EXPLORING VOLUMES WITH GEOGEBRA DISSECTION MODELS

Thomas E. Cooper

University of North Georgia

82 College Circle, Dahlonega, GA 30597

tom.cooper@ung.edu

Introduction

As a professor who teaches a variety of introductory mathematics courses as well as courses for pre-service K – 12 teachers, I have found the free dynamic mathematics software Geogebra to be extremely useful in my teaching. At the 28th ICTCM, I gave a presentation on the robustness of Geogebra across many branches of mathematics (Cooper, 2017). In this paper, I focus on the usefulness of Geogebra in visualizing three-dimensional solids. In particular, I demonstrate how Geogebra makes it possible to use dissection models to intuitively derive several volume formulas. While the methods could be extended to other solids, I have chosen to focus on pyramids, tetrahedra, and two shapes arising from recreational mathematics investigations, a square based frustum and a stellated burr puzzle.

I first learned the term "frustum" in 2006 when I was a graduate student at the University of Georgia. My professor, Thomas Banchoff, who was visiting from Brown University, presented my class with a Foxtrot comic (Amend, 2006) where the character Jason is asking his family if a cup is half-empty or half-full. His punch line is that it is 7/12 empty and 5/12 full. It turns out, Jason's cup, being two-dimensional is a trapezoid as shown in Figure 1, but the three-dimensional version would be a frustum of a cone, the shape formed by slicing the top off of a cone parallel to the base. We can also consider frustums of pyramids with any base. In this paper, I will explore the polyhedral version with square bases.

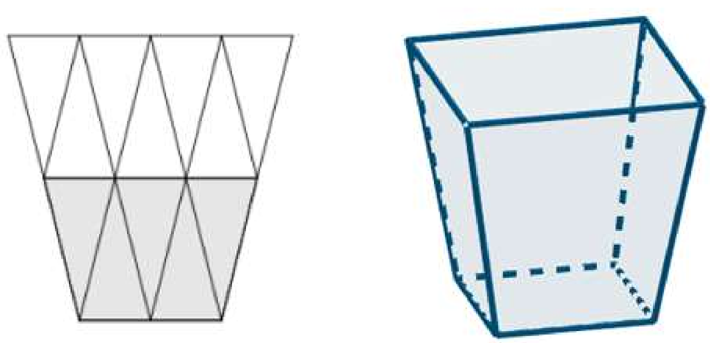

Figure 1. Jason's Trapezoid Cup and a Three-Dimensional Square-Based Frustum

Another object that I explore is called a stellated burr puzzle. Several companies produce wooden or plastic versions of this puzzle in which six congruent pieces can be assembled into a stellated burr or “star” shape. For those without a physical model, I created a virtual version with Geogebra that users can manipulate and break apart or assemble using a slider: https://www.Geogebra.org/classic/nhbhtf5b

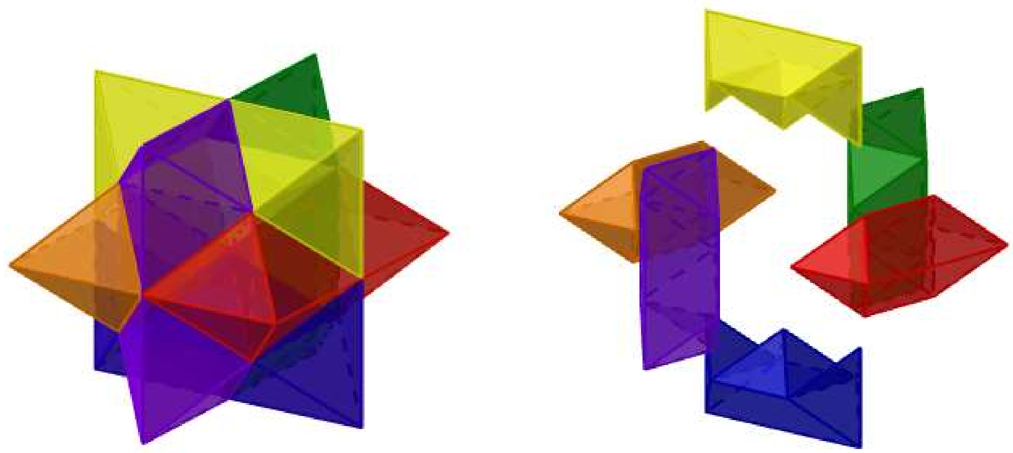

Figure 2. The Six Piece Stellated Burr Puzzle

Although the frustum and the stellated burr puzzle may seem unrelated, we can in fact explore the volume of each by dissecting the solids into a combination of pyramids and tetrahedra. The necessary tetrahedra, known as semi-orthocentric, have the property that they have a pair of perpendicular opposite edges. It turns out that this property greatly simplifies their volume formulas.

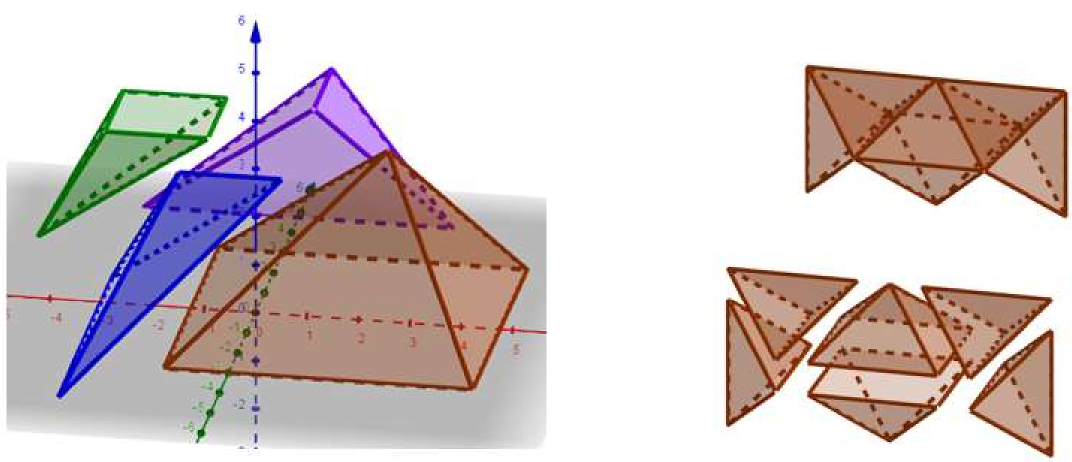

Figure 3. A Square-Based Frustum and a Stellated Burr Piece Dissected into Pyramids and Tetrahedra

