SOLVING DIOPHANTINE PROBLEMS BY *EXCEL*

Nadeem Aslam
Mathematics Department
Florida International University
11200 SW 8th Street
Miami, FL 33199
aslamn@fiu.edu

Jay Villanueva
Mathematics Department
Miami Dade College
11011 SW 104th Street
Miami, FL 33176
jvillanu@mdc.edu

Diophantus lived in Alexandria around 250 CE. Of 13 books he wrote on *Arithmetica*, only 6 remain, where he invented a system of notation complete with unknowns that allowed him to address algebra problems. The only source of details about his life is an epigram found in a collection called *Greek Anthology*: "Diophantus passed one sixth of his life in childhood, one twelfth in youth, and one seventh as a bachelor. Five years after his marriage was born a son who died four years before his father, at half his father's age." How long did Diophantus live?

$$x = \frac{x}{6} + \frac{x}{12} + \frac{x}{7} + 5 + \frac{x}{2} + 4 \qquad [x = 84\ years]$$

A Diophantine equation is an indeterminate equation that admits only integer solutions, of the form $f(x) + g(y) = h(z)$, f, g, h are polynomials. The linear case: $ax + by = c$, admits many interesting problems.

There are at least three methods of solving the linear case: by formula, by using congruence equations, and by *Excel*.

(1) By Formula

The equation $ax + by = c$, $\quad a,\ b,\ c,\ x,\ y$ integers, is indeterminate because there are two unknowns in one equation. We will find multiple solutions *x, y* given the constants *a, b, c.*

We know that if we let $\gcd(a, b) = d$, if $d \nmid c$, then there are no solutions; if $d \mid c$, there are unique or multiple solutions.

If $(x_0,\ y_0)$ is a solution:

$$ax_0 + by_0 = c$$
$$\Rightarrow a\left(x_0 + \frac{b}{d}n\right) + b\left(y_0 - \frac{a}{d}n\right) = c$$
$$x = x_0 + \frac{b}{d}n, \qquad y = y_0 - \frac{a}{d}n,$$

These are the solutions for integer *n.*

Example1: A professor returning from Europe changes her euros and pounds into US dollars, receiving a total of $125.78. If the exchange rate is €1 = $1.31 and £1 = $1.61, how many euros and pounds did she exchange?

Method 1. By Formula.

Let $x =$ number of pounds, $y =$ number of euros.

$$1.61x + 1.31y = 125.78 \Rightarrow 161x + 131y = 12578.$$
$$a = 161, b = 131,\ c = 12578.$$
$$d = (a, b) = (161\ ,\ 131) = 1: \qquad d|c\ \therefore\ \exists \text{ solutions!}$$

Using by Euclidean algorithm:

$$as + bt = d; s,\ t \text{ are integers.}$$

$161s + 131t = 1$, to find *s*, *t*:

$161 = 131 \cdot 1 + 30 \qquad 1 = 3 - 2\ .\ 1 = 3 - (8 - 3.2).1 = 3\ .\ 3\ - 8\ .\ 1.$

$131 = 30\ .\ 4 + 11 \qquad = (11 - 8.1).\ 3\ - 8\ .\ 1 = 11\ .\ 3\ - 8\ .\ 4$

$30 = 11\ .\ 2 + 8 \qquad = 11\ .\ 3\ - (30 - 11\ .\ 2).\ 4 = 11\ .\ 11\ - 30\ .\ 4$

$11 = 8\ .\ 1 + 3 \qquad = (131 - 30\ .\ 4).\ 11\ - 30\ .\ 4 = 131\ .\ 11\ - 30\ .\ 48.$

$8 = 3\ .\ 2 + 2 \qquad = 131\ .\ 11\ -\ (161 - 131\ .\ 1).\ 48$

$3 = 2\ .\ 1 + 1 \qquad = 131\ .\ 59\ - 161\ .\ 48 = 161(-48) + 131(59)$

$2 = 1\ .\ 2 + 0 \qquad \Rightarrow s = -48\ \&\ t = 59$

$$e(as) + e(bt) = ed = c$$
$$e(1) = 12578 \implies e = 12578$$
$$x_0 = es = 12578(-48) = -603744$$
$$y_0 = et = 12578(59) = 742102$$
$$x = x_0 + \frac{b}{d}n, \qquad y = y_0 - \frac{a}{d}n$$
$$= -603744 + \frac{131}{1}n, \qquad = 742102 - \frac{161}{1}n$$
$$x, y \geq 0 \implies 4608.73 \leq n \leq 4609.33$$
$$\therefore n = 4609$$
$$x = -603744 + 131(4609) = 35 \text{ pounds}$$
$$y = 742102 - 161(4609) = 53 \text{ euros}$$

Here, since *n* has only one value, the solution is unique.

