Exploring the "Reverse" Lucas Sequence 3, 1, 4, 5, 9, …

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Abstract: The reverse Lucas sequence will be explored using CAS technology to determine divisibility, prime outputs, periodicity and palatable number tricks.

1. Introductory Observations.

We initially examine the role played by the graphing calculator handhelds (models VOYAGE 200 and TI-89) when we enter the recursive sequence. See **FIGURES 1-4:**

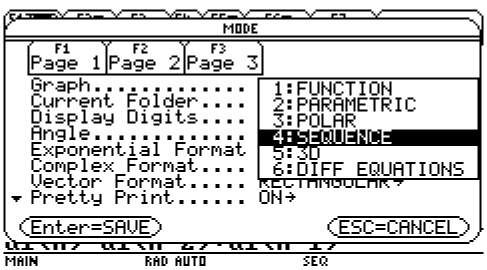

FIGURE 1: Sequence Mode

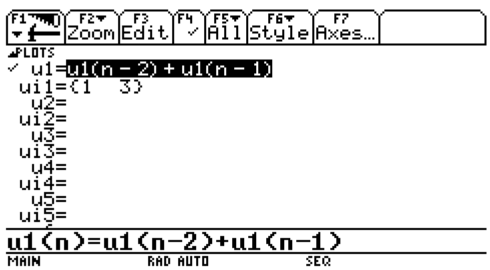

FIGURE 2: Entering the Sequence

FIGURE 3: Entering the Sequence

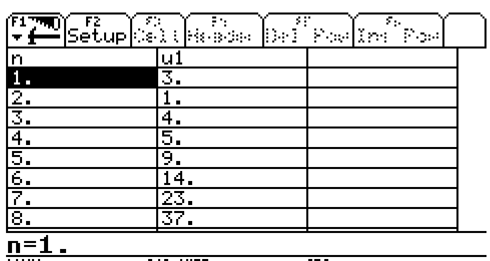

FIGURE 4: Table Illustrated

We note that the first ten terms in the table are 3, 1, 4, 5, 9, 14, 23, 37, 60 and 97. The calculator reads the second term followed by the first terms when we enter 1, 3. Hence the expected Lucas sequence is not generated. Instead the sequence we obtain will heretofore be classified as the reverse Lucas sequence. Let us examine in **TABLE 1** the initial one hundred terms in the sequence together with their prime factorizations with prime outputs underline

TABLE 1: PRIME FACTORIZATIONS OF THE FIRST ONE HUNDERED TERMS IN THE REVERSE LUCAS SEQUENCE.

n:	S_n:	**Prime Factorization of** S_n:
1	**3**	3
2	**1**	1
3	**4**	2^2
4	**5**	5
5	**9**	3^2
6	**14**	$2\cdot7$
7	**23**	23
8	**37**	37
9	**60**	$2^2\cdot3\cdot5$
10	**97**	97
11	**157**	157
12	**254**	$2\cdot127$
13	**411**	$3\cdot137$
14	**665**	$5\cdot7\cdot19$
15	**1076**	$2^2\cdot269$
16	**1741**	1741
17	**2817**	$3^2\cdot313$
18	**4558**	$2\cdot43\cdot53$
19	**7375**	$5^3\cdot59$
20	**11933**	11933
21	**19308**	$2^2\cdot3\cdot1609$
22	**31241**	$7\cdot4463$
23	**50549**	50549
24	**81790**	$2\cdot5\cdot8179$
25	**132339**	$3\cdot31\cdot1423$
26	**214129**	214129
27	**346468**	$2^2\cdot37\cdot2341$
28	**560597**	560597
29	**907065**	$3^3\cdot5\cdot6719$
30	**1467662**	$2\cdot7\cdot79\cdot1327$
31	**2374727**	$23\cdot223\cdot463$
32	**3842389**	$19\cdot202231$
33	**6217116**	$2^2\cdot3\cdot379\cdot1367$
34	**10059505**	$5\cdot227\cdot8863$
35	**16276621**	16276621
36	**26336126**	$2\cdot641\cdot20543$
37	**42612747**	$3\cdot1637\cdot8677$
38	**68948873**	$7\cdot181\cdot54419$
39	**111561620**	$2^2\cdot5\cdot5578081$

40	**180510493**	180510493
41	**292072113**	$3^2 \cdot 32452457$
42	**472582606**	$2 \cdot 1109 \cdot 213067$
43	**764654719**	$67 \cdot 2083 \cdot 5479$
44	**1237237325**	$5^2 \cdot 49489493$
45	**2001892044**	$2^2 \cdot 3 \cdot 53 \cdot 3147629$
46	**3239129369**	$7^2 \cdot 37 \cdot 1786613$
47	**5241021413**	$71 \cdot 3613 \cdot 20431$
48	**8480150782**	$2 \cdot 167 \cdot 3607 \cdot 7039$
49	**13721172195**	$3 \cdot 5 \cdot 914744813$
50	**22201322977**	$19 \cdot 83 \cdot 14078201$
51	**35922495172**	$2^2 \cdot 3371 \cdot 2664083$
52	**58123818149**	$129631 \cdot 448379$
53	**94046313321**	$3^2 \cdot 2671 \cdot 3912239$
54	**152170131470**	$2 \cdot 5 \cdot 7 \cdot 2173859021$
55	**246216444791**	$23 \cdot 31 \cdot 345324607$
56	**398386576261**	398386576261
57	**644603021052**	$2^2 \cdot 3 \cdot 619 \cdot 86780159$
58	**1042989597313**	1042989597313
59	**1687592618365**	$5 \cdot 97 \cdot 499 \cdot 6973091$
60	**2730582215678**	$2 \cdot 118967 \cdot 11476217$
61	**4418174834043**	$3 \cdot 1472724944681$
62	**7148757049721**	$7 \cdot 43 \cdot 929 \cdot 4967 \cdot 5147$
63	**11566931883764**	$2^2 \cdot 487 \cdot 947 \cdot 6270169$
64	**18715688933485**	$5 \cdot 919 \cdot 5023 \cdot 810881$
65	**30282620817249**	$3^3 \cdot 37 \cdot 401 \cdot 75593351$
66	**48998309750734**	$2 \cdot 305353 \cdot 80232239$
67	**79280930567983**	$102301 \cdot 774977083$
68	**128279240318717**	$19 \cdot 6751538964143$
69	**207560170886700**	$2^2 \cdot 3 \cdot 5^2 \cdot 179 \cdot 8017 \cdot 482123$
70	**335839411205417**	$7 \cdot 94343 \cdot 508538617$
71	**543399582092117**	$577 \cdot 683 \cdot 1378868287$
72	**879238993297534**	$2 \cdot 53 \cdot 363581 \cdot 22813919$
73	**1422638575389651**	$3 \cdot 229 \cdot 1194671 \cdot 1733363$
74	**2301877568687185**	$5 \cdot 460375513737437$
75	**3724516144076836**	$2^2 \cdot 367 \cdot 2537136337927$
76	**6026393712764021**	$271 \cdot 563 \cdot 156677 \cdot 252101$
77	**9750909856840857**	$3^2 \cdot 59 \cdot 103 \cdot 23917 \cdot 7454297$
78	**15777303569604878**	$2 \cdot 7 \cdot 257 \cdot 547 \cdot 57727 \cdot 138869$
79	**25528213426445735**	$5 \cdot 23 \cdot 221984464577789$
80	**41305516996050613**	41305516996050613
81	**66833730422496348**	$2^2 \cdot 3 \cdot 5569477535208029$