Volumes of Prisms

To derive the volume of a square-based frustum and the stellated burr puzzle, we need to begin with the most basic volume and work our way up through pyramids and tetrahedra. The simplest solid in terms of volume is known as a cuboid or right rectangular prism. For these shapes, we can literally count the number of unit cubes that fill the shape using the familiar $V = LWH$. This can be demonstrated for students with physical models of unit cubes, or with the virtual version that I created with Geogebra:
https://www.Geogebra.org/m/qdr9m9gq

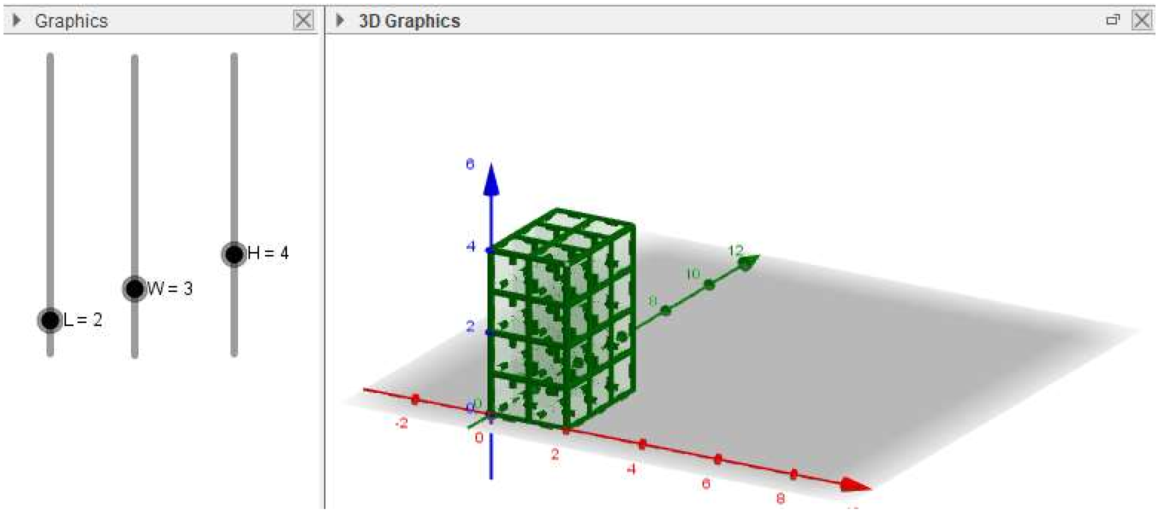

Figure 4. Geogebra Demonstration that $V = LWH$ for a Right Rectangular Prism

In the applet, I have created a sequence of cubes, using Geogebra's prism tool, with the number of cubes in each direction controlled by sliders. Using this applet, we can see that the length times the width gives the number of unit squares in the plane, and this value times the height gives the number of cubes in the solid.

To understand the volume formulas for oblique prisms or prisms with non-rectangular bases, we need the important result known as Cavalieri's Principle, which asserts that any pair of solids will have the same volume if for any intersecting plane, the cross sections have equal areas. This is often modeled for cylinders with stacks of coins, and I have designed a Geogebra applet with three different prisms:
https://www.Geogebra.org/m/sgrvjvrm

