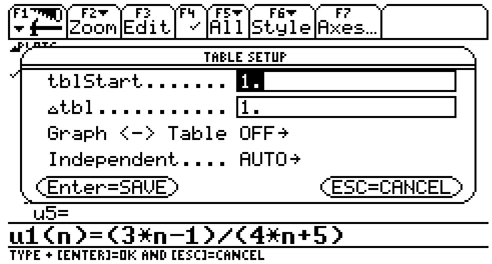


FIGURE 4: The Table Setup

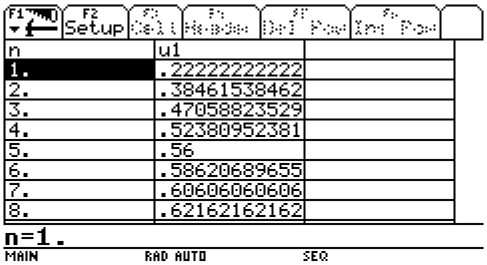


FIGURE 5: The Table

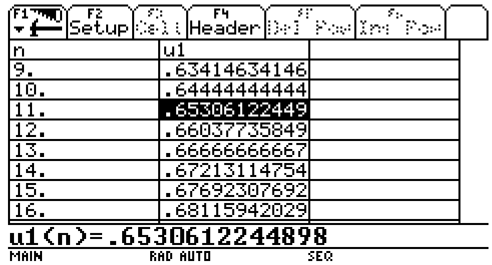


FIGURE 6: The Table

Next consider the harmonic series $\sum_{n=1}^{+\infty} \frac{1}{n}$ which diverges despite $\lim_{n\to+\infty} \frac{1}{n} = 0.$ An infinite series is the limiting value of a sequence of partial sums. For example, the first five terms of the harmonic series are $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \textit{and } \frac{1}{5}$ and the first five partial sums are $1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12}, \frac{137}{60}.$ The calculator enables one to form a sequence as well as a sequence of partial sums accessible via the command Cumulative Sum (cumSum) (**FIGURE 7). FIGURE 8** shows the sum of the first five terms of the harmonic series and represents the first five partial sums. The sequence command is accessible via the keystrokes 2nd 5 (MATH) $\Downarrow 3: List \Rightarrow 1: seq($.

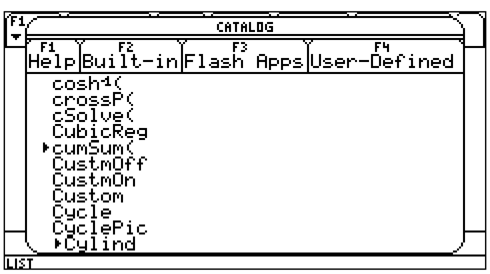


FIGURE 7: Cumulative Sum Command

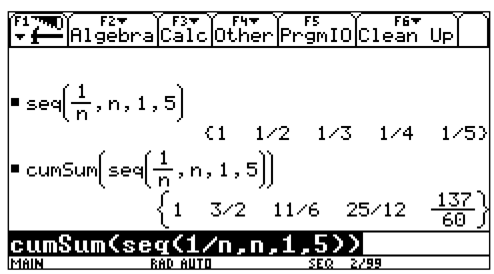


FIGURE 8: The Cumulative Sums

We next show that the minimal number of terms required for the sum of the terms in the harmonic series to exceed each of the first ten positive integers are respectively 2, 4, 11, 31, 83, 227, 616, 1674, 4550 and 12367. See **FIGURES 9-16.** (**FIGURE 8** clearly indicates that 2 terms are required for the sum to exceed one and four terms are needed for the sum to exceed two.)

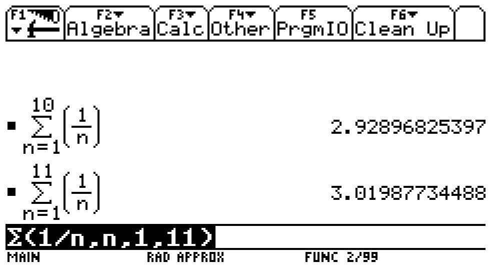


FIGURE 9: Sum Exceeds Three

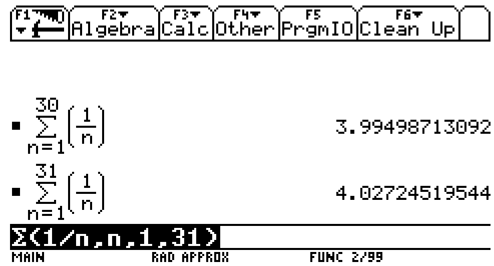


FIGURE 10: Sum Exceeds Four

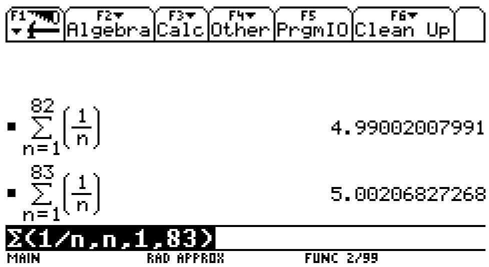


FIGURE 11: Sum Exceeds Five

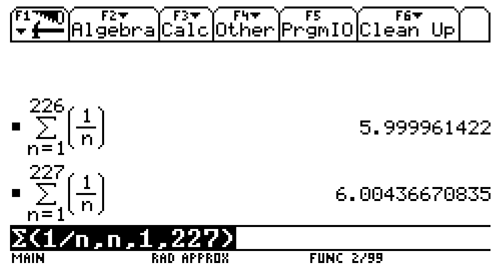


FIGURE 12: Sum Exceeds Six

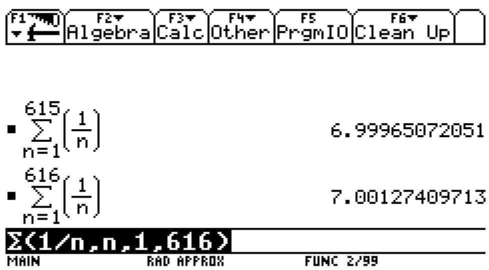


FIGURE 13: Sum Exceeds Seven

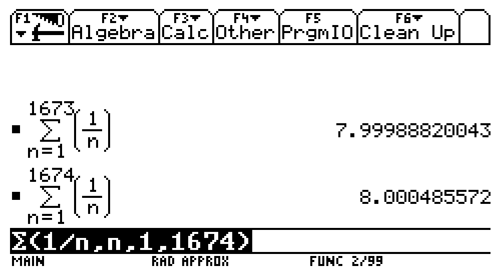


FIGURE 14: Sum Exceeds Eight

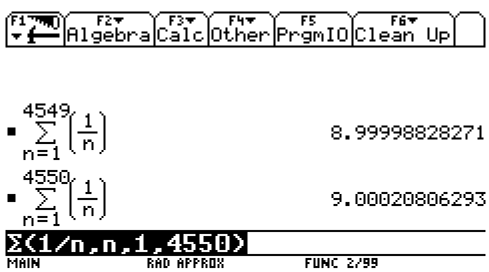


FIGURE 15: Sum Exceeds Nine

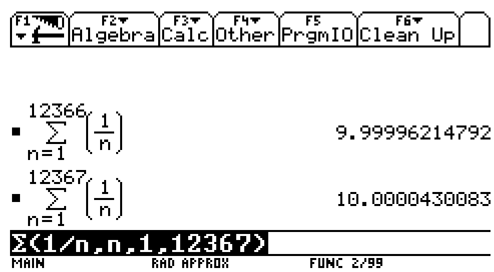


FIGURE 16: Sum Exceeds Ten

Eric Weisstein of Wolfram Research and owner of the acclaimed website MathWorld verified the accuracy of the minimum number of terms needed in the harmonic series for the sum to exceed each of the first twenty-eight positive integers while Tony Noe extended this to the first one hundred positive integers. The harmonic series diverges extremely slowly and is asymptotic $\log n$ where the logarithm being referred to is the natural logarithm. It can be shown that the harmonic series satisfies the inequality $\ln(n) < \sum_{n=1}^{+\infty} \frac{1}{n} < \ln(n)+1$. For example,