82	**1081392474185469610**	$137 \cdot 269 \cdot 2934340417837$
83	**1749729778410433309**	1749729778410433309
84	**2831122252595902700**	$2 \cdot 5 \cdot 37 \cdot 1223 \cdot 625648549777$
85	**4580852031006335790**	$3 \cdot 31 \cdot 2333 \cdot 14221 \cdot 148463071$
86	**7411974283602238490**	$7 \cdot 19 \cdot 18749 \cdot 297237879097$
87	**1199282631460857428**	$2^2 \cdot 1567 \cdot 6473 \cdot 29558810227$
88	**1940480059821081277**	$3671 \cdot 26903 \cdot 109541 \cdot 179369$
89	**3139762691281938705**	$3^2 \cdot 5 \cdot 69772504250709749$
90	**5080242751103019982**	$2 \cdot 157 \cdot 2063 \cdot 4483 \cdot 1749390847$
91	**8220005442384958687**	$279677689 \cdot 29390994583$
92	**13300248193487978669**	13300248193487978669
93	**21520253635872937356**	$2^2 \cdot 3 \cdot 647 \cdot 2357 \cdot 1175986337947$
94	**34820501829360916025**	$5^2 \cdot 7 \cdot 283 \cdot 703089385751861$
95	**56340755465233853381**	$311 \cdot 7417 \cdot 3749849 \cdot 6513587$
96	**91161257294594769406**	$2 \cdot 58921 \cdot 773588850279143$
97	**147502012759828622787**	$3 \cdot 631 \cdot 101807971 \cdot 765359629$
98	**238663270054423392193**	238663270054423392193
99	**386165282814252014980**	$2^2 \cdot 5 \cdot 53 \cdot 3643068705794830 33$
100	**624828552868675407173**	624828552868675407173

2. Some Conjectures and Further Analysis.

A brief perusal of the Table 1 leads us to the following true conjectures where S_n represents the n^{th} term in the sequence.

Conjecture 1: Every third term in the sequence is even. Hence $S_3, S_6, S_9, S_{12}, ...$ are even.

Conjecture 2: Every term that is congruent to one modulo four is divisible by three. Hence the terms $S_1, S_5, S_9, S_{13}, ...$ are divisible by three.

Conjecture 3: Every term that is congruent to three modulo six is divisible by four. Hence the terms $S_3, S_9, S_{15}, S_{21}, ...$ are divisible by four which are the odd integer multiples of three.

Conjecture 4: Every term that is congruent to four modulo five is divisible by five. Hence the terms $S_4, S_9, S_{14}, S_{19}, ...$ are divisible by five.

Conjecture 5: Every term that is congruent to six modulo eight is divisible by seven. Hence the terms $S_6, S_{14}, S_{22}, S_{30}, ...$ are divisible by seven.

Conjecture 6: Every term that is congruent to nine modulo fifteen is divisible by ten. Hence the terms $S_9, S_{24}, S_{39}, S_{54}, ...$ are divisible by ten.

The Principle of Mathematical Induction or Modular Arithmetic can be utilized to establish the truth of these conjectures. For example, we Prove Conjecture 2.

We are asserting that $3 \mid S_{4\cdot n-3}\ \forall n \in \mathbb{N}$. (The symbol | means divides or is a factor of.)

Step 1: The statement is true for $n=1$:

Observe that $3 \mid S_{4\cdot 1-3} \leftrightarrow 3 \mid S_{4-3} \leftrightarrow 3 \mid S_1 \leftrightarrow 3 \mid 3$ since $3 = 3\cdot 1$. (1)

Step 2: Assume the statement is true for $n=k$:

We assume that $3 \mid S_{4\cdot k-3}$. (2)

Step 3: We prove that the statement is true for $n=k+1$ given that the statement is assumed true for $n=k$:

We prove that $3 \mid S_{4\cdot(k+1)-3} \leftrightarrow 3 \mid S_{4\cdot k+4-3} \leftrightarrow 3 \mid S_{4\cdot k+1}$ is true given that $3 \mid S_{4\cdot k-3}$ is assumed true. (3)

$$S_{4\cdot k+1} = S_{4\cdot k-1} + S_{4\cdot k} = \left(S_{4\cdot k-3} + S_{4\cdot k-2}\right) + \left(S_{4\cdot k-2} + S_{4\cdot k-1}\right) = S_{4\cdot k-3} + 2\cdot S_{4\cdot k-2} + S_{4\cdot k-1} =$$

$$S_{4\cdot k-3} + 2\cdot S_{4\cdot k-2} + \left(S_{4\cdot k-3} + S_{4\cdot k-2}\right) = 2\cdot S_{4\cdot k-3} + 3\cdot S_{4\cdot k-2}.$$

Clearly $3 \mid [3\cdot S_{4\cdot k-2}]$. By the induction hypothesis (2), $3 \mid S_{4\cdot k-3} \to 3 \mid 2\cdot S_{4\cdot k-3}$. Hence $3 \mid \left[2\cdot S_{4\cdot k-3} + 3\cdot S_{4\cdot k-2}\right] \leftrightarrow 3 \mid S_{4\cdot k+1}$. Hence (3) is true and thus $P(k) \to P(k+1)$. Note that we have utilized the recursion relation in The Fibonacci sequence and elementary divisibility properties.