Method 2. By Congruence Equations.
The linear equation $ax + by = c$ can be written as: $ax \equiv c \bmod b$, and we are solving for *x* an integer.
(a) $161x + 131y = 12578$
$$161x \equiv 12578 \, mod(131)$$
$$30x \equiv 2 \, mod(131)$$
We can try $x = 0,1,2, \ldots ,130$:
$$\therefore \; x = 35, \; y = 53,$$
the solution is unique.

The linear equation above can also be written as: $by \equiv c \bmod a$.
Or (b): $131y \equiv 12578 \, mod(161)$
$$131y \equiv 20 \, mod(161)$$
We can try y= $0, \; 1, \ldots ,160$:
$$\therefore y = 53, \; x = 35.$$

Method 3. By *Excel*
First rewrite the equation in integer form: $131x + 161y = 12578$.
Constraint1= maximum integer value of *x* = $12578/161 = 78 \, R \, 20$;
Constraint2= maximum integer value of *y* = $12578/131 = 96 \, R \, 2$;
Constraint3= minimum integer value of *x* = 0;
Constraint4= minimum integer value of *y* = 0.
We will use the *Solver* function of *Excel* where we enter in the objective cell the RHS of the linear equation. Extra constraints should be integer or decimal values. When we run *Solver*, we get the answers *x* = 35 & *y* = 53 instantaneously as follows:

Table 1. Professor Problem.

Currency Exchange

$1.61x+1.31y = 125.78$	
# Pounds $x =$	35
# Euros $y =$	53
Objective	125.78
Constraints	
Cons1	78
Cons2	96
Cons3	0
Cons4	0

Example 2: Dinner Problem

A group dinner costs \$96.00. If a lobster order costs \$11.00 and a chicken order costs \$8.00, how many ordered lobsters and chicken?

Let x # lobsters, y # $chickens.$ $11x + 8y = 96,\ a = 11,\ b = 8,\ c = 96.$

Method 1. By Formula

$d = (a, b) = (11,8) = 1$: $d|c$ $\therefore$ $\exists$ solutions!

$as + bt = d; s,\ t$ are integers.

$11s + 8t = 1$, to find s, t:

$11 = 8 \cdot 1 + 3$ $\quad 1 = 3 - 2 \cdot 1 = 3 - (8 - 3 \cdot 2) \cdot 1$

$8 = 3 \cdot 2 + 2$ $\quad = 3 \cdot 3 - 8 \cdot 1 = (11 - 8 \cdot 1) \cdot 3 - 8 \cdot 1$

$3 = 2 \cdot 1 + 1$ $\quad = 11 \cdot 3 - 8 \cdot 4$

$2 = 1 \cdot 2 + 0$ $\quad = 11(3) + 8(-4) \Longrightarrow s = 3,\ t = -4.$

$e(as) + e(bt) = ed = c$

$e(1) = 96 \Longrightarrow e = 96$

$x_0 = es = 96(3) = 288$

$y_0 = et = 96(-4) = -384$

$x = x_0 + \frac{b}{d}n,$ $\quad y = y_0 - \frac{a}{d}n$

$= 288 + \frac{8}{1}n,$ $\quad = -384 - \frac{11}{1}n$

$x, y \geq 0 \Longrightarrow -36 \leq n \leq -34 \therefore n = -36\ or\ -35$

$\therefore (i)\ x = 288 + 8(-36)$ $= 0$ lobster order

$y = -384 - 11(-36)$ $= 12$ chicken orders;

$\therefore (ii)\ x = 288 + 8(-35)$ $= 8$ lobster orders

$y = -384 - 11(-35) \quad = 1$ chicken order.

Method 2. By Congruence Equations
$(a)\ 11x + 8y = 96$

$$11x \equiv 96\, mod(8)$$
$$3x \equiv 0\, mod(8)$$

We can try $x = 0,\ 1, \dots, 7$:
$\therefore\ (i)\ x = 0,\ y = 12 \quad \& \quad$ (ii) $x = 8,\ y = 1.$

Or (b): $8y \equiv 96\, mod(11)$

$$8y \equiv 8\, mod(11)$$

We can try y$=\ 0,\ 1, \dots, 10$:
$\therefore\ (i)\ y = 1,\ \ x = 8 \quad \& \quad (ii)\ x = 0,\ y = 12.$

Method 3. By *Excel*
Here, the difference from the first example is that we have multiple solutions. When we enter the constraints, when the *x* constrained value is maximum, then we keep the corresponding *y* constrained value minimum, or vice versa. When we run *Solver*, both solutions of the problem show up.