$\ln 12367 < \sum_{i=1}^{12367} \frac{1}{i} < \ln 12367 + 1 \Leftrightarrow 9.42279 < 10.00004 < 10.42279.$ **FIGURES 17-18** illustrate the bounds. For example, 12367 is the minimal number of terms required for the sum of the terms in the harmonic series to exceed ten for the first time as displayed in **FIGURE 16.**

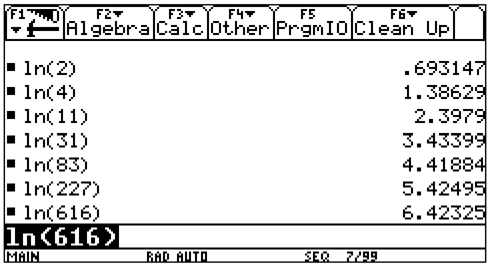


FIGURE 17: Examples of the Bound

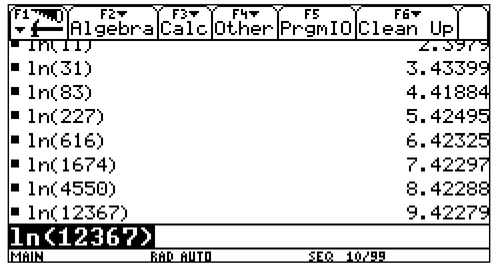


FIGURE 18: Examples of the Bound

There are several accessible proofs justifying the divergence of the harmonic series including the integral test, the p series test and the polynomial test.

We next thin the series and only consider the harmonic series of primes and seek the minimal number of terms required for the sum of the terms in the harmonic series of primes to exceed each of the initial two positive integers. Euler proved that this series likewise diverges, but incredibly slowly. The minimal number of terms required for the sum to exceed each of the first three counting integers are 3, 59 and 361139 respectively. In 2005, Eric Bach and Jon Sorenson discovered that an incredible 43922730588128390 terms are needed for the sum to exceed 4 for the initial time. This new extension was achieved using a variant of the Lagarias-Miller-Odlyzko algorithm for $\pi(x)$. For $p \le 1801241230056600467$, $\sum \frac{1}{p} = 3.99999999999999999966$ while for $p \le 1801241230056600523$, $\sum \frac{1}{p} = 4.00000000000000000021.$ Moreover, there are no primes between 1801241230056600467 and 1801241230056600523. We refer to **FIGURE 19.** Total computing time was about two weeks, equally divided between two workstations.

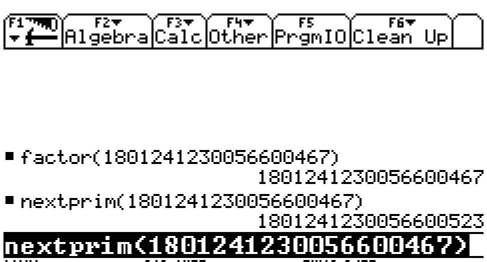


FIGURE 19: Using the Next Prime Program in FIGURE 22

While the harmonic series is asymptotic $\log(n)$, the harmonic series of primes is asymptotic $\log\log(n)$. We next seek the minimum number of terms required for the sum of the terms in the harmonic series of primes to exceed 1 and 2 respectively. Consider the series $\sum_{p=1}^{+\infty}\frac{1}{p}$ where p is prime. One can verify that the initial sixty primes are as follows:

$$\left\{\begin{matrix} 2,3,5,7,11,13,17,19,23,29,31,37,41,43,47,53,59,61,67,71,73,79,83,89,97, \\ 101,103,107,109,113,127,131,137,139,149,151,157,163,167,173,179,181,191, \\ 193,197,199,211,223,227,229,233,239,241,251,257,263,269,277,281 \end{matrix}\right\}.$$

Page 435 of the TI-89 guidebook provides a program entitled Next Prime enabling one to secure the next prime after any given positive integer. Note that the Program Editor is depicted in **FIGURES 20-21**, the Program in **FIGURE 22,** the VARIABLES LINK folder in **FIGURE 23.** The successive primes can hence be calculated. A capsule of the technology for this series is displayed in **FIGURES 24-25:**

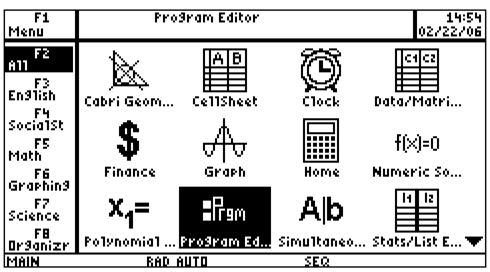


FIGURE 20: The Program Editor

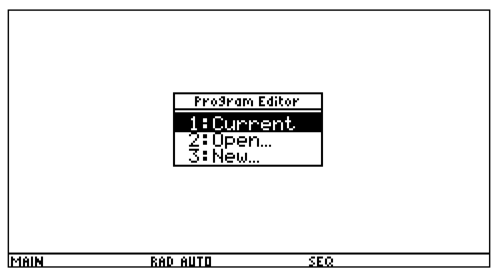


FIGURE 21: The Current Program

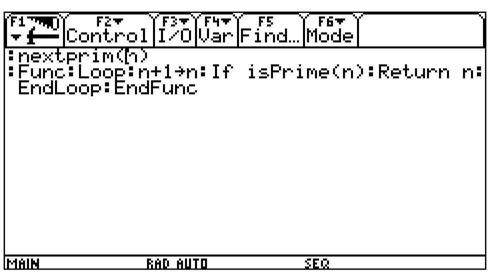


FIGURE 22: The Next Prime Program

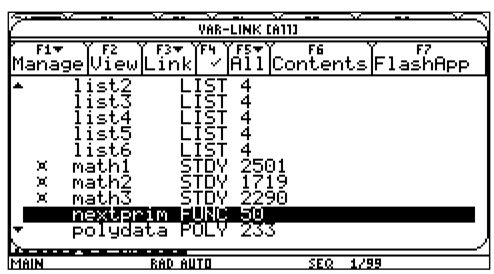


FIGURE 23: The Variables Link Folder

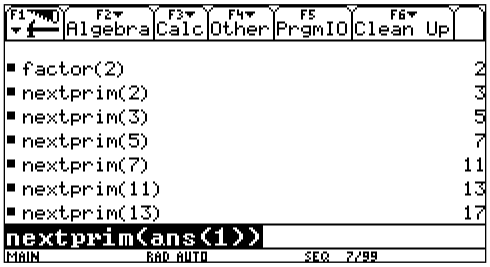


FIGURE 24: Examples of the Next Prime

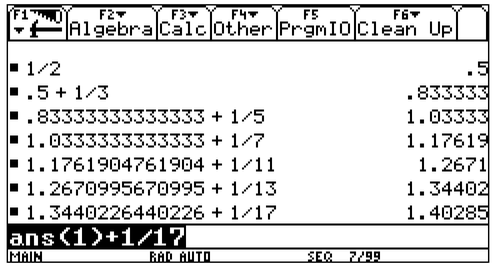


FIGURE 25: The Harmonic Series of Primes

FIGURE 26 illustrates the following with regards to the divergent harmonic series of primes and the number of terms for the sum to exceed 1, 2, and 3 for the estimate:

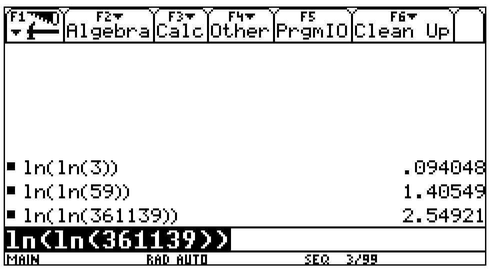


FIGURE 26: An Illustration of the Estimate

Note that ans (1) represents the previous answer on the HOME SCREEN. Next we calculate the sequence of partial sums. It will be shown that one requires three terms for the sum to exceed 1 and 59 terms for the sum to exceed 2. A formidable 361139 terms are required for the sum to exceed 3. See **FIGURES 27-38:**

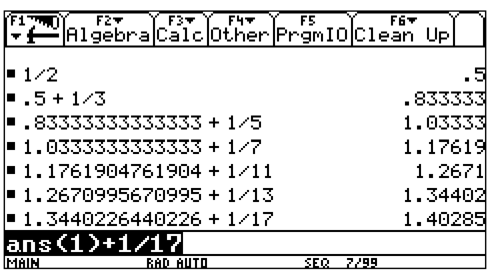


FIGURE 27: Sums of Prime Reciprocals

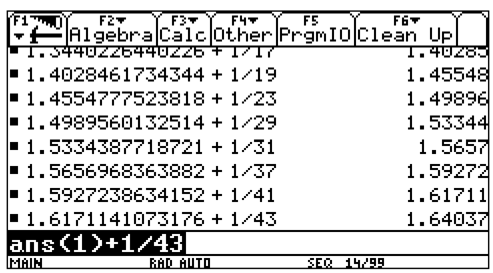


FIGURE 28: Sums of Prime Reciprocals

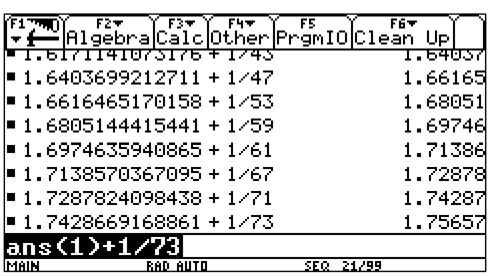


FIGURE 29: Sums of Prime Reciprocals

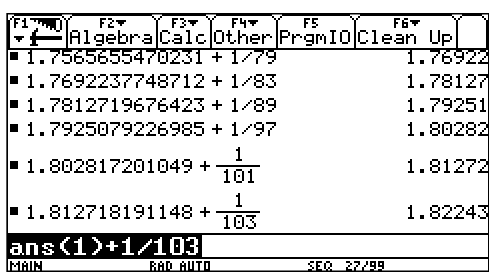


FIGURE 30: Sums of Prime Reciprocals

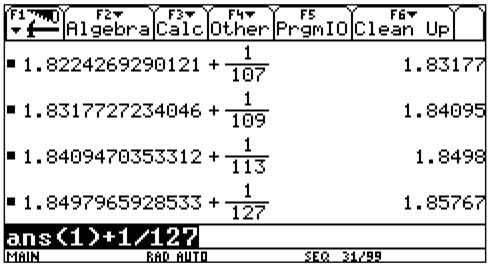


FIGURE 31: Sums of Prime Reciprocals

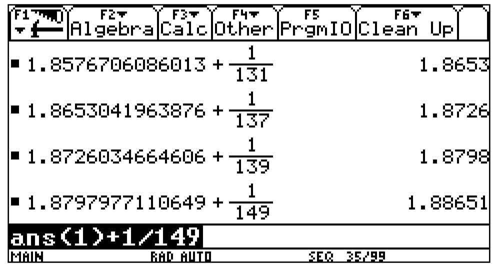


FIGURE 32: Sums of Prime Reciprocals

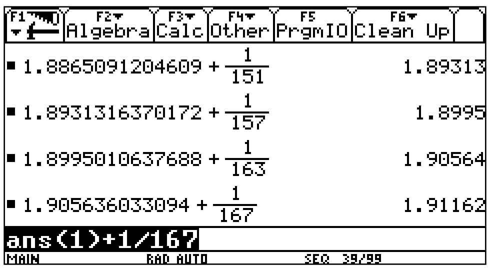


FIGURE 33: Sums of Prime Reciprocals

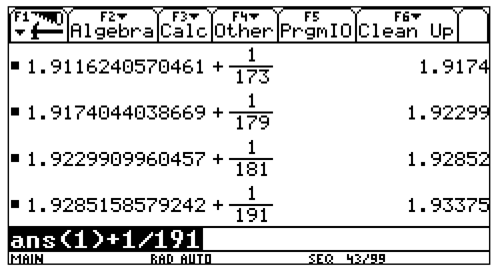


FIGURE 34: Sums of Prime Reciprocals

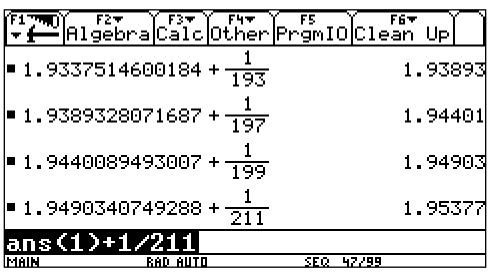


FIGURE 35: Sums of Prime Reciprocals

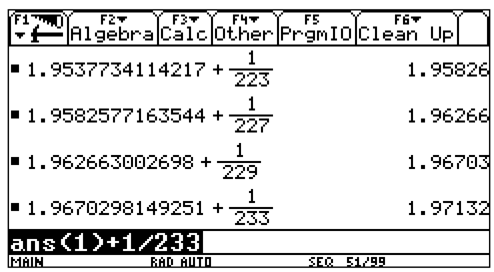


FIGURE 36: Sums of Prime Reciprocals

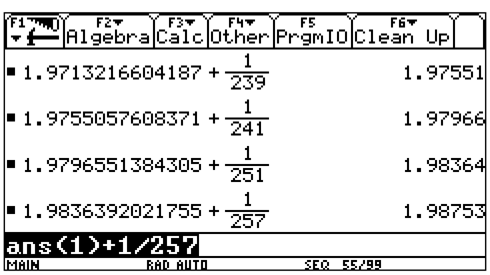


FIGURE 37: Sums of Prime Reciprocals

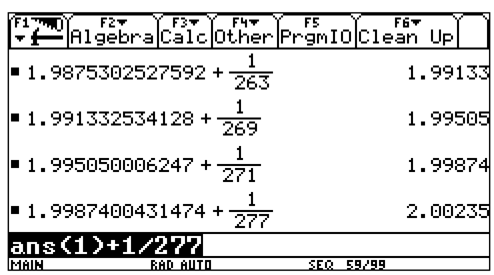


FIGURE 38: Sums of Prime Reciprocals

In addition, **FIGURE 39** illustrates the estimate with regards to the divergent harmonic series of primes and the number of terms for the sum to exceed 1, 2, and 3:

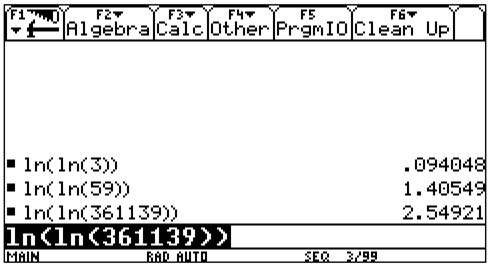


FIGURE 39: An Illustration of the Estimate

We next consider the ratios of the successive terms 2, 4, 11, 31, 83, 227, 616, 1674, 4550, and 12367 (the minimal number of terms needed for the sum of the terms in the harmonic series to exceed each of the initial ten counting integers). See **FIGURES 40-41:**

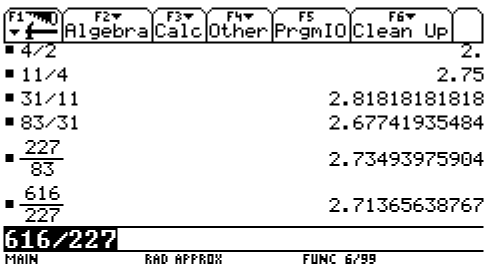


FIGURE 40: Ratios of Successive Terms

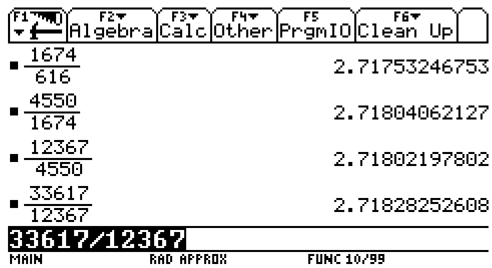


FIGURE 41: Ratios of Successive Terms

One might conjecture that these ratios approach the famous transcendental number $e \approx 2.71828182846.$ This is indeed the case. The OEIS gives the minimal number of terms required for the sum of the harmonic series to exceed each of the initial 28 counting integers. This sequence is A002387 with terms 2, 4, 11, 31, 83, 227, 616, 1674, 4550, 12367, ….

In contrast, the harmonic series of squares converges and it is well known that $\sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.6449340668482264365.$ We consider the sum to three terms in **FIGURES 42-43:**

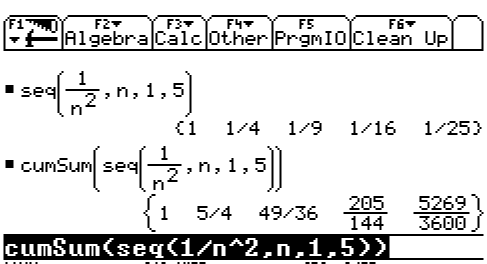


FIGURE 42: First Five Partial Sums

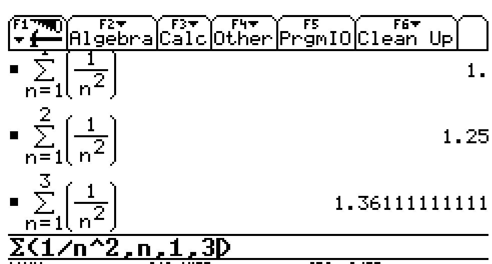


FIGURE 43: The Sum Up to Three Terms

We now take the partial sums of the series is to fifty, one hundred, five hundred, one thousand, five thousand and ten thousand terms in **FIGURES 44-45:**

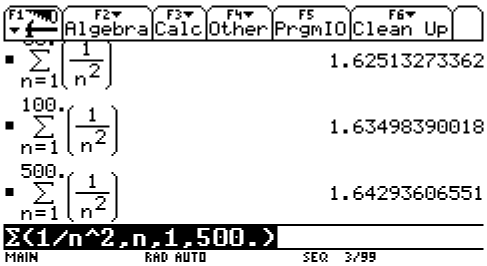


FIGURE 44: Larger Partial Sums

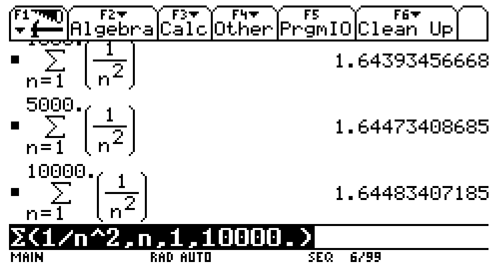


FIGURE 45: Larger Partial Sums

The more powerful package MATHEMATICA enables one to easily determine the partial sums of this series to fifty thousand, one hundred thousand, five hundred thousand and one million terms respectively to twenty decimal places. These respective partial sums are 1.64491406704822510031, 1.64492406689822629981, 1.64493206685022643351 and 1.64493306684872643363. In **FIGURE 46,** we compute the sum to infinitely many terms:

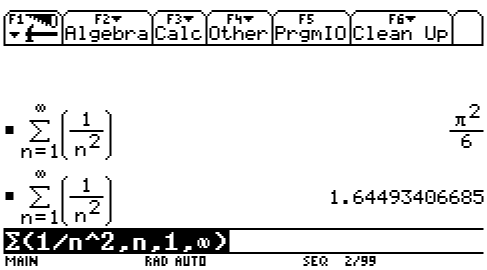


FIGURE 46: The Sum of the Harmonic Series of Squares

We next embark on other variations of the harmonic series. For example, consider the harmonic series of twin primes where twin primes are pairs of odd primes that differ by two such as 3 and 5. Unlike the classical harmonic series and the harmonic series of primes, the harmonic series of twin primes (HSTP) actually converges to a constant first discovered by Viggo Brun in 1919 and is known as Brun's constant 1.9021605831. Let us explore the harmonic series of twin primes with our TI-89 hand-held. In order to achieve this, consider the following functions in **FIGURE 47,** the TABLE SETUP in **FIGURE 48** and the TABLE in **FIGURES 49-79.** We are seeking those outcomes where the entries headed by columns $y1$ *and* $y3$ are both true and seek all twin prime pairs ≤ 500.

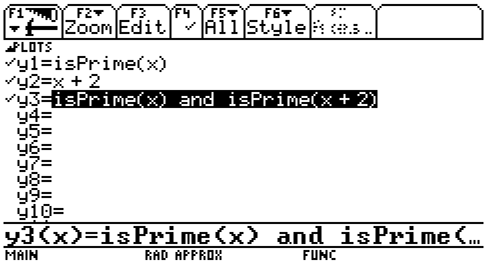


FIGURE 47: The Defining Functions

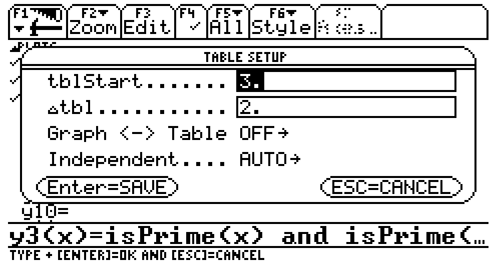
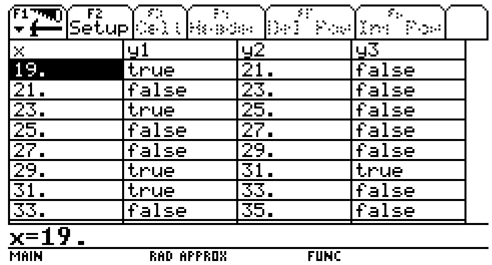


FIGURE 48: The Table Setup

x	y1	y2	y3
3.	true	5.	true
5.	true	7.	true
7.	true	9.	false
9.	false	11.	false
11.	true	13.	true
13.	true	15.	false
15.	false	17.	false
17.	true	19.	true

x=3.

