LAGRANGE MULTIPLIERS AND THE CALCULUS OF VARIATION IN GAME DESIGN

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Abstract:

In this paper we will consider a generalization of the brachistochrone problem using Lagrange multipliers. Our goal is to design a level in a video game in which the player moves an avatar from one point on a computer screen to another point on the screen in a minimal amount of time. The speed at which the avatar can move depends on where on the screen the avatar is.

Key words: Euler-Lagrange equation, Beltrami equation, Lagrange multipliers

1 Introduction

The simplest version of this problem is to break the screen into a finite number, n, of rectangles each of which stretch horizontally across the screen. The height of the rectangles l_1, l_2 …l_n are known as are the corresponding speeds s_1, s_2 …s_n with which the avatar can move through each rectangle. The player must create a path from a point A on the screen to a point B through the rectangles which minimizes the time of transversal. The problem reduces to a constrained minimization problem which can be solved by the method of Lagrange multipliers [1]. The same technique used can also be used to approximate the solution of the brachistochrone problem from the calculus of variations [2].

2 Creating the Fastest Path Between Two Points

In Figure 1 below we illustrate our problem. For sake of example we will look at a computer screen defined by [0, 1000] x [0, 1000]. In this example the domain is broken into n=10 rectangles: [0, 100]x[0,1000]…[900, 1000]x[0,1000]. On the i^{th} rectangle the avatar can move at a speed of s_i =10*i pixels/time unit.

The objective is to create a path of shortest time from a given point A=(0,0) to another point B=(XE, YE) ((1000,1000) in this example). Note: As is customary with this type of problem, positive y will be denoted in a downwards direction. Here, on the rectangle y=0 to y=100 x=0..1000 an avatar can travel at s=10 pixels/time unit. On y=100 to y=200 x=0..1000 the avatar can travel at s=20 pixels/time unit and so on to on the rectangle y=900 to y=1000 x=0..1000 the avatar can travel at a speed of 100 pixels/time unit.