Step 4: By The Principle of Mathematical Induction, since the statement is true for $n=1$, the statement must be true for $n = 1+1 = 2, n = 2+2 = 3, \ldots, i.e.\ \forall n \in \mathbb{N}.$ □

The other properties can be verified similarly. When the modulus is large, it is much easier to employ modular arithmetic. To check for divisibility by seven, we examine the sequence modulo seven and secure the remainders. When we see a term of the sequence with zero remainder, we arrive at our desired goal. The sequence of remainders modulo seven are respectively 3, 1, 4, 5, 2, 0, 2, 2, 4, 6, 3, 2, 5, 0, 5, 5, 3, 1, …. We notice that the sixth, fourteenth, twenty-second and thirtieth terms are divisible by seven and the pattern continues. Once the sequence of remainders enters with successive terms of 3 and 1, we know that the period modulo that integer has been completed. Here the sequence of remainders modulo seven has a period of length sixteen as can easily be viewed. The VOYAGE 200 displays this in **FIGURES 5-8:**

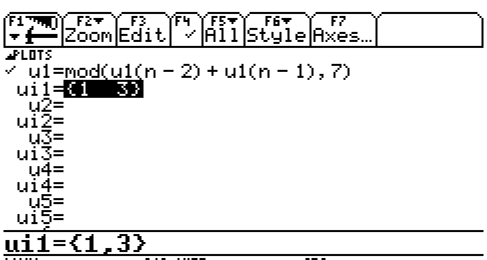

FIGURE 5: Entering the Sequence

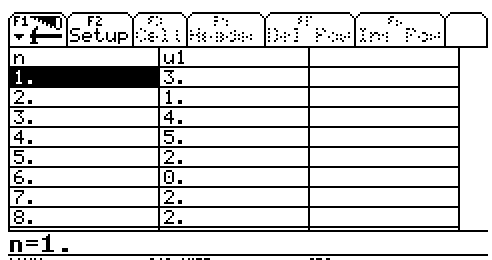

FIGURE 6: Table Displayed

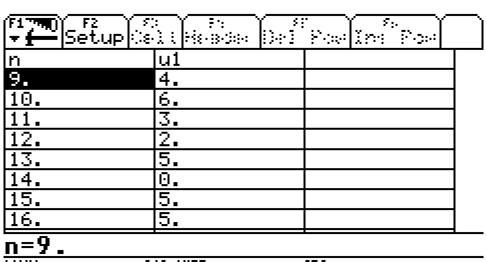

FIGURE 7: Table Displayed

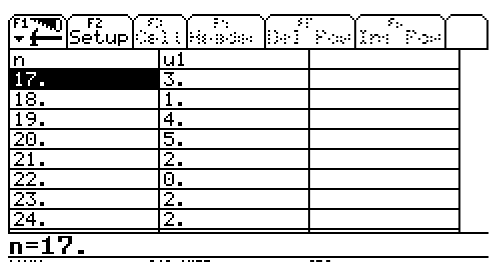

FIGURE 8: Table Displayed

We are asserting the following congruences are valid:

$$3 \equiv 3 \pmod{7},\ 1 \equiv 1 \pmod{7},\ 4 \equiv 4 \pmod{7},\ 5 \equiv 5 \pmod{7},\ 9 \equiv 2 \pmod{7}, 14 \equiv 0 \pmod{7},$$
$$23 \equiv 2 \pmod{7}, 37 \equiv 2 \pmod{7}, 60 \equiv 4 \pmod{7},\ 97 \equiv 6 \pmod{7},\ 157 \equiv 3 \pmod{7}, 254 \equiv 2 \pmod{7},$$
$$411 \equiv 5 \pmod{7}, 665 \equiv 0 \pmod{7}, 1076 \equiv 5 \pmod{7},\ 1741 \equiv 5 \pmod{7},\ 2817 \equiv 3 \pmod{7}, 4558 \equiv 1 \pmod{7},....$$

We notice that the prime 11 does not occur as a factor of any term of the reverse Lucas sequence. This is immediate from modular arithmetic; for the sequence of remainders modulo eleven are respectively 3, 1, 4, 5, 9, 3, 1, … and the sequence of remainders recycles after five terms which is the length of the period modulo eleven. This can be seen below using congruences:

$$3 \equiv 3 \pmod{11},\ 1 \equiv 1 \pmod{11},\ 4 \equiv 4 \pmod{11},\ 5 \equiv 5 \pmod{11},\ 9 \equiv 9 \pmod{11}, 14 \equiv 3 \pmod{11},\ 23 \equiv 1 \pmod{11},....$$

Also see **FIGURES 9-10** using the VOYAGE 200 screen captures.

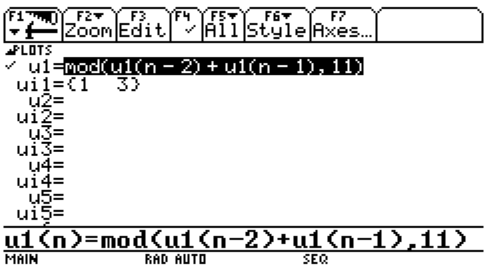

FIGURE 9: Entering the Sequence

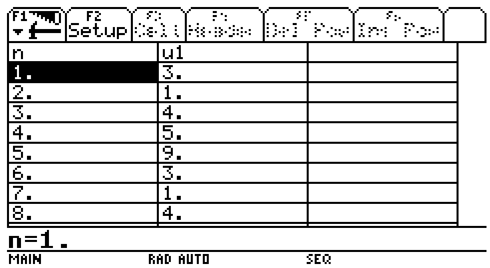

FIGURE 10: Table Displayed

Determining the length of the period of a prime as well as when the prime enters the sequence as a factor (if it does) is best achieved by resorting to modular arithmetic and technology.