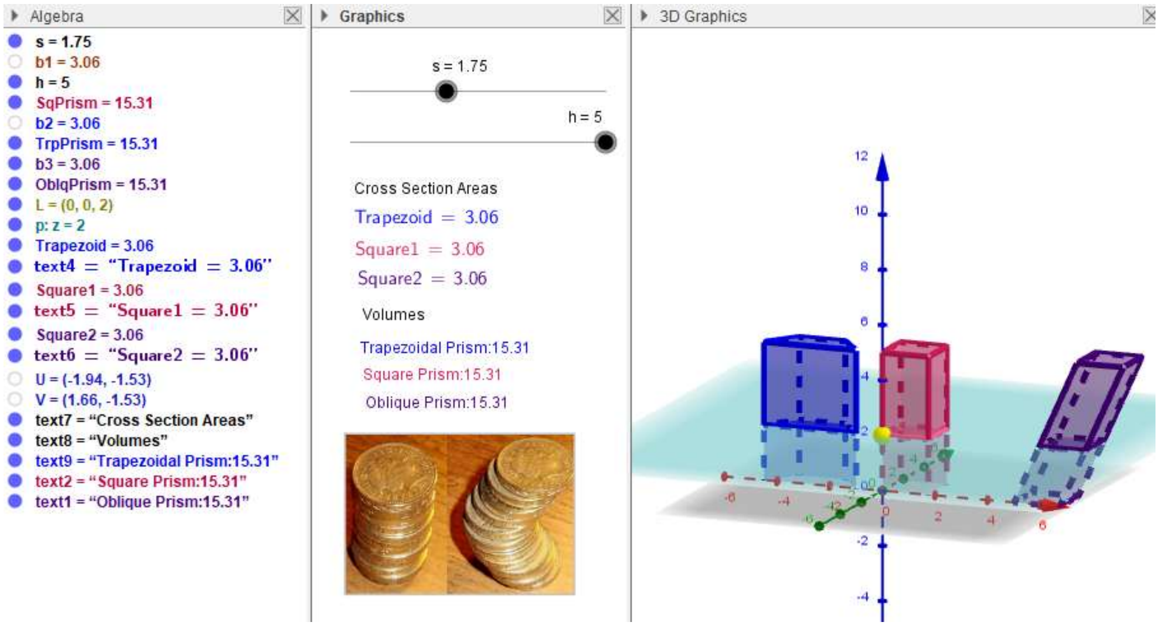

Figure 5. Geogebra Demonstration of Cavalieri's Principle for Prisms

I constructed three prisms with equal heights and equal area bases in the xy-plane. The slider s controls the length of the sides of the square base of the center prism, and h controls the height. A second prism is constructed with a congruent square base, but it is oblique with parallelogram sides instead of rectangles. A third prism is constructed with a trapezoid base of the same area. I have added an intersecting plane parallel to the xy-plane that can be moved up and down with a draggable point on the z-axis. Dragging the point, shows that for each prism, the cross sections are congruent to the respective bases, and thus, they all have the same area. The volumes, which are automatically computed by Geogebra for prisms, are all the same, verifying Cavalieri's Principle. Note that for a rectangular prism, the volume formula is equivalent to computing the area of the base times the height. Therefore, by Cavalieri's Principle, the volume of any prism, including cylinders or prisms with non-polygonal bases, can be computed as the height times the area of the base.

$$V_{\text{prism}} = (\text{height}) \times (\text{area of the base})$$

Volumes of Pyramids

We can also use Cavalieri's Principle to see that the volume of a pyramid is determined by the area of the base and the height, and with other constructions, we can derive that the volume of a pyramid, including cones and pyramids with non-polygonal bases, is one-third the area of the base times the height or one-third times the area of the circumscribed prism. To reach this result, I start with a Geogebra construction for pyramids that is similar to the one for prisms, demonstrating Cavalieri's Principle:
https://www.Geogebra.org/m/zskwjznv

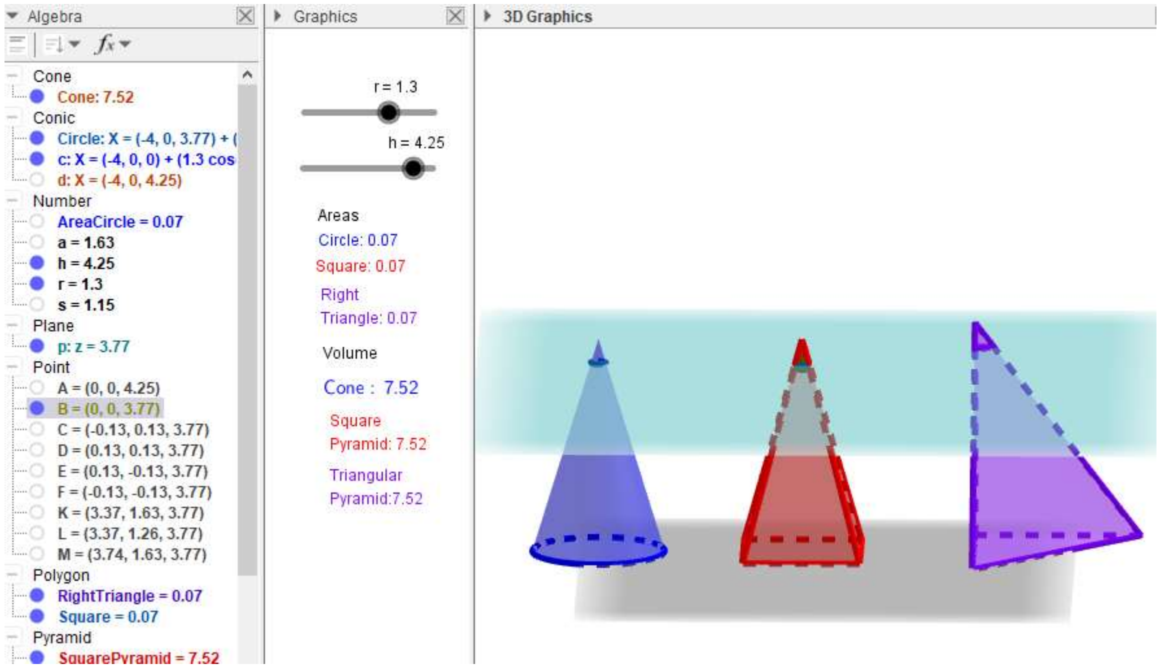

Figure 6. Geogebra Demonstration of Cavalieri's Principle for Pyramids

In this applet, I have constructed pyramids with equal height and equal area bases in the shape of a circle, a square, and a right triangle. The slider *h* controls the common height, and the slider *r* controls the radius of the circular base of the cone. The square and right triangle bases are constructed to have the same area as the circular base. By dragging the intersecting plane, controlled by a draggable point, we see that while the areas of the cross section change as the height of the plane changes, the three cross-section areas are always the same. The computed volumes verify that by Cavalier's Principle the three solids all have the same volume. Therefore, any two pyramids with the same height and bases of equal area will have the same volume.

There is a classic demonstration that the volume of a pyramid is one-third the height times the area of the base for a very specific case, using a trisection of a cube. Slicing a cube as shown in Figure 7 creates three congruent pyramids. These pyramids with square bases and apexes perpendicular over one corner of the base were called yangma by Chinese Mathematician Liu Hui (263 AD). I have shown this dissection of the cube into three congruent yangma in Geogebra: https://www.Geogebra.org/classic/agk6edhf