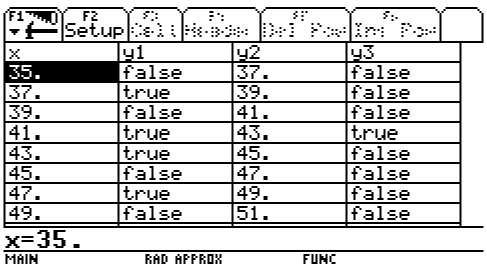
FIGURE 49: The Table Revealed



x	y1	y2	y3
19.	true	21.	false
21.	false	23.	false
23.	true	25.	false
25.	false	27.	false
27.	false	29.	false
29.	true	31.	true
31.	true	33.	false
33.	false	35.	false

x=19.

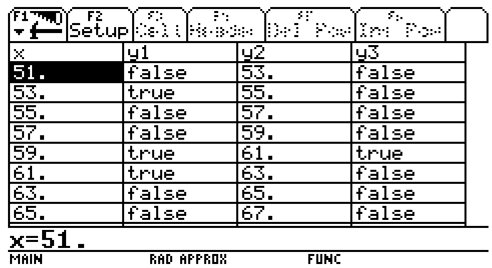
FIGURE 50: The Table Revealed



x	y1	y2	y3
35.	false	37.	false
37.	true	39.	false
39.	false	41.	false
41.	true	43.	true
43.	true	45.	false
45.	false	47.	false
47.	true	49.	false
49.	false	51.	false

x=35.

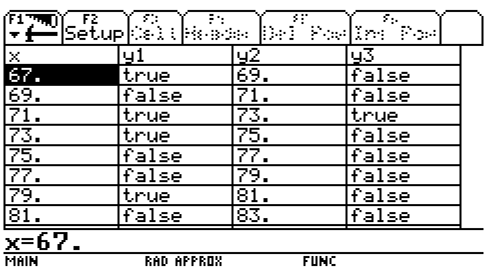
FIGURE 51: The Table Revealed



x	y1	y2	y3
51.	false	53.	false
53.	true	55.	false
55.	false	57.	false
57.	false	59.	false
59.	true	61.	true
61.	true	63.	false
63.	false	65.	false
65.	false	67.	false

x=51.

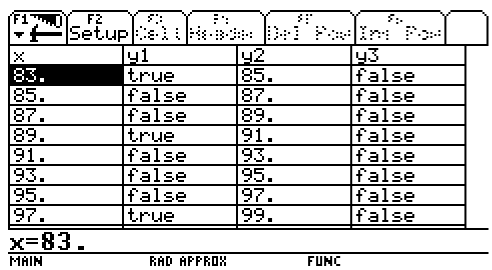
FIGURE 52: The Table Revealed



x	y1	y2	y3
67.	true	69.	false
69.	false	71.	false
71.	true	73.	true
73.	true	75.	false
75.	false	77.	false
77.	false	79.	false
79.	true	81.	false
81.	false	83.	false

x=67.

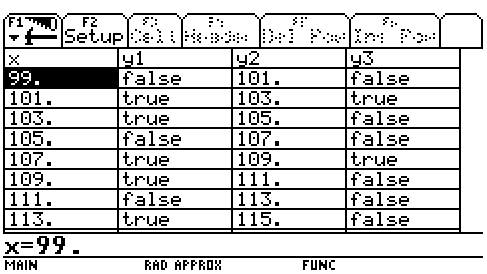
FIGURE 53: The Table Revealed



x	y1	y2	y3
83.	true	85.	false
85.	false	87.	false
87.	false	89.	false
89.	true	91.	false
91.	false	93.	false
93.	false	95.	false
95.	false	97.	false
97.	true	99.	false

x=83.

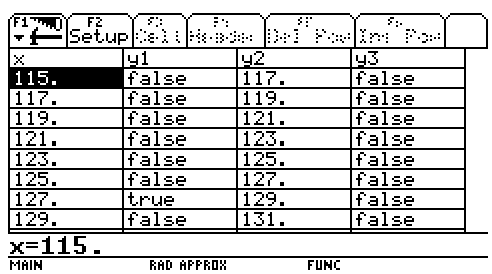
FIGURE 54: The Table Revealed



x	y1	y2	y3
99.	false	101.	false
101.	true	103.	true
103.	true	105.	false
105.	false	107.	false
107.	true	109.	true
109.	true	111.	false
111.	false	113.	false
113.	true	115.	false

x=99.

FIGURE 55: The Table Revealed



x	y1	y2	y3
115.	false	117.	false
117.	false	119.	false
119.	false	121.	false
121.	false	123.	false
123.	false	125.	false
125.	false	127.	false
127.	true	129.	false
129.	false	131.	false

x=115.

FIGURE 56: The Table Revealed

x	y1	y2	y3
131.	true	133.	false
133.	false	135.	false
135.	false	137.	false
137.	true	139.	true
139.	true	141.	false
141.	false	143.	false
143.	false	145.	false
145.	false	147.	false

x=131.

MAIN RAD APPROX FUNC

FIGURE 57: The Table Revealed

x	y1	y2	y3
147.	false	149.	false
149.	true	151.	true
151.	true	153.	false
153.	false	155.	false
155.	false	157.	false
157.	true	159.	false
159.	false	161.	false
161.	false	163.	false

x=147.

MAIN RAD APPROX FUNC

FIGURE 58: The Table Revealed

x	y1	y2	y3
163.	true	165.	false
165.	false	167.	false
167.	true	169.	false
169.	false	171.	false
171.	false	173.	false
173.	true	175.	false
175.	false	177.	false
177.	false	179.	false

x=163.

MAIN RAD APPROX FUNC

FIGURE 59: The Table Revealed

x	y1	y2	y3
179.	true	181.	true
181.	true	183.	false
183.	false	185.	false
185.	false	187.	false
187.	false	189.	false
189.	false	191.	false
191.	true	193.	true
193.	true	195.	false

x=179.

MAIN RAD APPROX FUNC

FIGURE 60: The Table Revealed

x	y1	y2	y3
195.	false	197.	false
197.	true	199.	true
199.	true	201.	false
201.	false	203.	false
203.	false	205.	false
205.	false	207.	false
207.	false	209.	false
209.	false	211.	false

x=195.

MAIN RAD APPROX FUNC

FIGURE 61: The Table Revealed

x	y1	y2	y3
211.	true	213.	false
213.	false	215.	false
215.	false	217.	false
217.	false	219.	false
219.	false	221.	false
221.	false	223.	false
223.	true	225.	false
225.	false	227.	false

x=211.

MAIN RAD APPROX FUNC

FIGURE 62: The Table Revealed

x	y1	y2	y3
227.	true	229.	true
229.	true	231.	false
231.	false	233.	false
233.	true	235.	false
235.	false	237.	false
237.	false	239.	false
239.	true	241.	true
241.	true	243.	false

x=227.

MAIN RAD APPROX FUNC

FIGURE 63: The Table Revealed

x	y1	y2	y3
243.	false	245.	false
245.	false	247.	false
247.	false	249.	false
249.	false	251.	false
251.	true	253.	false
253.	false	255.	false
255.	false	257.	false
257.	true	259.	false

x=243.

MAIN RAD APPROX FUNC

FIGURE 64: The Table Revealed

x	y1	y2	y3
259.	false	261.	false
261.	false	263.	false
263.	true	265.	false
265.	false	267.	false
267.	false	269.	false
269.	true	271.	true
271.	true	273.	false
273.	false	275.	false

x=259.

MAIN RAD APPROX FUNC

FIGURE 65: The Table Revealed

x	y1	y2	y3
275.	false	277.	false
277.	true	279.	false
279.	false	281.	false
281.	true	283.	true
283.	true	285.	false
285.	false	287.	false
287.	false	289.	false
289.	false	291.	false

x=275.