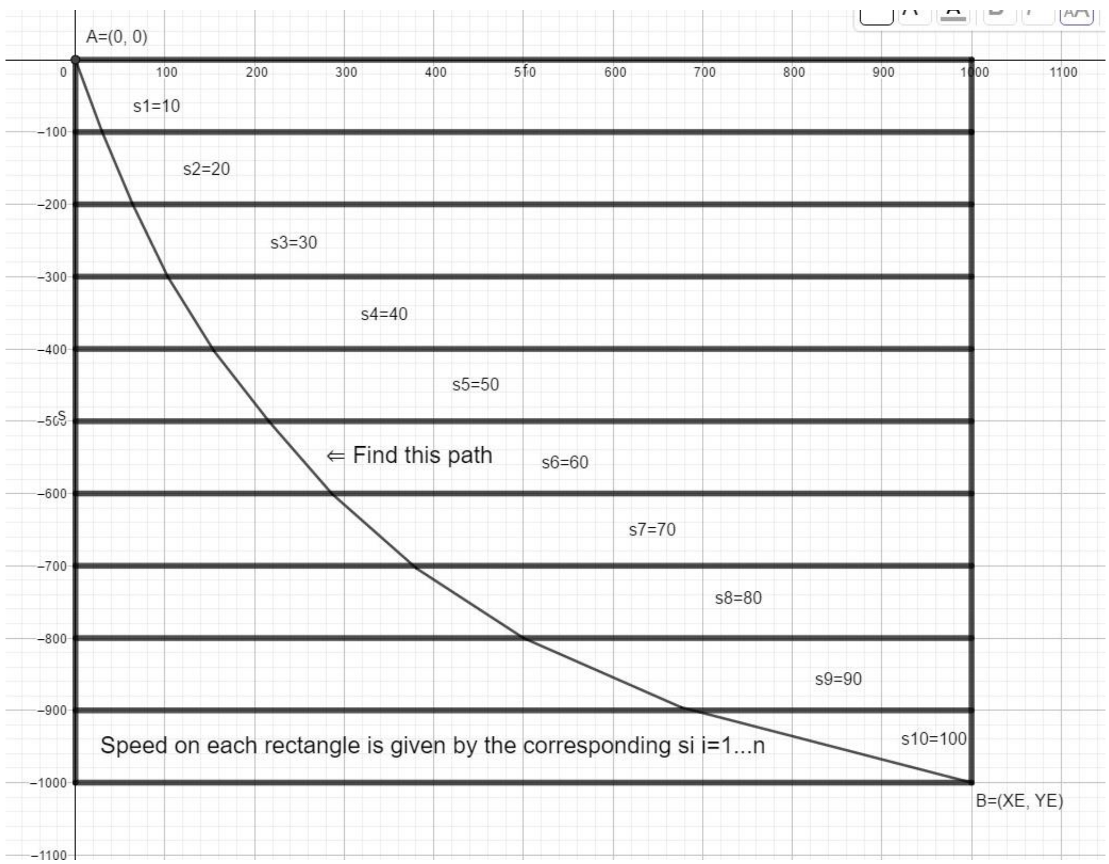

Figure 1-Finding the Fastest Path

We shall approximate our optimal path by breaking the optimal curve into n line segments-each of which is the hypotenuse of a right triangle-see Figure 2 below. As we know the length of the vertical sides, the li, we only need to find the lengths of the horizontal components, the di, which minimizes the sum of the times is takes to traverse the line segments. As we are dealing with a minimization problem subject to the constraint that the sum of the horizontal displacements must equal XE (1000 in our example), we use the classic method of Lagrange multipliers.

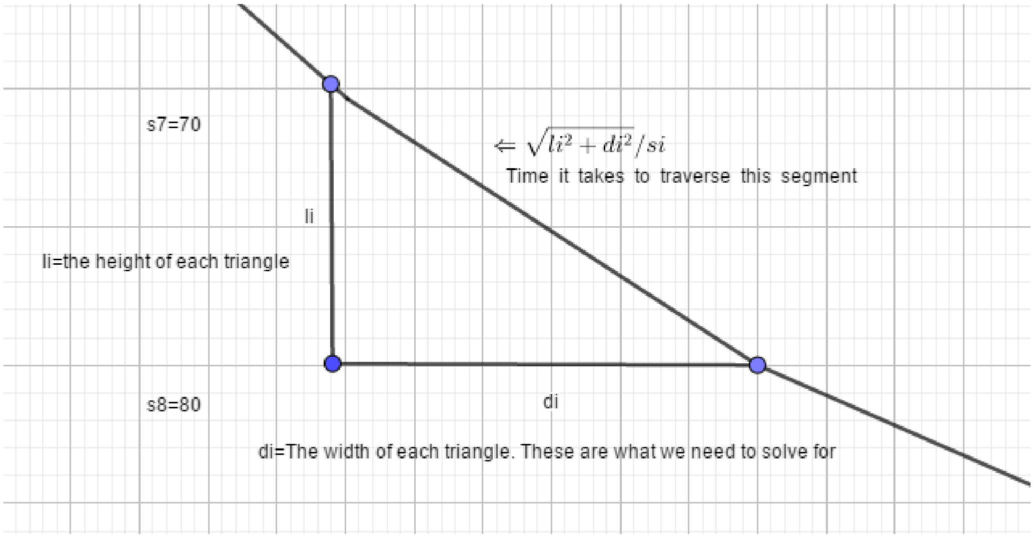

Figure 2-What we Need to Minimize

The function we need to minimize is the sum of the times it takes to transverse each of the n rectangles-which is given by:

$$f(d1, d2, \dots, dn) = \sum_{i=1}^{n} \frac{\sqrt{di^2 + li^2}}{si} \tag{2.1}$$

subject to the constraint:

$$g(d1, d2, \dots, dn) = d1 + d2 + \cdots + dn = X_E \tag{2.2}$$

Hence, we must find a λ for which satisfies [1]

$$\nabla f = \lambda \nabla g = \lambda < 1, 1, \dots, 1 > \tag{2.3}$$

Solving for (2.3) yields

$$\frac{di}{si * \sqrt{li^2 + di^2}} = \lambda \quad i = 1 \dots n \; where \; \lambda > 0 \tag{2.4}$$

Solving (2.4) for the d_i we get

$$di = \frac{\lambda * li}{(\frac{1}{si^2} - \lambda^2)^{.5}} \quad i = 1 \ldots n \tag{2.5}$$

By (2.2) and (2.5) λ must satisfy

$$\sum_{i=1}^{n} \frac{\lambda * li}{(\frac{1}{si^2} - \lambda^2)^{.5}} = X_E \tag{2.6}$$

where $\lambda < \min\left\{\frac{1}{s_i}\right\} i = 1 \ldots n.$

As the left-hand side of (2.6) is an increasing function of λ, (2.6) has a unique solution on (0, min{1/si}). Using any simple numerical tool allows us to approximate the solution λ with any degree of accuracy desired.

Example:

Let the starting point A= (0, 0) and the ending point B= (XE, YE)=(1000, 1000) as illustrated in Figure 1.