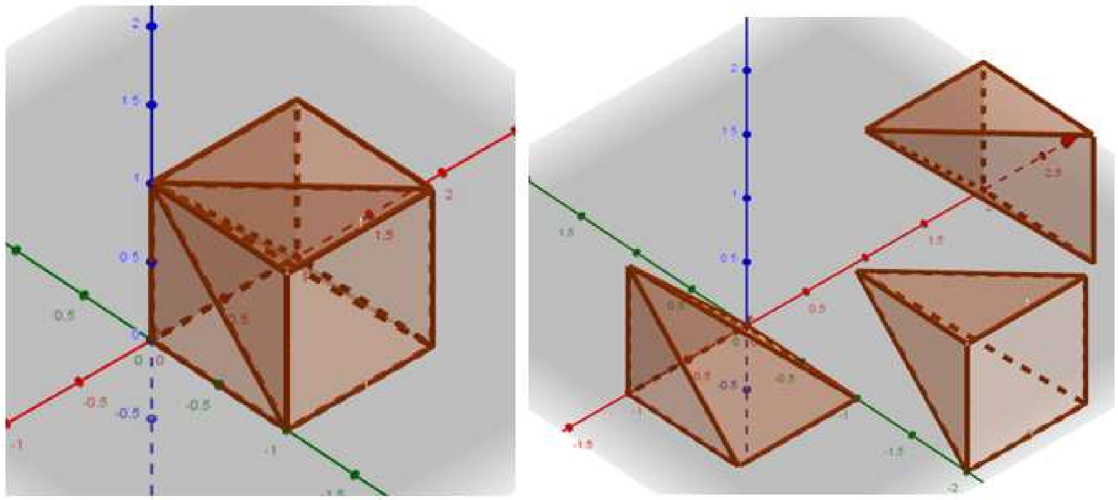

Figure 7. A Cube Trisected into Three Congruent Yangma

For more general cases, we can use Cavalieri's Principle and pyramids with right triangle bases. By Cavalier's Principle, if we need to compute the volume of a given pyramid, we may instead compute the volume of a pyramid with the same height and a right triangle base with the same area as the base of the given pyramid. We can then construct a triangular prism with this pyramid and two others with the same volume.
https://www.Geogebra.org/m/qbabburu

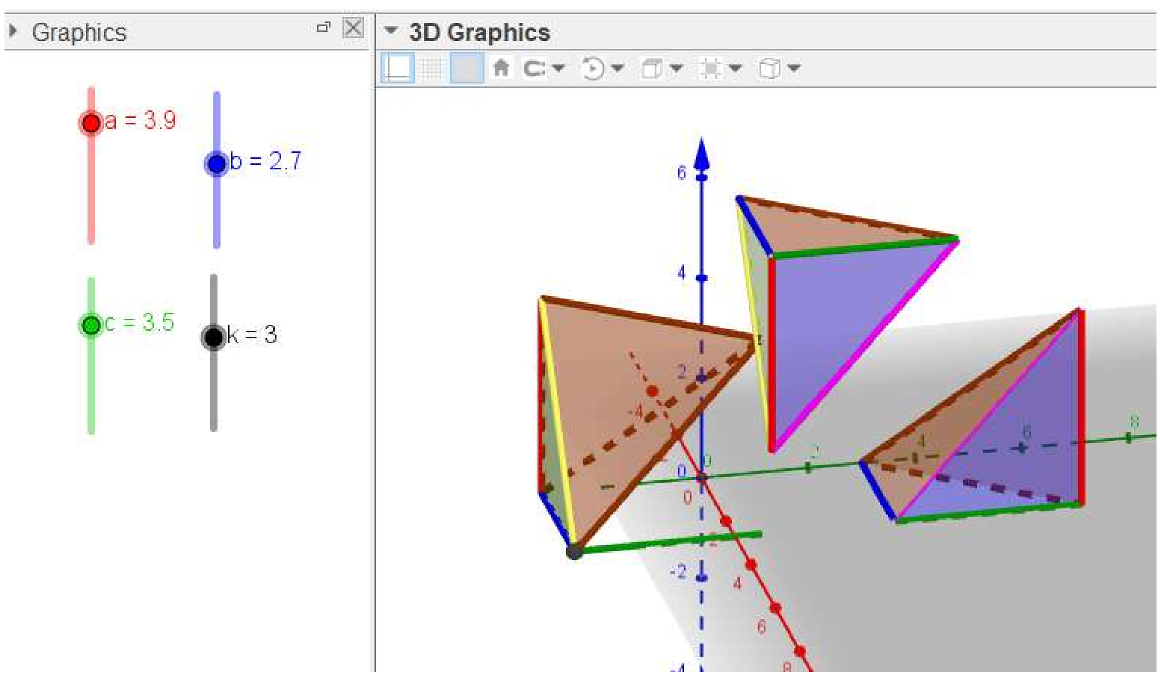

Figure 8. Constructing a Prism with Three Pyramids of Equal Volume

For my Geogebra construction, I began with a right triangle base and created a pyramid with the apex perpendicular over one of the acute angles. Three color-coded sliders control the lengths of the two legs of the base and the edge that is the height of the pyramid. I then used vectors and translations to create two other pyramids with right triangle bases having one leg that is the same length as the height of the original pyramid and another leg that is the same length as one of the legs of the base of the original pyramid.

A slider can be used to move the pyramids apart or slide them together to create a triangular prism whose bases are right triangles with the same dimensions as the original pyramid base. The volume of this prism is, therefore, the height of the original pyramid times the area of its base. If the three pyramids have the same volume, then each is one-third the area of the original base times the original height. Which in turn, means that the volume of any pyramid is one-third its height times the area of its base. Exploring the applet, we can observe that any pair of the pyramids can be seen to have congruent bases and the same heights. So again, by Cavalieri's Principle, the three all have the same volume. Note in Figure 8, that I added an additional green segment onto the vertex of one of the pyramids to show that the height matches the lengths of the other green segments. Thus, we have established the familiar formula for pyramids, which includes cones and pyramids with any type of base.

$$V_{\text{pyramid}} = \frac{1}{3} \times (\text{area of the base}) \times \text{height}$$

Volumes of Tetrahedra

Note that a tetrahedron is just a special case of a pyramid where the base is a triangle. So, the pyramid formula holds for a tetrahedron, though it may be hard to measure the height of a given tetrahedron. It is common in multivariable calculus to note that when using vectors, the volume of a tetrahedron is one-sixth the volume of a parallelepiped defined by the same vectors. This can be computed with dot products and cross products.