MAIN RAD APPROX FUNC

FIGURE 66: The Table Revealed

x	y1	y2	y3
291.	false	293.	false
293.	true	295.	false
295.	false	297.	false
297.	false	299.	false
299.	false	301.	false
301.	false	303.	false
303.	false	305.	false
305.	false	307.	false

x=291.

MAIN RAD APPROX FUNC

FIGURE 67: The Table Revealed

x	y1	y2	y3
307.	true	309.	false
309.	false	311.	false
311.	true	313.	true
313.	true	315.	false
315.	false	317.	false
317.	true	319.	false
319.	false	321.	false
321.	false	323.	false

x=307.

MAIN RAD APPROX FUNC

FIGURE 68: The Table Revealed

x	y1	y2	y3
323.	false	325.	false
325.	false	327.	false
327.	false	329.	false
329.	false	331.	false
331.	true	333.	false
333.	false	335.	false
335.	false	337.	false
337.	true	339.	false

x=323.

MAIN RAD APPROX FUNC

FIGURE 69: The Table Revealed

x	y1	y2	y3
339.	false	341.	false
341.	false	343.	false
343.	false	345.	false
345.	false	347.	false
347.	true	349.	true
349.	true	351.	false
351.	false	353.	false
353.	true	355.	false

x=339.

MAIN RAD APPROX FUNC

FIGURE 70: The Table Revealed

x	y1	y2	y3
355.	false	357.	false
357.	false	359.	false
359.	true	361.	false
361.	false	363.	false
363.	false	365.	false
365.	false	367.	false
367.	true	369.	false
369.	false	371.	false

x=355.

MAIN RAD APPROX FUNC

FIGURE 71: The Table Revealed

x	y1	y2	y3
371.	false	373.	false
373.	true	375.	false
375.	false	377.	false
377.	false	379.	false
379.	true	381.	false
381.	false	383.	false
383.	true	385.	false
385.	false	387.	false

x=371.

MAIN RAD APPROX FUNC

FIGURE 72: The Table Revealed

x	y1	y2	y3
387.	false	389.	false
389.	true	391.	false
391.	false	393.	false
393.	false	395.	false
395.	false	397.	false
397.	true	399.	false
399.	false	401.	false
401.	true	403.	false

x=387.

MAIN RAD APPROX FUNC

FIGURE 73: The Table Revealed

x	y1	y2	y3
403.	false	405.	false
405.	false	407.	false
407.	false	409.	false
409.	true	411.	false
411.	false	413.	false
413.	false	415.	false
415.	false	417.	false
417.	false	419.	false

x=403.

MAIN RAD APPROX FUNC

FIGURE 74: The Table Revealed

x	y1	y2	y3
419.	true	421.	true
421.	true	423.	false
423.	false	425.	false
425.	false	427.	false
427.	false	429.	false
429.	false	431.	false
431.	true	433.	true
433.	true	435.	false

x=419.

MAIN RAD APPROX FUNC

FIGURE 75: The Table Revealed

x	y1	y2	y3
435.	false	437.	false
437.	false	439.	false
439.	true	441.	false
441.	false	443.	false
443.	true	445.	false
445.	false	447.	false
447.	false	449.	false
449.	true	451.	false

x=435.

MAIN RAD APPROX FUNC

FIGURE 76: The Table Revealed

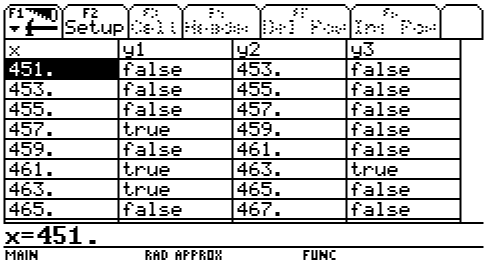


FIGURE 77: The Table Revealed

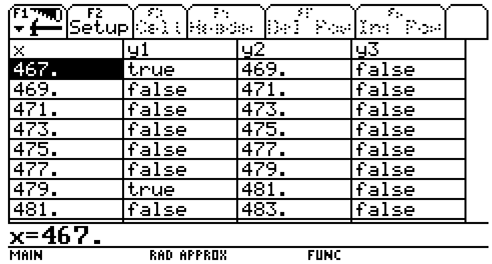


FIGURE 78: The Table Revealed

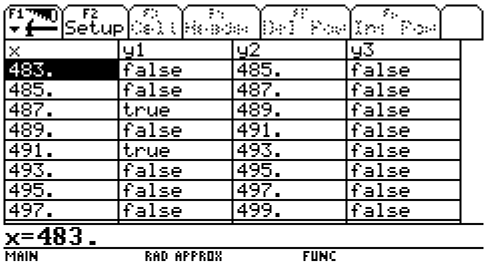


FIGURE 79: The Table Revealed

One has a total of two dozen twin prime pairs; namely

$$\left\{\begin{array}{l}(3,5),(5,7),(11,13),(17,19),(29,31),(41,43),(59,61),(71,73),\\ (101,103),(107,109),(137,139),(149,151),(179,181),(191,193),(197,199),\\ (227,229),(239,241),(269,271),(281,283),(311,313),(347,349),(419,421),\\ (431,433),(461,463)\end{array}\right\}.$$

We now set the hand-held in **APPROXIMATE MODE in FIGURE 80** and view the calculations in **FIGURES 81-88:**