The least common multiple greatly enhances obtaining the entry points of divisibility of a term in the sequence and the length of the period modulo the term for composite integer indices. For example, if one desires to determine the entry point for the composite integer 12 as a factor of the terms of the sequence and the length of the period modulo 12, we note that $[3,4]=12.$ We utilize $[a,b]$ as the standard notation for the least common multiple (lcm) of two non-negative integers a and b. The reader can easily check that the sequence of remainders modulo four is respectively 3, 1, 0, 1, 1, 2, 3, 1, …. The length of the period modulo four for this sequence is six. Now since the sequence of remainders modulo three is respectively 0, 1, 1, 2, 0, 2, 2, 1, 0, 1, … , we see that the length of the period modulo three is eight. We note that $3 \equiv 0 \pmod{3}$ (3 and 0 have the same remainder of 0 upon division by three).

Now $[6,8]=24.$ Hence the length of the period modulo twelve for this sequence is 24. This is indeed the case. The reader can check that the sequence of remainders modulo 24 is respectively 3, 1, 4, 5, 9, 2, 11, 1, 0, 1, 1, 2, 3, 5, 8, 1, 9, 10, 7, 5, 0, 5, 5, 10, 3, 1, ….

Observe that since the first, fifth, ninth and thirteenth terms are divisible by three while the third, ninth, fifteenth and twenty-first terms are divisible by four, it is immediate that the ninth term is divisible by twelve (since it is divisible by the relatively prime pair of term integers 3 and 4) which is the least common multiple of three and four. The next three terms that are divisible by twelve are hence the twenty-first, the thirty-third and the forty-fifth.

On the other hand, no term of the sequence is divisible by twenty-two; for an integer to be divisible by twenty-two, the integer must be divisible by both 2 and 11. We have seen that no term in the sequence is divisible by 11. No term of the sequence is divisible by eight; for the sequence of remainders modulo eight are respectively 3, 1, 4, 5, 1, 6, 7, 5, 4, 1, 5, 6, 3, 1, … and recycles after twelve terms so that none of the remainders is ever 0.

No term of the sequence is likewise divisible by twenty-eight; for a term to be divisible by twenty-eight, the term would have to divisible by both four and seven. We note that $S_3, S_9, S_{15}, S_{21}, \ldots$ which are odd numbered terms are divisible by four while the even numbered terms $S_6, S_{14}, S_{22}, S_{30}, \ldots$ are divisible by seven which yields no possible intersection; for no term number can be both even and odd.

In working with the first one thousand counting integers, I found that within the length of the period, if there are factors that enter the sequence, such factors enter the sequence either once, twice or four times. For example, the prime 19 enters the sequence once (at the fourteenth term) within its period of length eighteen modulo nineteen. The prime 7 enters the sequence twice (at the sixth and fourteenth terms) within its period of length sixteen modulo seven. Meanwhile the prime 5 enters the sequence four times (at the fourth, ninth, fourteenth and nineteenth terms) within its period of length twenty modulo five.

We present **TABLE 2** which represents the divisibility and periodicity patterns for the first one hundred terms in the sequence:

TABLE 2: DIVISIBILITY AND PERIODICITY IN THE REVERSE LUCAS SEQUENCE TABLE FOR THE FIRST FIFTY INTEGERS

n:	**Length of Period in Z_n:**	**Values of S_n when n is a Factor:**
1	**---**	**ALL**
2	**3**	$S_3, S_6, S_9, S_{12}, \ldots$
3	**8**	$S_1, S_5, S_9, S_{13}, \ldots$
4	**6**	$S_3, S_9, S_{15}, S_{21}, \ldots$
5	**20**	$S_4, S_9, S_{14}, S_{19}, \ldots$

6	**24**	$S_9, S_{21}, S_{33}, S_{45}, \ldots$
7	**16**	$S_6, S_{14}, S_{22}, S_{30}, \ldots$
8	**12**	**NONE**
9	**24**	$S_5, S_{17}, S_{29}, S_{41}, \ldots$
10	**60**	$S_9, S_{24}, S_{39}, S_{54}, \ldots$
11	**5**	**NONE**
12	**24**	$S_9, S_{21}, S_{33}, S_{45}, \ldots$
13	**28**	**NONE**
14	**48**	$S_6, S_{30}, S_{54}, S_{78}, \ldots$
15	**40**	$S_9, S_{29}, S_{49}, S_{69}, \ldots$
16	**24**	**NONE**
17	**36**	**NONE**
18	**24**	**NONE**
19	**18**	$S_{14}, S_{32}, S_{50}, S_{68}, \ldots$
20	**60**	$S_9, S_{39}, S_{69}, S_{99}, \ldots$
21	**16**	**NONE**
22	**15**	**NONE**
23	**48**	$S_7, S_{31}, S_{55}, S_{79}, \ldots$
24	**24**	**NONE**
25	**100**	$S_{19}, S_{44}, S_{69}, S_{94}, \ldots$
26	**84**	**NONE**
27	**72**	$S_{29}, S_{65}, S_{101}, S_{137}, \ldots$

28	**48**	**NONE**
29	**14**	**NONE**
30	**120**	$S_9, S_{69}, S_{129}, S_{189}, \ldots$
31	**30**	$S_{25}, S_{55}, S_{85}, S_{115}, \ldots$
32	**48**	**NONE**
33	**40**	**NONE**
34	**36**	**NONE**
35	**80**	$S_{14}, S_{54}, S_{94}, S_{134}, \ldots$
36	**24**	**NONE**
37	**76**	$S_8, S_{27}, S_{46}, S_{65}, \ldots$
38	**18**	**NONE**
39	**56**	**NONE**
40	**60**	**NONE**
41	**40**	**NONE**
42	**48**	**NONE**
43	**88**	$S_{18}, S_{62}, S_{106}, S_{150}, \ldots$
44	**30**	**NONE**
45	**120**	$S_{29}, S_{89}, S_{149}, S_{209}, \ldots$
46	**48**	**NONE**
47	**32**	**NONE**
48	**24**	**NONE**
49	**112**	$S_{46}, S_{102}, S_{158}, S_{214}, \ldots$
50	**300**	$S_{69}, S_{144}, S_{219}, S_{294}, \ldots$

In **TABLE 3,** we show the divisibility and periodicity patterns for all primes less than five hundred with regards to this sequence.