Table 2. Dinner Problem

$11x + 8y = 96$			
Sol1:		Sol2:	
# Lobsters x =	0	# Lobsters	8
# Chickens y =	12	# Chickens	1
Objective	96	Objective	96
Cons1	0	Cons1	8
Cons2	12	Cons2	1
Cons3	0	Cons3	0.1
Cons4	0	Cons4	0

Example 3. A farmer has chickens and pigs such that the number of legs and heads total 70. How many chickens and pigs does the farmer have?
Let *x* = # pigs, *y* = # chickens. Our linear equation is:
$5x + 3y = 70.$

Method 1. By Formula:

$$5x + 3y = 70, \qquad a = 5,\ b = 3,\ c = 70$$

$d = (a, b) = (5,3) = 1$: $d|c$ $\therefore$ $\exists$ solutions!

$as + bt = d; s,\ t$ are integers.

$5s + 3t = 1$, to find s, t:

$5 = 3 \cdot 1 + 2$ $\quad 1 = 3 - 2 \cdot 1 = 3 - (5 - 3.1).1$

$3 = 2.1 + 1$ $\quad = 3.2 - 5.1 = 5(-1) + 3(2)$

$2 = 1.2 + 0$ $\quad \Longrightarrow s = -1$ & $t = 2$.

$e(as) + e(bt) = ed = c$

$e(1) = 70 \ \Longrightarrow e = 70$

$x_0 = es = 70(-1) = -70$

$y_0 = et = 70(2) \quad = 140$

$x = x_0 + \frac{b}{d}n, \qquad y = y_0 - \frac{a}{d}n$

$= -70 + \frac{3}{1}n, \qquad = 140 - \frac{5}{1}n$

$x, y \geq 0 \ \Longrightarrow 23.33 \leq n \leq 28$

$\therefore n = 24,\ 25,\ 26,\ 27,$ or $28.$

There are five solutions.

$\therefore x = -70 + 3(24) = 2$ pigs

$y = 140 - 5(24) = 20$ chickens;

$\therefore x = -70 + 3(25) = 5$ pigs

$y = 140 - 5(25) = 15$ chickens;

$\therefore x = -70 + 3(26) = 8$ pigs

$y = 140 - 5(26) = 10$ chickens;

$\therefore x = -70 + 3(27) = 11$ pigs

$y = 140 - 5(27) = 5$ chickens;

$\therefore x = -70 + 3(28) = 14$ pigs

$y = 140 - 5(28) = 0$ chickens.

Table3. Summary of Cases: Chickens and Pigs.

n	***x***	***y***	***N***
24	2	20	70
25	5	15	70
26	8	10	70
27	11	5	70
28	14	0	70

Method 2. By Congruence Equations:

Writing the equation as a congruence relation, we have: (a) $5x \equiv 70 \ mod(3)$

$2x \equiv 1 \ mod(3)$

$\therefore x = 2, y = 20;\ x = 5,\ y = 15;\ x = 8,\ y = 10;\ x = 11,\ y = 5;\ x = 14,\ y = 0.$

Or, (b) $3y \equiv 70\ mod(5)$,

$3y \equiv 0\ mod(5)$

$\therefore y = 0,\ x = 14;\ y = 5,\ x = 11;\ y = 10,\ x = 8;\ y = 15,\ x = 5;\ y = 20,\ x = 2.$
Which are the same solutions as case (a).

Method 3. By *Excel*:
First rewrite the equation in integer form: $5x + 3y = 70$.
Constraint1= maximum integer value of $x = 70/5 = 14\ R\ 0$;
Constraint2= maximum integer value of $y = 70/3 = 23\ R\ 1$;
Constarint3= minimum integer value of $x = 0$;
Constarint4= minimum integer value of $y = 0$.

We will use the *Solver* function of *Excel* where we enter in the objective cell the RHS of the linear equation. Extra constraints should be integer or decimal values. When we run Solver, we get the answers $x = 35$, $y = 53$ instantaneously as follows:

Table 4. Chickens and Pigs Problem by *Excel*

5*x*+3y=70									
# pigs *x* =	2	*x* =	5	*x* =	8	*x* =	11	*x* =	14
# Chickens *y* =	20	*y* =	15	*y* =	10	*y* =	5	*y* =	0
Objective	70	Obj	70	Obj	70	Obj	70	Obj	70
Cons1	2	Cons5	5	Cons9	8	Cons13	11	Cons17	14
Cons2	23	Cons6	18	Cons10	13	Cons14	8	Cons18	3
Cons3	0	Cons7	2.1	Cons11	5.1	Cons15	8.1	Cons19	11.1
Cons4	0	Cons8	0	Cons12	0	Cons16	0	Cons20	0

Notice that the results come instantaneously.