$$V_{\text{tetrahedron}} = \frac{1}{6} |\boldsymbol{a} \times \boldsymbol{b} \cdot \boldsymbol{c}|$$

Geometrically, this result is true because as we have seen, a tetrahedron is one-third the volume of the prism that is defined by its vectors, and this can be doubled to get a parallelepiped. I made a Geogebra construction to demonstrate this idea that is much like the one for more general pyramids. In this applet, shown in Figure 9, the three tetrahedra can be moved by dragging the three green points, and the reflection of the triangular prism can be turned off and on with a checkbox. The slider controls the height.
https://www.Geogebra.org/m/z7akxsrm

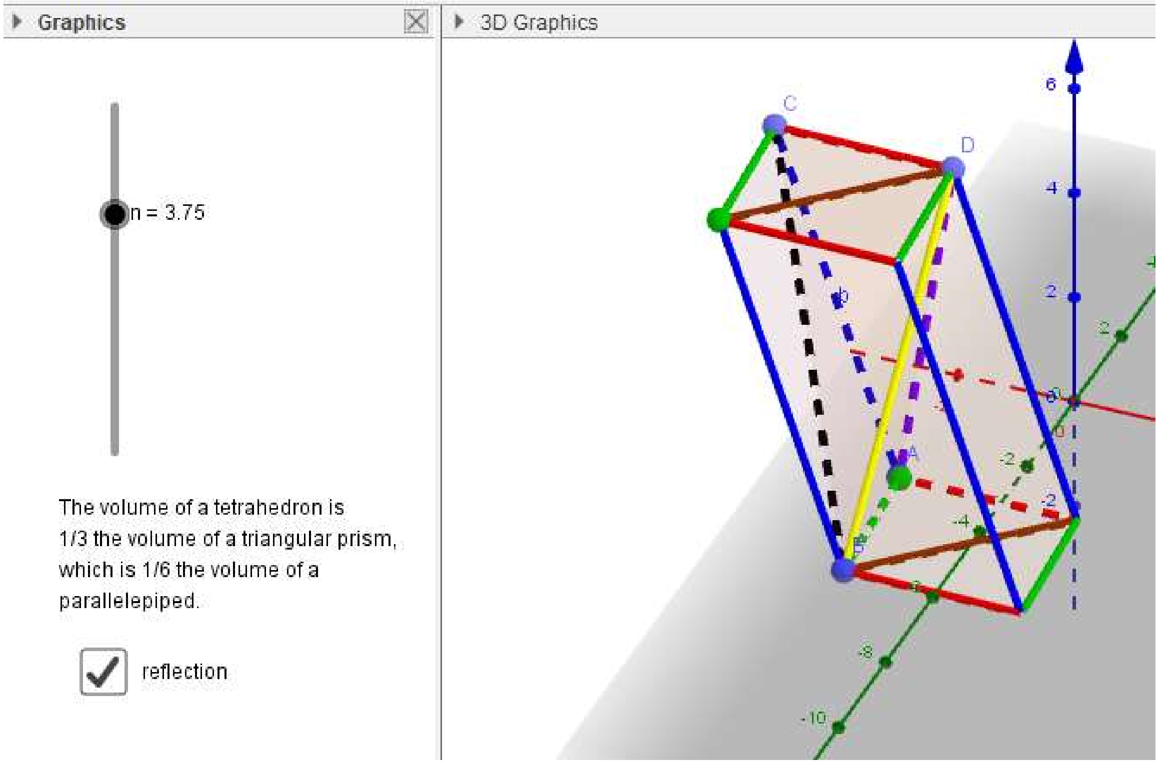

Figure 9. Geogebra Demonstration that the Volume of a Tetrahedron is 1/6 the Volume of a Parallelepiped

While investigating the frustum and the burr puzzle, I stumbled onto the special family of tetrahedra known as "semi-orthocentric." An orthocentric tetrahedron has perpendicular opposite edges, and a semi-orthocentric tetrahedron has one pair of perpendicular opposite edges. When a pair of opposite edge are perpendicular, the tetrahedron is one-third of a prism with right triangular bases as shown in Figure 10. In this case, the area of the base of this prism is $\frac{1}{2}ab$, where a and b are the lengths of the perpendicular opposite edges of the tetrahedron. Therefore, the volume of the semi-orthocentric tetrahedron is one-third times the volume of the prism, which is simply $\frac{1}{6}abh$, where the height of the prism h is the distance between the two perpendicular opposite edges.
https://www.Geogebra.org/classic/ydcegax6