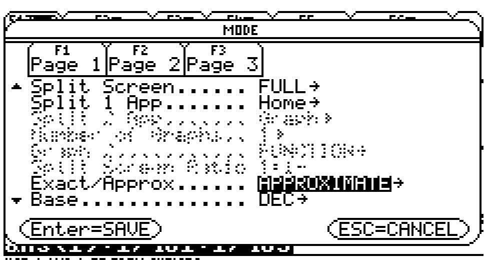


FIGURE 80: APPROXIMATE MODE

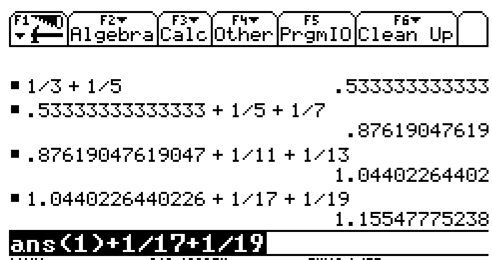


FIGURE 81: The HSTP

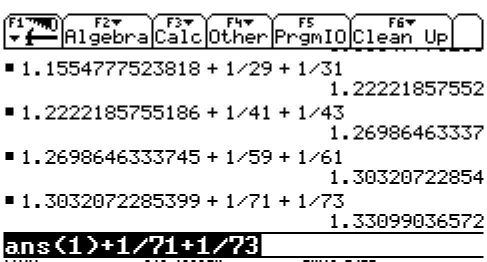


FIGURE 82: The HSTP

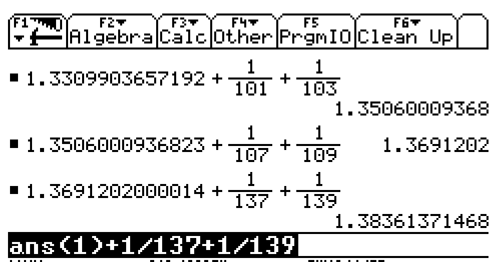


FIGURE 83: The HSTP

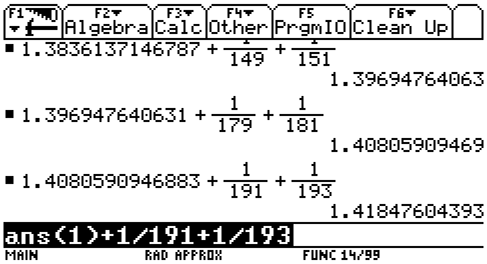


FIGURE 84: The HSTP

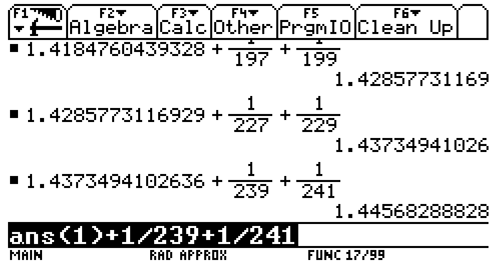


FIGURE 85: The HSTP

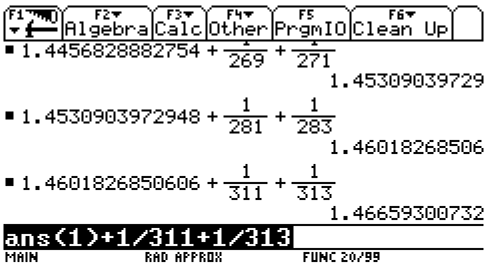


FIGURE 86: The HSTP

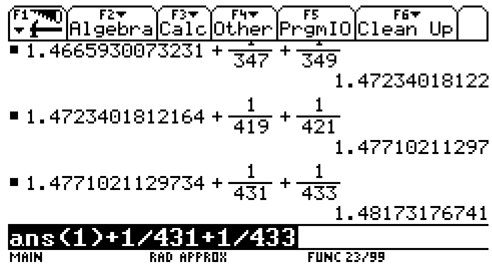


FIGURE 87: The HSTP

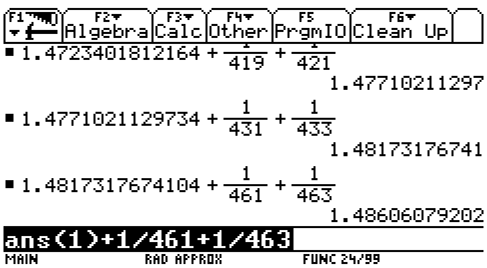


FIGURE 88: The HSTP

This is only a small sample and one must go much further into the harmonic series of twin primes to see anything close to the sum of the terms in the series approaching the value of Brun's constant. The more powerful MATHEMATICA package enables one to achieve our desired goal more readily. With the aid of MATHEMATICA, I arrived at the following for the sums in the harmonic series of twin primes:

Range:	Sum:
$[1,10]$	0.87619
$[1,100]$	1.33099
$[1,1000]$	1.51803
$[1,10000]$	1.61689

$[1,100000]$	1.6728
$[1,1000000]$	1.71078
$[1,10000000]$	1.73836
$[1,100000000]$	1.75882

Our next goal is to explore additional harmonic series including the harmonic series of selected figurative numbers such as the harmonic series of triangular numbers and the harmonic series of tetrahedral numbers. We note that the harmonic series of square numbers which converges were previously considered in **FIGURES 42-46** above.

Recall that the closed for the triangular numbers is $\frac{n\cdot(n+1)}{2}=1+2+3+...+k=\sum_{k=1}^{n}k.$ The TI-89 graphing calculator hand-held enables one to secure this sum as illustrated in **FIGURE 89:**

Four arguments are required; namely the formula, the independent variable, the lower limit of summation and the upper limit of summation:

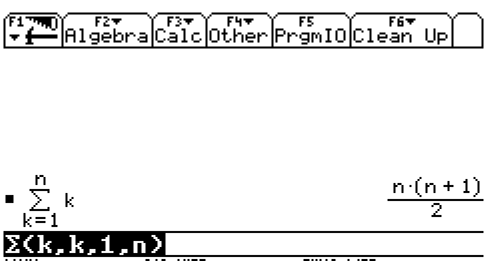


FIGURE 89: The Closed Formula for the Triangular Numbers.

This formula is readily established using The Principle of Mathematical Induction as well as by Gauss' Formula among other proofs.

The sum of the harmonic series of triangular numbers $\sum_{N=1}^{+\infty}\frac{1}{\frac{n\cdot(n+1)}{2}}=\sum_{n=1}^{+\infty}\frac{2}{n\cdot(n+1)}=2$ can readily be demonstrated using partial fractions to obtain a telescoping series. We appeal to **FIGURES 90-91:**

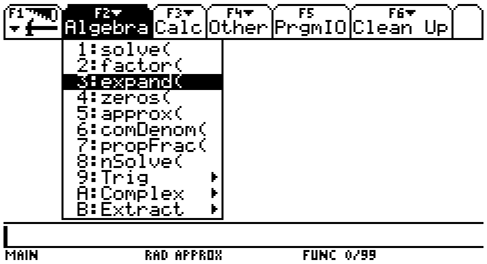


FIGURE 90: The Expand Command.

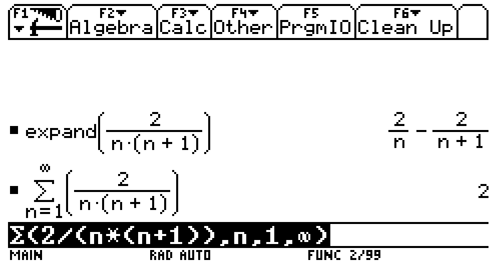


FIGURE 91: The Decomposition and Sum.

The expand command from the algebra menu enables one to decompose into a telescoping series, one in which all of the terms with the exception of the first and last cancel in pairs so that we have collapsing behavior akin to a folding telescope which accounts for the name telescoping series. We observe the following:

$$\sum_{n=1}^{+\infty}\frac{2}{n\cdot(n+1)}=\sum_{n=1}^{+\infty}\left[\frac{2}{n}-\frac{2}{n+1}\right]=\left[\frac{2}{1}-\frac{2}{2}\right]+\left[\frac{2}{2}-\frac{2}{3}\right]+\left[\frac{2}{3}-\frac{2}{4}\right]+\left[\frac{2}{4}-\frac{2}{5}\right]+\ldots+\left[\frac{2}{n}-\frac{2}{n+1}\right]+\ldots=2-\frac{2}{n+1}.$$

Now $\lim_{n\to+\infty}\left[2-\frac{2}{n+1}\right]=2-0=2.$

The set of tetrahedral numbers 1, 4, 10, 20, 35, 56, 84, … form the outline of a tetrahedron and has the closed form $TH(n)=\frac{n\cdot(n+1)\cdot(n+2)}{6}$. Hence the reciprocals of the tetrahedral numbers has the closed form $\frac{6}{n\cdot(n+1)\cdot(n+2)}$. We thus seek $\sum_{n=1}^{+\infty}\frac{6}{n\cdot(n+1)\cdot(n+2)}$. A resolution into partial fractions is possible; for while $\frac{6}{n\cdot(n+1)\cdot(n+2)}=\frac{3}{n+2}-\frac{6}{n+1}+\frac{3}{n}$, the calculator cannot find this sum of this series which indeed converges using the limit comparison test. To see this, let $u_n=\frac{6}{n\cdot(n+1)\cdot(n+2)}$ *and* $v_n=\frac{1}{n^3}$, the convergent p series with $p=3>1$. Now

$\sum_{n=1}^{+\infty}u_n=\sum_{n=1}^{+\infty}\frac{6}{n\cdot(n+1)\cdot(n+2)}$ *and* $\sum_{n=1}^{+\infty}v_n=\sum_{n=1}^{+\infty}\frac{1}{n^3}$. Since

$\lim_{n\to+\infty}\frac{u_n}{v_n}=\lim_{n\to+\infty}\left[\frac{\frac{6}{n\cdot(n+1)\cdot(n+2)}}{\frac{1}{n^3}}\right]=\lim_{n\to+\infty}\left[\frac{6\cdot n^3}{n\cdot(n+1)\cdot(n+2)}\right]=6\neq 0,$ the series

$\sum_{n=1}^{+\infty}\frac{6}{n\cdot(n+1)\cdot(n+2)}$ converges by the Limit Comparison Test.