Example 4: Stamps Problem
We wish to mail a package; total cost is 83¢. Only 6¢ and 15¢ stamps are available. What combinations of stamps can be used to mail the package?

Method 1. By Formula
Let x = number of 15¢ stamps and y = number of 6¢ stamps.
$15x + 6y = 83$, $a = 15,\ b = 6,\ c = 83$
$d = (a, b) = (15,6) = 3$: $d \nmid c\ \therefore \exists$ no solution.
Since 84 is divisible by 3, we have to spend an extra 1¢ to mail the package. In that case,

$$15x + 6y = 84 \implies 5x + 2y = 28; \; a = 5, \; b = 2, \; c = 28.$$
$d = (a, b) = (5,2) = 1$: $\quad d/c \;\therefore\; \exists$ solutions.

Using the Euclidean algorithm, to find *s* and *t:*

$as + bt = d$; $\quad s, \, t$ are integers.

$5s + 2t = 1$. Note that *s* = 1 & *t* = *-2* will work.

$$e(as) + e(bt) = ed = c$$
$$e(1) = 28 \implies e = 28$$
$$x_0 = es = 28(1) = 28$$
$$y_0 = et = 28(-2) = -56$$

$$x = x_0 + \frac{b}{d}n, \qquad y = y_0 - \frac{a}{d}n$$
$$= 28 + \frac{2}{1}n, \qquad = -56 - \frac{5}{1}n$$

$x, y \geq 0 \implies -14 \leq n \leq -11.2$

$\therefore n = -14, \; -13, \; -12$. There are three solutions:

$\therefore (i)\; x = 28 + 2(-14) = 0 \quad 15¢\; stamps.$

$y = -56 - 5(-14) = 14 \quad$ 6¢ stamps.

$(ii)\; x = 28 + 2(-13) = 2 \quad$ 15¢ stamps.

$y = -56 - 5(-13) = 9 \quad$ 6¢ stamps.

$(iii)\; x = 28 + 2(-12) = 4 \quad$ 15¢ stamps.

$y = -56 - 5(-12) = 4 \quad$ 6¢ stamps.

Method 2. By Congruence Equations

$$ax + by = c$$

(a) $5x + 2y = 28$

$$5x \equiv 28\; mod(2)$$
$$x \equiv 0\; mod(2)$$
$$\therefore x = 0, \; y = 14; x = 2, \; y = 9; x = 4, \; y = 4;$$

(b)

$$2y \equiv 28\; mod(5)$$
$$2y \equiv 3\; mod(5)$$
$$\therefore y = 4\; x = 4; y = 9, \; x = 2; y = 14, \; x = 0.$$

Method 3: By *Excel*

Table 5. Solutions by *Excel*

$15x + 6y = 84$					
$x =$	0	$x =$	2	$x =$	4
$y =$	14	$y =$	9	$y =$	4
Objective	84	Obj	84	Obj	84
Cons1	1	Cons5	3	Cons9	5
Cons2	14	Cons6	9	Cons10	4
Cons3	0	Cons7	1.1	Cons11	3.1
Cons4	0	Cons8	0	Cons12	0

Example 5. Museum-Tickets Problem
Tickets to the museum are \$2.25 for adults and \$1.00 for children, with a total of 60 seats. One day the total amount collected was \$117.25. How many adults and children attended?

In a regular algebra problem, we usually fill the total number of seats. If we use the elimination method, solving a 2 x 2 system of equations:

$$x + y = 60 \quad (1)$$
$$2.25x + 1.00y = 117.25 \quad (2)$$
$$\therefore \quad x = 45.8 \text{ adults}$$
$$y = 14.2 \text{ children.}$$

Since we get fractional number of people, this means that not all 60 seats were taken.

Method1. By Formula
Let x = number of adults, y = number of children.
$2.25x + 1.00y = 117.25 \implies 225x + 100y = 11725 \implies 9x + 4y = 469$,
$a = 9, b = 4, \ c = 469$.