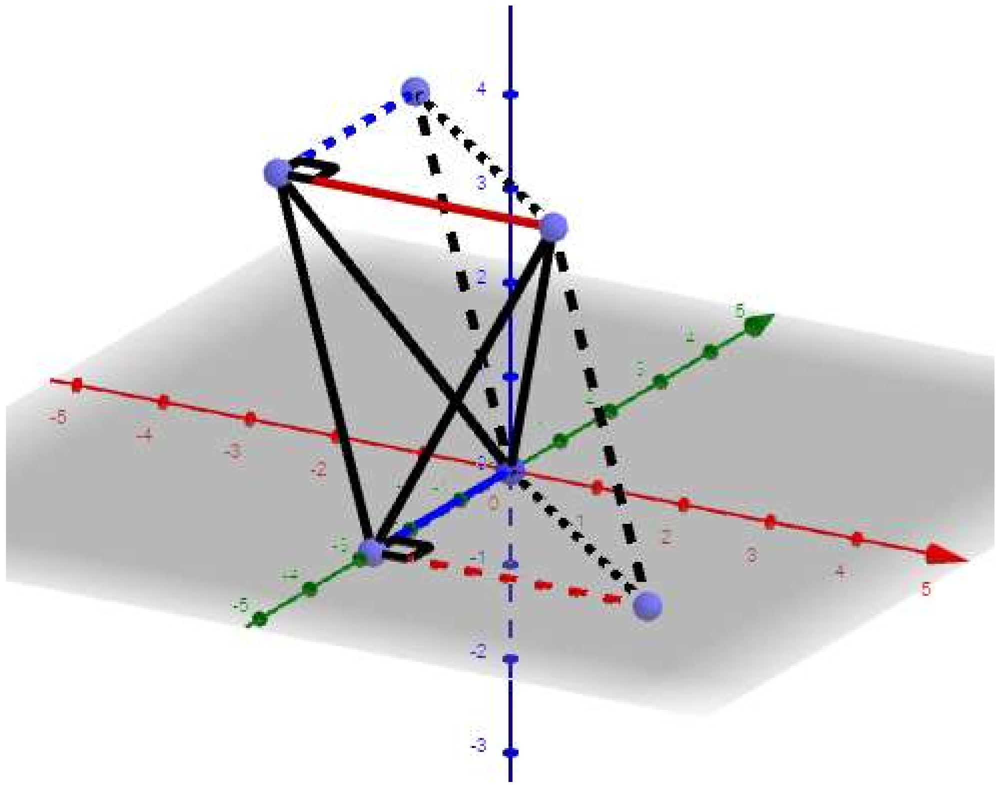

Figure 10. A Semi-Orthocentric Tetrahedron and the Circumscribed Prism with Right Triangle Bases.

Volume of a Square-Based Frustum

There is a classic decomposition of a square-based frustum that dates at least as far back as Liu Hui's work (263 AD). Hui decomposed a square frustum into a center cuboid, four wedge shaped triangular prisms, and four yangma pyramids. To compute the volume, suppose that the smaller base has side lengths b, the larger base has side lengths a, and the height of the frustum is h. Then the cuboid has a base with area b^2 and height h. Thus, the cuboid has volume hb^2. The yangma have height h and square bases with side lengths $\frac{a-b}{2}$. Together the yangma have a total volume of $\frac{4}{3}h\left(\frac{a-b}{2}\right)^2 = \frac{h}{3}(a-b)^2$. The triangular prisms have height b and right triangle bases with leg lengths h and $\frac{a-b}{2}$. Combined the triangular prisms have volume $4\left(\frac{1}{2}\right)\left(\frac{a-b}{2}\right)bh = (a-b)bh$. Adding these together gives the volume of the frustum.

$$V_{frustum} = hb^2 + \frac{h}{3}(a-b)^2 + (a-b)bh$$
$$= \frac{1}{3}h(a^2 + ab + b^2)$$

I have modeled Hui's decomposition in Geogebra, using a slider to break apart the frustum: https://www.Geogebra.org/classic/hagbkp78

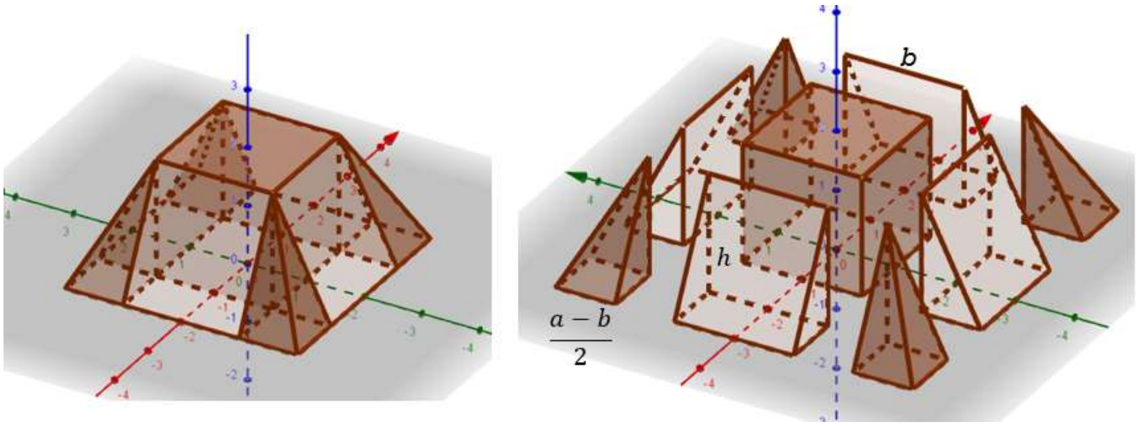

Figure 11. Liu Hui's Decomposition of a Square Frustum

This is a nice derivation, but it does not show a clear correspondence between the pieces and the final three terms of the volume formula. While preparing a discussion of the FoxTrot comic for the Gathering for Gardner conference, Tom Banchoff came up with an interesting decomposition based on Jason's cup (Banchoff & Cooper, 2018). Tom envisioned placing the 12 triangle decomposition of a trapezoid shown in Figure 1 onto a three dimensional frustum and slicing the object along the lines. I made a Geogebra construction that focuses on the bottom of "full" portion of Jason's cup: https://www.Geogebra.org/classic/g8rmspkk

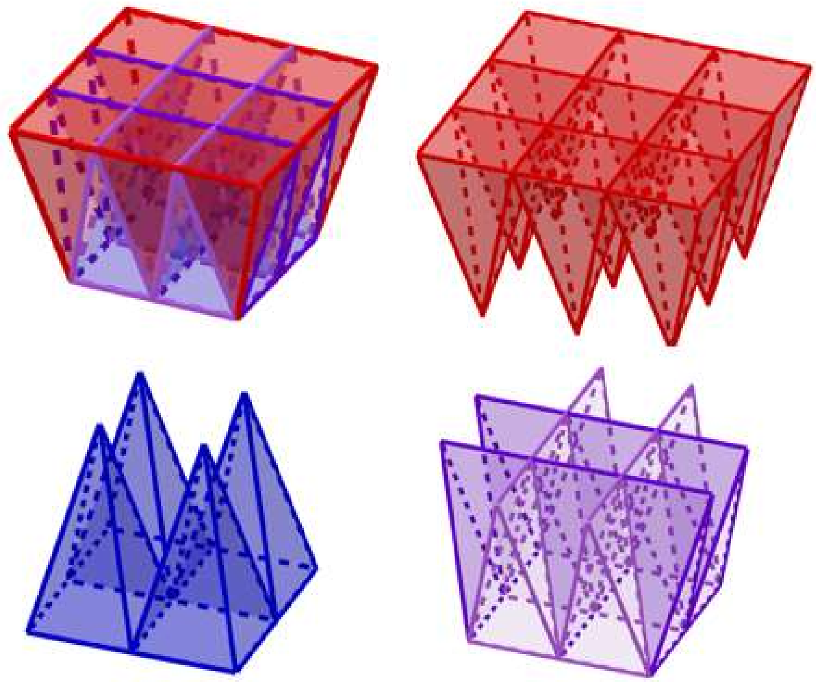

Figure 12. Banchoff's Decomposition of a Square-Based Frustum into Square Pyramids and Semi-Orthocentric Tetrahedra