TABLE 3: TABLE OF DIVISIBILITY AND PERIODICITY FOR THE REVERSE LUCAS SEQUENCE OF PRIMES <500.

p:	**Length of Period in** Z_p:	**Values of** S_p **where** p **is a Factor:**
2	**3**	$S_3, S_6, S_9, S_{12}, \ldots$
3	**8**	$S_1, S_5, S_9, S_{13}, \ldots$
5	**20**	$S_4, S_9, S_{14}, S_{19}, \ldots$
7	**16**	$S_6, S_{14}, S_{22}, S_{30}, \ldots$
11	**5**	**NONE**
13	**28**	**NONE**
17	**36**	**NONE**
19	**18**	$S_{14}, S_{32}, S_{50}, S_{68}, \ldots$
23	**48**	$S_7, S_{31}, S_{55}, S_{79}, \ldots$
29	**14**	**NONE**
31	**30**	$S_{25}, S_{55}, S_{85}, S_{115}, \ldots$
37	**76**	$S_8, S_{27}, S_{46}, S_{65}, \ldots$
41	**40**	**NONE**
43	**88**	$S_{18}, S_{62}, S_{106}, S_{150}, \ldots$
47	**32**	**NONE**
53	**108**	$S_{18}, S_{45}, S_{72}, S_{99}, \ldots$
59	**58**	$S_{19}, S_{77}, S_{135}, S_{193}, \ldots$
61	**60**	**NONE**

67	**136**	$S_{43}, S_{111}, S_{179}, S_{247}, \ldots$
71	**70**	$S_{47}, S_{117}, S_{187}, S_{257}, \ldots$
73	**148**	**NONE**
79	**78**	$S_{30}, S_{108}, S_{186}, S_{264}, \ldots$
83	**168**	$S_{50}, S_{134}, S_{218}, S_{302}, \ldots$
89	**44**	**NONE**
97	**196**	$S_{10}, S_{59}, S_{108}, S_{157}, \ldots$
101	**50**	**NONE**
103	**208**	$S_{77}, S_{181}, S_{285}, S_{389}, \ldots$
107	**72**	**NONE**
109	**108**	**NONE**
113	**76**	**NONE**
127	**256**	$S_{12}, S_{140}, S_{268}, S_{396}, \ldots$
131	**130**	$S_{122}, S_{252}, S_{382}, S_{512}, \ldots$
137	**276**	$S_{13}, S_{82}, S_{151}, S_{220}, \ldots$
139	**46**	**NONE**
149	**148**	**NONE**
151	**50**	**NONE**
157	**316**	$S_{11}, S_{90}, S_{169}, S_{248}, \ldots$
163	**328**	$S_{113}, S_{277}, S_{441}, S_{605}, \ldots$
167	**336**	$S_{48}, S_{216}, S_{384}, S_{552}, \ldots$
173	**348**	**NONE**

179	**178**	$S_{69}, S_{247}, S_{425}, S_{603}, \ldots$
181	**90**	$S_{38}, S_{128}, S_{218}, S_{308}, \ldots$
191	**190**	$S_{123}, S_{313}, S_{503}, S_{693}, \ldots$
193	**388**	**NONE**
197	**396**	**NONE**
199	**22**	**NONE**
211	**42**	**NONE**
223	**448**	$S_{31}, S_{255}, S_{479}, S_{703}, \ldots$
227	**456**	$S_{34}, S_{262}, S_{490}, S_{718}, \ldots$
229	**114**	$S_{73}, S_{187}, S_{301}, S_{415}, \ldots$
233	**52**	**NONE**
239	**238**	$S_{138}, S_{376}, S_{614}, S_{852}, \ldots$
241	**240**	**NONE**
251	**250**	$S_{113}, S_{363}, S_{613}, S_{863}, \ldots$
257	**516**	$S_{78}, S_{207}, S_{336}, S_{465}, \ldots$
263	**176**	**NONE**
269	**268**	$S_{15}, S_{82}, S_{149}, S_{216}, \ldots$
271	**270**	$S_{76}, S_{346}, S_{616}, S_{886}, \ldots$
277	**556**	**NONE**
281	**56**	**NONE**
283	**568**	$S_{94}, S_{378}, S_{662}, S_{946}, \ldots$
293	**588**	**NONE**

307	**88**	**NONE**
311	**310**	$S_{95}, S_{405}, S_{715}, S_{1025}, \ldots$
313	**628**	$S_{17}, S_{174}, S_{331}, S_{488}, \ldots$
317	**636**	$S_{144}, S_{303}, S_{462}, S_{621}, \ldots$
331	**110**	**NONE**
337	**676**	**NONE**
347	**232**	**NONE**
349	**174**	**NONE**
353	**236**	**NONE**
359	**358**	$S_{108}, S_{466}, S_{824}, S_{1182}, \ldots$
367	**736**	$S_{75}, S_{443}, S_{811}, S_{1179}, \ldots$
373	**748**	**NONE**
379	**378**	$S_{33}, S_{411}, S_{789}, S_{1167}, \ldots$
383	**768**	$S_{293}, S_{677}, S_{1061}, S_{1445}, \ldots$
389	**388**	**NONE**
397	**796**	$S_{166}, S_{365}, S_{564}, S_{763}, \ldots$
401	**200**	$S_{65}, S_{265}, S_{465}, S_{665}, \ldots$
409	**408**	**NONE**
419	**418**	$S_{257}, S_{675}, S_{1093}, S_{1511}, \ldots$
421	**84**	**NONE**
431	**430**	$S_{320}, S_{750}, S_{1180}, S_{1610}, \ldots$
433	**868**	$S_{200}, S_{417}, S_{634}, S_{851}, \ldots$