$d = (a, b) = (9,4) = 1 \qquad d|c \ \therefore \ \exists$ solutions.
$as + bt = d; s, \ t$ are integers.
$9s + 4t = 1$, to find s, t
$9 = 4 \cdot 2 + 1 \qquad 1 = 9 - 4 \cdot 2 = 9(1) + 4(-2)$
$4 = 1 \, . \, 4 + 0 \qquad \implies s = 1 \;\&\; t = -2$
$e(as) + e(bt) = ed = c$

$$e(1) = 469 \implies e = 469$$

$$x_0 = es = 469(1) = 469$$
$$y_0 = et = 469(-2) = -938$$
$$x = x_0 + \frac{b}{d}n, \quad y = y_0 - \frac{a}{d}n$$
$$= 469 + \frac{4}{1}n, \quad = -938 - \frac{9}{1}n$$

$x, y \geq 0 \implies -117.25 \leq n \leq -104.22$

∴ There are 13 possible solutions:

$n = -117, \ -116, \ -115, \ \ldots, \ -105.$

Table 6. Summary of Solutions

n	$x = 469 + 4n$	$y = -938 - 9n$	$ amount	# Seats
-117	1	115	$117.25	116
-116	5	106	$117.25	111
-115	9	97	$117.25	106
-114	13	88	$117.25	101
-113	17	79	$117.25	96
-112	21	70	$117.25	91
-111	25	61	$117.25	86
-110	29	52	$117.25	81
-109	33	43	$117.25	76
-108	37	34	$117.25	71
-107	41	25	$117.25	66
-106	45	16	$117.25	61
-105	49	7	$117.25	56*

Looking at the answers in the last column, there is only one actual solution out of the possible 13 solutions. So only 56 seats were filled, confirming our statement in the algebraic solution.

Method 2. By Congruence Equations

$$ax + by = c$$
$$(a)\ 9x + 4y = 469.$$

Or in congruence form:
$$9x \equiv 469 \ mod(4)$$
$$x \equiv 1 \ mod(4)$$

∴ $x = 1, \ y = 115.$

The other solutions are:

Table 7. Other Solutions

x	y		x	y
5	106		29	52
9	97		33	43
13	88		37	34
17	79		41	25
21	70		45	16
25	61		49	7

Again, in these solutions only the last entry satisfies the condition that $x + y \leq 60$.

Or, (b) $(a)\ 9x + 4y = 469.$

In congruence form: $4y \equiv 469\ mod(9)$

$$4y \equiv 1\ mod(9)$$

$\therefore\ y = 7, x = 49.$

So we get the same solutions as in part (a).

Method 3: By *Excel*

Here, our spreadsheet appears as follows.

Table 8. Solutions By *Excel*

x	49
y	7
	469
Cons1	52
Cons2	117
Cons3	0
Cons4	0
Cons5	60
Cons6	56

Notice that there is an extra constraint, namely, that the maximum number seats is 60. After calling *Solver*, our solution is $x = 49$, $y = 7$, and 56 seats are occupied. The last line of our solution is the total number of seats occupied.

Conclusions

A linear Diophantine equation is often encountered in our teaching and research. It can be solved in at least three ways: by use of a formula, by using congruence equations, and by *Excel* in a computer. Our examples showed that the first two ways are easy to apply and they give us both the unique solutions or multiple solutions depending on the case of the problem. We also found that the solution by computer is just as easy through *Solver* in *Excel.* In fact, *Excel* gives us instantaneous results.

- A Diophantine equation is of the form: $ax + by = c$, where *a, b* and *c* are integer constants and the solutions *x* and *y* are also integers.
- A first method of solution is by formula, *n* = an integer, with

$$x = x_0 + \frac{b}{d}n, \qquad y = y_0 - \frac{a}{d}n.$$

- A second method is by use of congruences: $ax + by = c$;

equivalent expressions are: $ax \equiv b \, mod(c)$ and $by \equiv a \, mod(c)$.

- A final method is by *Excel,* where we use the tool *Solver. Solver* works whether there are unique or multiple solutions.
- The advantage of this last method is that it works very fast.

References

1. J Agnew, 1972. *Explorations in Number Theory*. CA: Wadsworth Publishing Co..

2. RG Archibald, 1970. *An Introduction to the Theory of Numbers*. OH: Merrill Publishing Co.

3. EP Armindariz & SJ McAdam, 1980. *Elementary Number Theory*. NY: Macmillan Publishing Co.

4. IA Barnett, 1972. *Elements of Number Theory*. MA: Prindle, Weber & Schmidth, Inc.

5. DM Bourg, 2006. *Excel Scientific and Engineering Cookbook.* CA: O'Reilly Media Inc.

6. KH Rosen, 2011. *Elementary Number Theory*. MA: Addison Wesley Publishing Co.

7. FW Stevenson 2000. *Exploring the Real Numbers*. NJ: Prentice-Hall, Inc.