Break the region [0, 1000]x[0, 1000] into n=10 rectangles each having height l_i=100 and let the speed on the i^{th} rectangle be given by s_i=10*i i=1…n. Solving (2.6) for λ we can solve for d_i i=1…10 using (2.5):

The values of d_i are presented in the following table:

	Where $xi = d1 + \cdots + di$
d1 is 9.717592076941	x1 is 9.717592076941
d2 is 19.716469280426	x2 is 29.434061357368
d3 is 30.320551487255	x3 is 59.754612844623
d4 is 41.955199508211	x4 is 101.709812352834
d5 is 55.250555011642	x5 is 156.960367364476
d6 is 71.258726625095	x6 is 228.219093989570
d7 is 91.996416566653	x7 is 320.215510556223
d8 is 122.145576401652	x8 is 442.361086957875
d9 is 176.855999289317	x9 is 619.217086247193
d10 is 380.782915477145	x10 is 1000.000001724338

Total time is 35.777931135649
Total distance is 1542.980226242951

The path is illustrated as follows (where the x-coordinate of A is 0, the x-coordinate of B is x1, the x-coordinate of C is x2…):

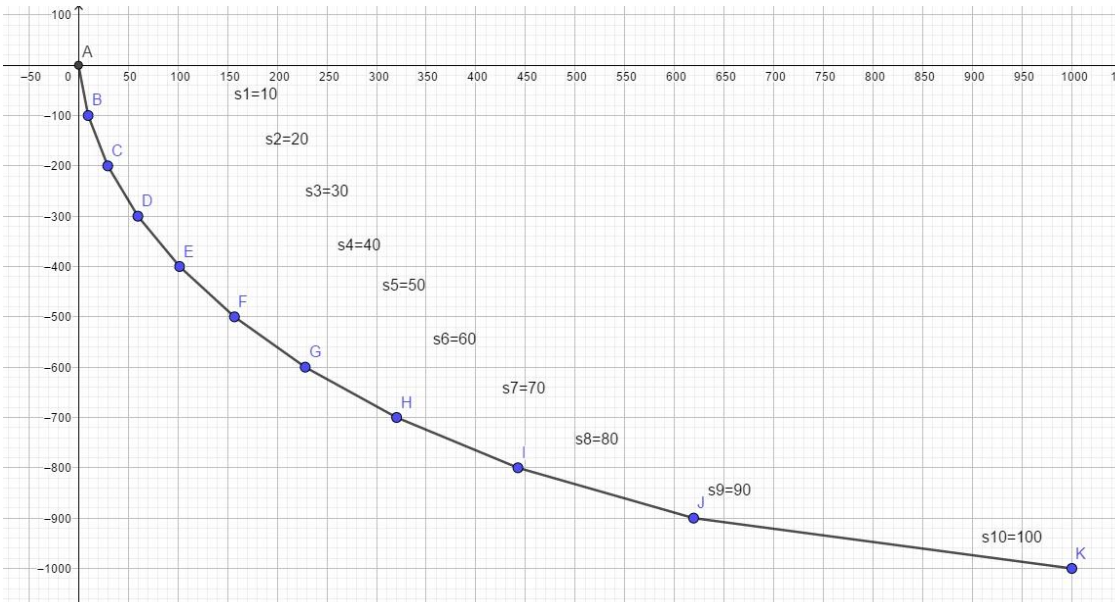

Figure 3-Our Optimal Path Timewise

3 A Special Case-The Brachistochrone Problem

Using the above method, we can approximate the solution to the classic brachistochrone problem [2]:

The brachistochrone problem was stated by Bernoulli in 1696: Given two points A and B, find the path along which an object would slide (without friction) in the shortest time from a point A to a point B, if it starts at A in rest and is only accelerated by gravity, denoted by g.

We shall once again define A= (0, 0) and B= (XE, YE). To solve this problem, we need to find the function y(x) which minimizes the time T where T is given by

$$T = \int_0^{X_E} \frac{\sqrt{1 + (\frac{dy}{dx})^2}}{\sqrt{2gy}} dx \tag{3.1}$$