439	**438**	$S_{372}, S_{810}, S_{1248}, S_{1686}, \ldots$
443	**888**	$S_{269}, S_{713}, S_{1157}, S_{1601}, \ldots$
449	**448**	$S_{212}, S_{436}, S_{660}, S_{884}, \ldots$
457	**916**	**NONE**
461	**46**	**NONE**
463	**928**	$S_{31}, S_{495}, S_{959}, S_{1423}, \ldots$
467	**936**	$S_{195}, S_{663}, S_{1131}, S_{1599}, \ldots$
479	**478**	$S_{190}, S_{668}, S_{1146}, S_{1624}, \ldots$
487	**976**	$S_{63}, S_{551}, S_{1039}, S_{1527}, \ldots$
491	**490**	$S_{350}, S_{840}, S_{1330}, S_{1820}, \ldots$
499	**498**	$S_{59}, S_{557}, S_{1055}, S_{1553}, \ldots$

3. Palatable Number Tricks Associated With The Reverse Lucas Sequence.

In light of the fact that the Reverse Lucas Sequence is a Fibonacci-style sequence (with the exception of the initial two terms, but follows the Fibonacci recursion rule), all of the palatable number tricks that work for the Fibonacci sequence likewise hold for this sequence including the following:

a. If one considers the sum of any ten consecutive terms in the sequence, forms the sum and divides the sum by eleven, then the quotient will always be the seventh term in the sequence.

b. Suppose one consider any four consecutive terms in the Fibonacci sequence and applies the following three simple steps. First form the product of the first and fourth terms. Take twice the product of the second and third terms. Finally take the sum of the squares of the second and third terms in your sequence. We relate this to plane geometry and observe that a Pythagorean triple is obtained.

c. We finally take the ratio of each even numbered term to the previous odd-numbered term and form a conjecture and then take the ratio of each odd numbered term to the previous even numbered term and form a conjecture. What appears to be happening to these ratios is remarkable. These sequence of ratios appear to be approaching the Golden Ratio!

Let us prove each of these assertions. The reader is invited to first gather some empirical evidence to see that the above conclusions are palatable. We first illustrate with two examples.

a. Suppose the ten consecutive terms in the sequence are $\{5, 9, 14, 23, 37, 60, 97, 157, 254, 411\}$. Using the VOYAGE 200 in **FIGURES 11-12,** we obtain the following:

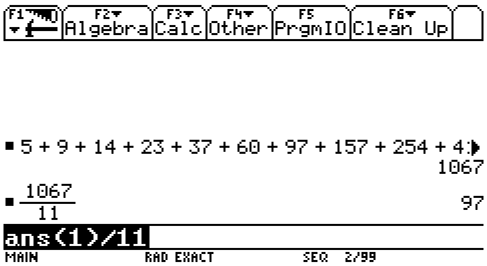

FIGURE 11: Entering The Terms and Dividing the Sum by 11

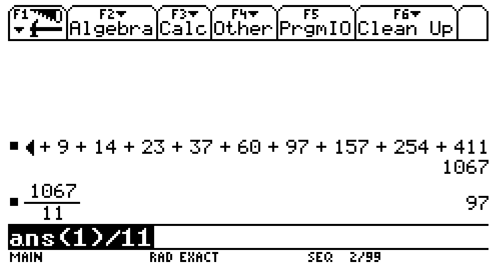

FIGURE 12: Entering the Terms and Dividing Dividing the Sum by 11

Note that 97 is the seventh term in the sequence.

As a second example, suppose the ten consecutive terms in the sequence are $\{3, 1, 4, 5, 9, 14, 23, 37, 60, 97\}$. Using the VOYAGE 200 in **FIGURE 13,** we obtain the following:

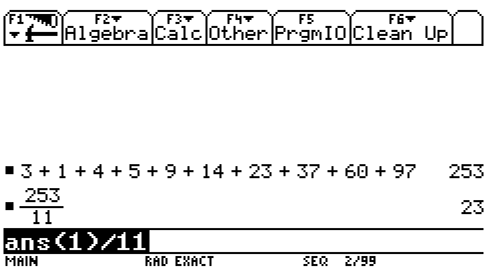

FIGURE 13: Entering the Terms and Dividing the Sum by 11

Note that 23 is the seventh term in the sequence.

Based on the empirical evidence supported by the above examples, one might desire to venture the following conjecture:

<u>Conjecture</u>: The sum of any ten consecutive terms in the sequence is always divisible by eleven. The quotient is always the *seventh term* in the sequence.

Proof: Suppose the initial two terms in the sequence are x and y respectively. Then the ten terms of the sequence (all of whose coefficients are indeed Fibonacci numbers) comprises the following set:

$\{x, y, x+y, x+2\cdot y, 2\cdot x+3\cdot y, 3\cdot x+5\cdot y, 5\cdot x+8\cdot y, 8\cdot x+13\cdot y, 13\cdot x+21\cdot y, 21\cdot x+34\cdot y\}$. Let us employ the VOYAGE 200 to furnish a proof of the above assertion. See **FIGURES 14-16:**

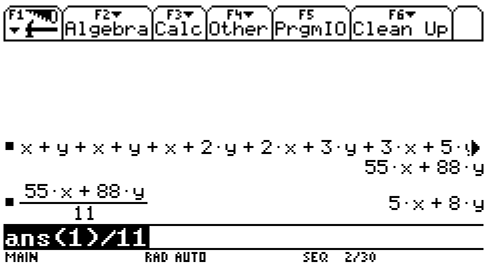

FIGURE 14: General Proof

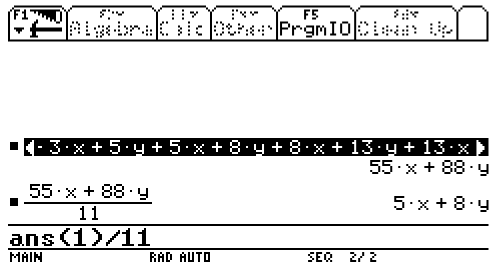

FIGURE 15: General Proof

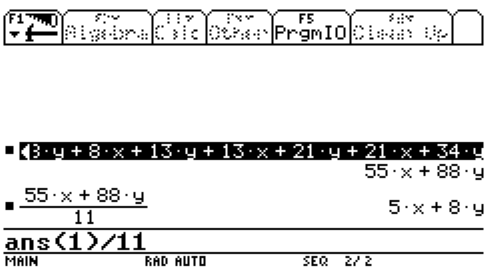

FIGURE 16: General Proof

Observe that $5\cdot x+8\cdot y$ constitutes the seventh term in the sequence.