While **FIGURE 92** yields the partial fractions decomposition, **FIGURE 93** illustrates the limitations of the calculator to compute the sum of this series:

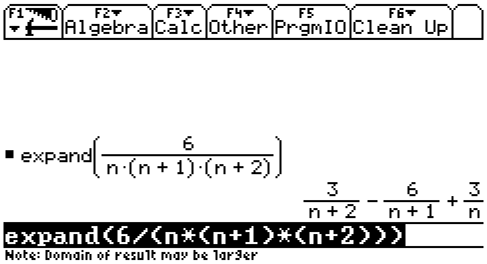


FIGURE 92: The Decomposition

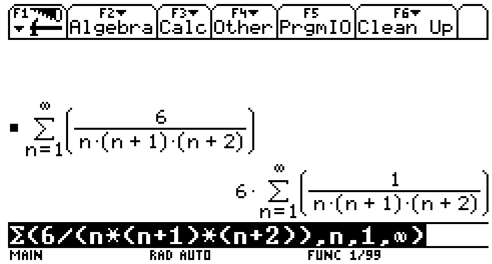


FIGURE 93: Limitations of the Calculator

More powerful technology such as MATHEMAITCA and its subsidiary Wolfram Alpha come to the rescue.

For the sum of the reciprocals of the tetrahedral numbers, one is considering

$\sum_{n=1}^{+\infty}\frac{6}{n\cdot(n+1)\cdot(n+2)}=\frac{3}{2}$. In fact, the sequence of partial sums has the form

$$\sum_{n=1}^{+\infty}\frac{6}{n\cdot(n+1)\cdot(n+2)}=\sum_{n=1}^{+\infty}\left[\frac{3}{n+2}-\frac{6}{n+1}+\frac{3}{n}\right]=\frac{3}{2}.$$

According to MATHEMATICA and Wolfram alpha, $\sum_{n=1}^{m}\frac{6}{n\cdot(n+1)\cdot(n+2)}=\frac{3\cdot\left(m^2+3\cdot m\right)}{2\cdot(m+1)\cdot(m+2)}$.

Hence $\lim_{m\to+\infty}\left[\frac{3\cdot\left(m^2+3\cdot m\right)}{2\cdot(m+1)\cdot(m+2)}\right]=\frac{3}{2}$. We view the initial five partial sums in **FIGURE 94:**

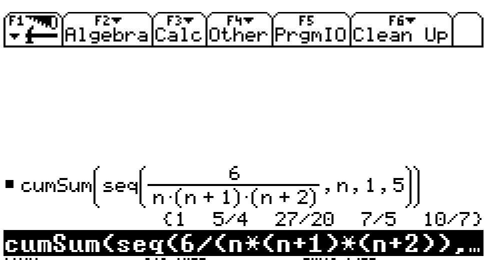


FIGURE 94: The Initial Five Partial Sums

We conclude by investigating the harmonic series of Fibonacci and Lucas numbers. Recall that the Fibonacci and Lucas sequences satisfy the following recursion relations:

For the Fibonacci sequence: $F_1=F_2=1$ *and* $F_n=F_{n-2}+F_{n-1};\ n\geq 3.$

For the Lucas sequence: $L_1=1,\ L_2=3$ *and* $L_n=L_{n-2}+L_{n-1};\ n\geq 3.$

For both the Fibonacci and Lucas Sequences, it is well known that

$$\lim_{n\to+\infty}\frac{F_{n+1}}{F_n}=\lim_{n\to+\infty}\frac{L_{n+1}}{L_n}=\Phi=\frac{1+\sqrt{5}}{2}\approx 1.61803398875...\text{ while}$$

$$\lim_{n\to+\infty}\frac{F_n}{F_{n+1}}=\lim_{n\to+\infty}\frac{L_n}{L_{n+1}}=\frac{1}{\Phi}=\frac{-1+\sqrt{5}}{2}\approx 0.61803398875...$$

Since the latter ratios are less than one in absolute value, both the harmonic series of Fibonacci and Lucas numbers converge absolutely by the Cauchy ratio test. These series converge to what are respectively known as the reciprocal Fibonacci and the reciprocal Lucas constants. We note

$$\sum_{n=1}^{+\infty}\frac{1}{F_k}=3.35988566...\ \textit{and}\ \sum_{n=1}^{+\infty}\frac{1}{L_k}=1.9628517....$$

While a calculator can furnish some empirical evidence, MATHEMATICA and/or Wolfram Alpha are needed to obtain much more powerful results. Let us compute the sum of the reciprocals of the Fibonacci numbers to ten, one hundred, one thousand and ten thousand terms. The results are provided in the following table:

Reciprocal Fibonacci Summation:	Sum:
$\sum_{n=1}^{10}\frac{1}{F_n}=\frac{1}{F_1}+\frac{1}{F_2}+\frac{1}{F_3}+\frac{1}{F_4}+\frac{1}{F_5}+\frac{1}{F_6}+\frac{1}{F_7}+\frac{1}{F_8}+\frac{1}{F_9}+\frac{1}{F_{10}}=$ $\frac{1}{1}+\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{8}+\frac{1}{13}+\frac{1}{21}+\frac{1}{34}+\frac{1}{55}$	3.3304690407631584102
$\sum_{n=1}^{100}\frac{1}{F_n}$	3.3598856662431775531
$\sum_{n=1}^{1000}\frac{1}{F_n}$	3.3598856662431775531
$\sum_{n=1}^{10000}\frac{1}{F_n}$	3.3598856662431775531

In similar fashion, we compute the sum of the reciprocals of the Lucas numbers to ten, one hundred, one thousand and ten thousand terms. The results are furnished in the following table:

Reciprocal Lucas Summation:	Sum:
$\sum_{n=1}^{10}\frac{1}{L_n}=\frac{1}{L_1}+\frac{1}{L_2}+\frac{1}{L_3}+\frac{1}{L_4}+\frac{1}{L_5}+\frac{1}{L_6}+\frac{1}{L_7}+\frac{1}{L_8}+\frac{1}{L_9}+\frac{1}{L_{10}}=$ $\frac{1}{1}+\frac{1}{3}+\frac{1}{4}+\frac{1}{7}+\frac{1}{11}+\frac{1}{18}+\frac{1}{29}+\frac{1}{47}+\frac{1}{76}+\frac{1}{123}$	1.9497024530581482747
$\sum_{n=1}^{100}\frac{1}{L_n}$	1.9628581732096457829
$\sum_{n=1}^{1000}\frac{1}{L_n}$	1.9628581732096457829
$\sum_{n=1}^{10000}\frac{1}{F_n}$	1.9628581732096457829

<u>Conclusion:</u> Additional harmonic series can be explored with figurative numbers including the harmonic series of pentagonal, hexagonal and octagonal numbers all of which converge by the several tests including the limit comparison, integral and polynomial tests. One can explore a world of possibilities leading to stimulating discovery.

References

1. The On-Line Encyclopedia of Integer Sequences, Sequence A002387, The OEIS Foundation, 2015

2. MathWorld - A Wolfram Resource, Wolfram Research, Inc., 2016

3. Texas Instruments - TI-89 Guidebook, Dallas, TX, 2016