The frustum shown in Figure 12 has 2×2 and 3×3 square bases, but Professor Banchoff's dissection would work for any frustum with $n \times n$ and $(n+1) \times (n+1)$ bases. The fascinating result is that the decomposition corresponds exactly to the three terms in the volume formula. There a n^2 upward pointing pyramids, each with volume $\frac{h}{3}\left(\frac{a}{n}\right)^2$. So, the upward pointing pyramids have a total volume of $\frac{h}{3}a^2$. There are $(n+1)^2$ downward pointing pyramids, each with volume $\frac{h}{3}\left(\frac{b}{n+1}\right)^2$. So, the downward pointing pyramids have a total volume of $\frac{h}{3}b^2$. Looking at the frustum's volume formula, it must be that the space in between the pyramids has a volume of $\frac{h}{3}ab$. In fact, this space is made up of $2n(n+1)$ semi-orthocentric tetrahedra, each with a volume of $\frac{1}{6}h\left(\frac{a}{n}\right)\left(\frac{b}{n+1}\right)$.

While working on the applet for Tom's decomposition, I wondered about a case with $n = 1$, and I came up with a nice decomposition that should work for any square based frustum. My decomposition is shown in Figure 13 and modeled with a Geogebra applet: https://www.Geogebra.org/classic/b6puzg3w

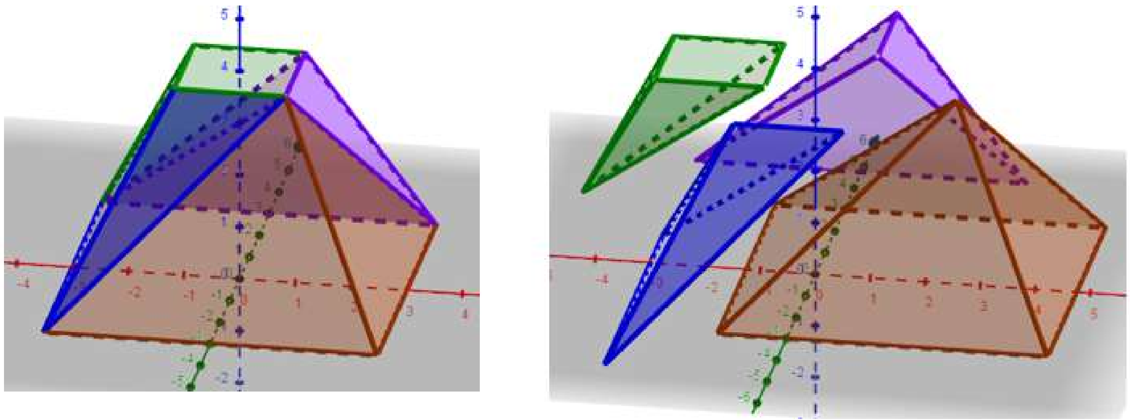

Figure 13. Decomposition of a Square-Based Frustum into Two Pyramids and Two Semi-Orthocentic tetrahedra.

With this smaller decomposition, I have given up much of the symmetry found in the other decompositions, but the formulas still hold. We get two pyramids with square bases that have volumes $\frac{h}{3}a^2$ and $\frac{h}{3}b^2$. The pyramids are oblique with the apexes not over the center of the bases, but by Cavalieri's Principle, the volumes are one-third the area of the base times the height. Also, while not as symmetric as the tetrahedra in Banchoff's dissection, the two tetrahedra formed are semi-orthocentric with the lengths of the opposite edges being a and b, and the distance between them being the height of the frustum. So the volume of the frustum is $\frac{h}{3}a^2 + \frac{h}{6}ab + \frac{h}{6}ab + \frac{h}{3}b^2$, giving a simple derivation in which the terms of the resulting formula match nicely to the volumes of the decomposition.

Volume of The Stellated Burr Puzzle

At first glance, the stellated burr puzzle with its many sharp corners may appear complicated in terms of volume; however, it turns out to be a simple result. The burr puzzle is made up of six congruent pieces, and each of these can be dissected into two square pyramids and four semi-orthocentric tetrahedra. As with the other decompositions, I have modeled this with a slider in Geogebra: https://www.Geogebra.org/m/gm2mrt9k

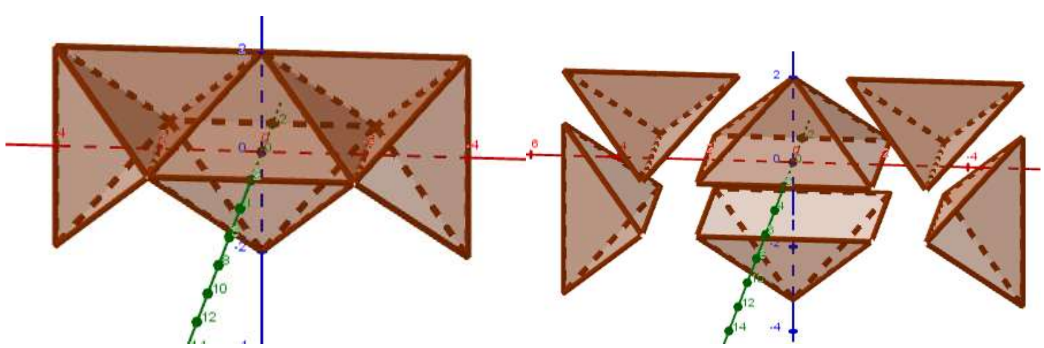

Figure 14. The Burr Puzzle Pieces Dissected into Pyramids and Semi-Orthocentric Tetrahedra