b. We now consider any four consecutive terms in the sequence. First form the product of the first and fourth terms. Take twice the product of the second and third terms. Finally take the sum of the squares of the second and third terms in your sequence. We relate this to a theorem in plane geometry by conjecturing based on several examples, and then substantiate our conjecture.

Example 1: Suppose the four terms are $\{5, 9, 14, 23\}$. **FIGURE 17** displays our data:

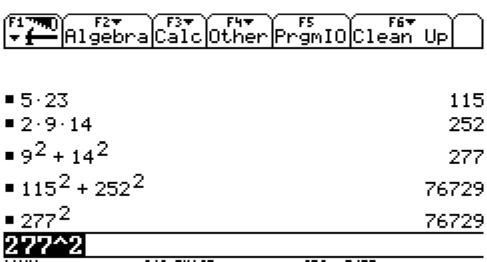

FIGURE 17: Four Consecutive Terms in the Sequence

Observe that the primitive Pythagorean triplet $\{115, 252, 277\}$ is formed. By primitive, we mean that no two integers in the triplet have a common factor higher than one.

Example 2: Suppose the four terms are $\{3,1,4,5\}$. See **FIGURE 18** for the relevant data:

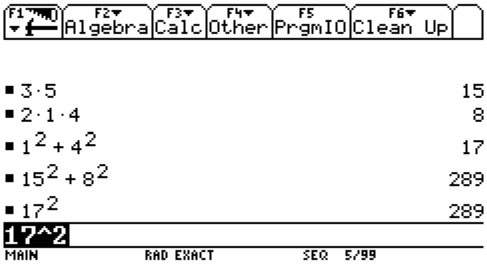

FIGURE 18: Four Consecutive Terms in the Sequence

Observe that the primitive Pythagorean triplet $\{15,8,17\}$ is formed which is likely recognizable to most readers.

Let us prove the conjecture is true in general by considering any four consecutive terms in the sequence. Let the terms of this generic sequence be as follows:

$\{x, y, x+y, x+2\cdot y\}$. **FIGURE 19** provides the expand key required to multiply algebraic expressions while **FIGURES 20-23** provides out inputs and outputs:

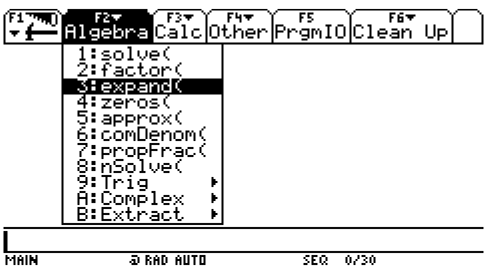

FIGURE 19: The Formal Proof

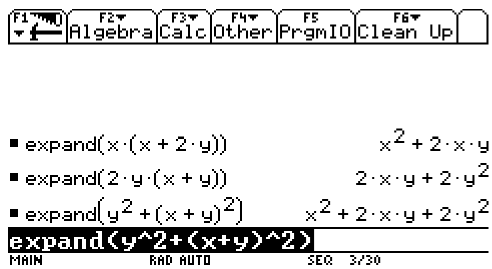

FIGURE 20: The Formal Proof

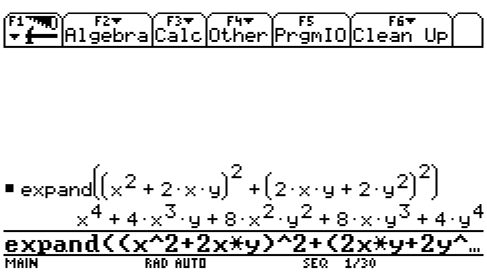

FIGURE 21: The Formal Proof

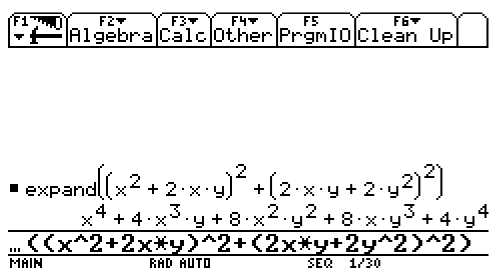

FIGURE 22: The Formal Proof

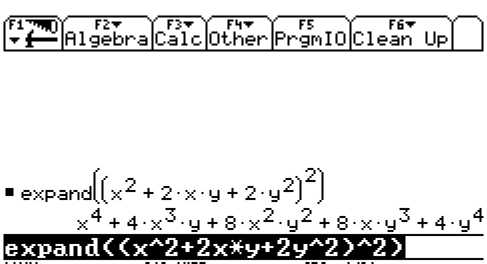

FIGURE 23: The Formal Proof

Hence we view the Pythagorean triplet $\left\{x^2+2\cdot x\cdot y, 2\cdot x\cdot y+2\cdot y^2, x^2+2\cdot x\cdot y+2\cdot y^2\right\}$.

c. We finally take the ratio of each even numbered term to the previous odd-numbered term in the sequence and form a conjecture as well as take the ratio of each odd numbered term to the previous even numbered term in the sequence and form a conjecture. We see what appears to be happening to these ratios. In **FIGURES 25-28,** we consider the ratios of even numbered terms to the previous odd numbered terms while in **FIGURES 29-32,** we consider the ratios of odd numbered terms to the previous even numbered terms. Our MODE is APPROXIMATE as in **FIGURE 24** and proceed on Page 2 down to 3: APPROXIMATE.