Define our integrand as follows

$$F(x, y, y') = \frac{\sqrt{1 + (\frac{dy}{dx})^2}}{\sqrt{2gy}} \tag{3.2}$$

To make T stationary, as F does not contain x we can use the special case of the Euler-Lagrange equation-The Beltrami Identity [3]

$$F - y' \frac{\partial F}{\partial y'} = C \tag{3.3}$$

Using separable equations and substitutions we get [2] a solution of (3.1)-(3.3) of the form

$$x(t)=a(t-\sin(t))$$

$$y(t)=a(1-\cos(t)) \tag{3.4}$$

Where the range $t=0..t_E$ comes from approximating the solution of

$$\frac{t - \sin{(t_E)}}{1 - \cos{(t_E)}} - \frac{x_E}{y_E} = 0 \tag{3.5}$$

Using the solution of (3.5) we can then solve for a:

$$a = \frac{x_E}{t_E - \sin{(t_E)}} = \frac{y_E}{1 - \cos{(t_E)}} \tag{3.6}$$

It is easy to show that the time T along the curve is given by

$$T = \frac{1}{\sqrt{2g}} \int_0^{x_E} \sqrt{\frac{1 + (\frac{dy}{dx})^2}{y}} dx = t_E \sqrt{\frac{a}{g}} \tag{3.7}$$

Example:

Letting A=(0, 0) be our starting point and B=(XE, YE)=(1000, 1000) we get the following equations for the brachistochrone problem:

$$x(t)=572.917(t-\sin(t))$$

$$y(t)=572.917(1-\cos(t)) \tag{3.8}$$

where t=0..2.412. We will graph this in a moment.

Using our method of Lagrange multipliers and breaking the y domain [0, 1000] into 1000 partitions and approximating gravity g to be a constant over each subinterval, we get the following result (sampling at every 100th point):

Lagrange Multiplier Method (n=1000 partitions)

x1=20.24 y1=100	x2=58.93 y2=200	x3=111.74 y3=300	x4=178.08 y4=400	x5=258.56 y5=500
x6=354.7 y6=600	x7=469.44 y7=700	x8=607.44 y8=800	x9=777.67 y9=900	x10=1000 y10=1000

The solution of the brachistochrone problem is given by the solid line and the points (x_i, y_i) i=1…n are from the Lagrange multiplier method as illustrated in Figure 4 below:

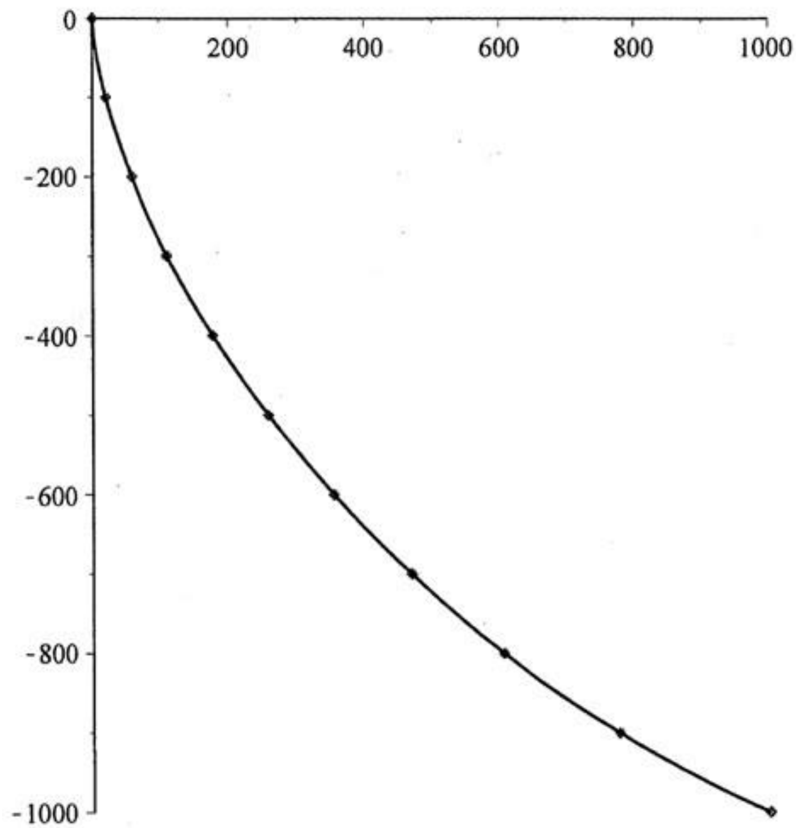

Figure 4-Comparing the Method of Lagrange Multipliers and the Calculus of Variation

As can be seen, the Lagrange multiplier method yields excellent results. However, the Lagrange multiplier method allows us to consider any speed, atmospheric modeling, acceleration, and decelerations in a very easy method.

4 References

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[3] Beltrami Identity https://en.wikipedia.org/wiki/Beltrami_identity
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