To compute the volume of the puzzle piece, let the length of the longest edge be $2s$. Then the square pyramids have base edge length s and height $\frac{s}{2}$. This gives a volume of $\frac{s^3}{6}$ for each pyramid. The four tetrahedra have opposite perpendicular edges of length s with a distance of $\frac{s}{2}$ between them. This gives a volume of $\frac{s^3}{12}$ for each tetrahedron and a total volume of $\frac{2}{3}s^3$ for the entire puzzle piece. Multiplying by six, gives a volume of $4s^3$ for the complete burr puzzle. This is precisely half of $(2s)^3$, or one half the volume of a cube with the side lengths matching the longest edge of the burr puzzle piece. In other words, the stellated burr puzzle is one half the cube that circumscribes it.

The relationship between the stellated burr and a cube was not immediately obvious to me until I computed it, but I later discovered another interesting object known as the Yoshimoto Cube. A simpler version, sometimes called an infinity cube consists of eight cubes hinged together to form a larger cube. The hinges allow the object to be folded repeatedly into larger cubes changing face colors. Naoki Yoshimoto created a version in which the eight smaller cubes are replaced by convex polyhedra with half the volume. This version allows the user to repeatedly transform the outside shell from a cube to a star matching the shape of the stellated burr puzzle. Some companies even sell pairs of Yoshimoto cubes that fit inside one another, showing that the stellated burr puzzle fits perfectly inside of a cube. I have also created a Geogebra model showing the transformation from cube to star: https://www.Geogebra.org/classic/ee4nvbxf

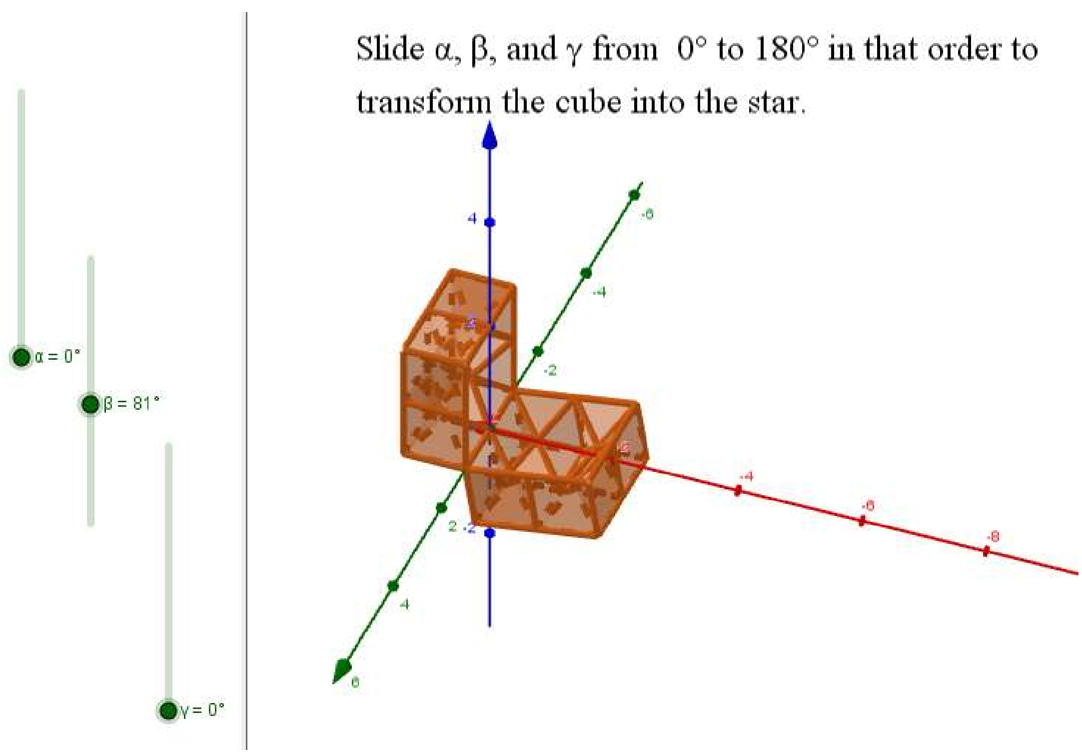

Figure 15. Geogebra Model of a Yoshimoto Cube

In summary, the dynamic capabilities of Geogebra combined with its ability to model three-dimensional solids has allowed me to take a journey from simple prisms to pyramids and tetrahedra, ultimately finding the volumes of square-based frustums and the star shape of the stellated burr puzzle. Although the results are not novel, my investigations with Geogebra led to my own self discovery of the volume of semi-orthocentric tetrahedra and the relationship between the stellated burr puzzle and the Yoshimoto cube. These are interesting relationships that I would not have made without the technology. For many of us, it is challenging to visualize complex three-dimensional objects mentally, but tools such as Geogebra can provide us with powerful models for exploration.

References

Amend, B. (2006, January 22). 'Foxtrot' [Comic Strip]. Retrieved from https://www.gocomics.com/foxtrot/2006/01/22

Banchoff, T. F., & Cooper T. E. (2018). The Foxtrot Half-Empty/Half-Full Problem, Including 13. Mathematical paper in *Gathering for Gardner 13 (G4G13) Gift Exchange.*

Cooper, T. E. (2017). An introduction to the robustness of Geogebra. *Proceedings of the 28th Annual International Conference on Technology in Collegiate Mathematics Proceedings.* Atlanta: ICTCM.

Hui, L. (263 AD). *The Nine Chapters on the Mathematical Art*.