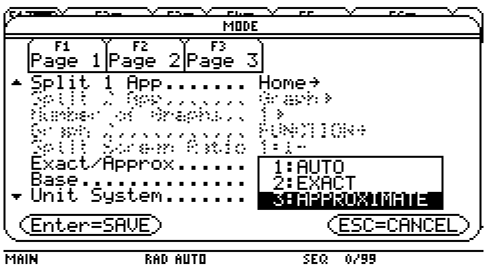

FIGURE 24: The Approximate Mode

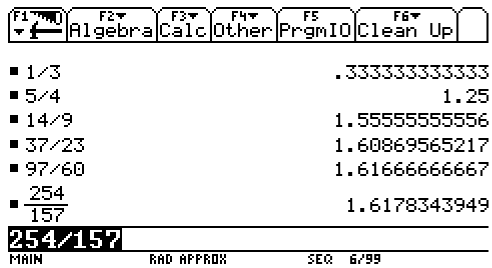

FIGURE 25: Ratios of Successive Terms

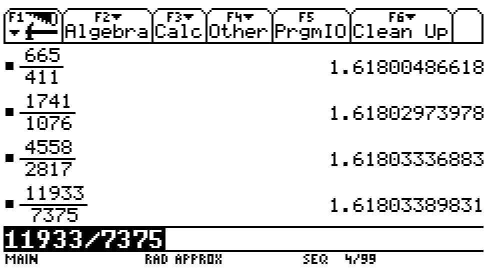

FIGURE 26: Ratios of Successive Terms

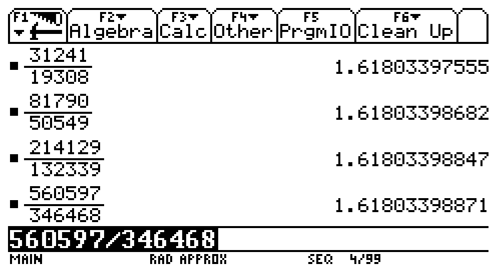

FIGURE 27: Ratios of Successive Terms

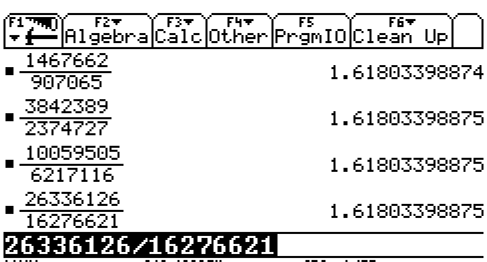

FIGURE 28: Ratios of Successive Terms

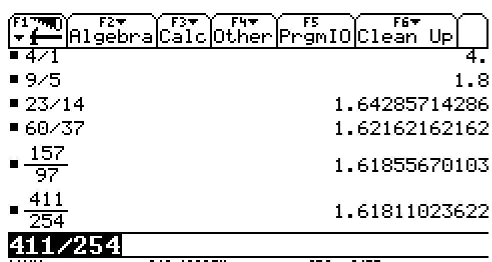

FIGURE 29: Ratios of Successive Terms

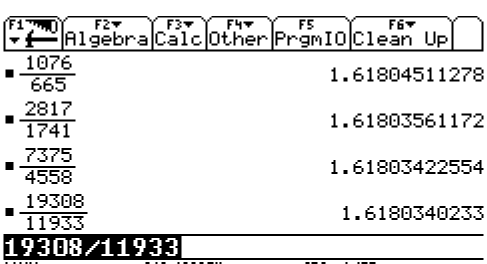

FIGURE 30: Ratios of Successive Terms

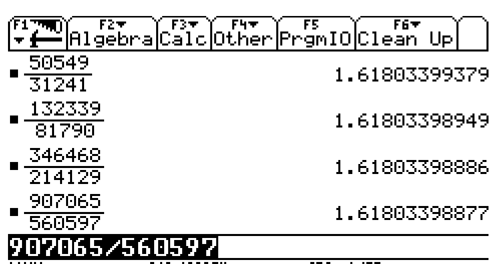

FIGURE 31: Ratios of Successive Terms

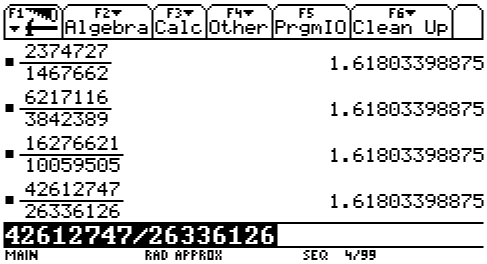

FIGURE 32: Ratios of Successive Terms

It appears that the ratio of the even numbered terms to the preceding odd numbered terms forms a monotonically increasing sequence while the ratio of the odd numbered terms to the preceding even numbered terms forms a monotonically decreasing sequence and both sequences are approaching the same number, namely $\phi = \frac{1+\sqrt{5}}{2} \approx 1.61803398875!$ Many of us will recognize that this constant represents ***The Golden Mean or The Golden Ratio.*** In fact, all of the number tricks in this section work for the Fibonacci sequence and any Fibonacci-like sequence in which the initial two terms may assume any value and the sequence then follows the Fibonacci recursion rule. Our sequence, of course, is a Fibonacci-like sequence. Let us prove that the ratio of successive terms in any Fibonacci-like sequence approaches the Golden Ratio constant.

<u>Proof:</u> Let the initial two terms of the Fibonacci-like sequence be $x \ and \ y$ respectively. Then the next terms are respectively

$x+y,\ x+2\cdot y,\ 2\cdot x+3\cdot y,\ 3\cdot x+5\cdot y,\ 5\cdot x+8\cdot y,\ \ldots,\ F_{n-1}\cdot x+F_n\cdot y,\ F_n\cdot x+F_{n+1}\cdot y,\ \ldots.$

Now if we take the ratio r of two consecutive terms, we obtain $r = \frac{F_n \cdot x + F_{n+1}\cdot y}{F_{n-1}\cdot x + F_n \cdot y}$. Let us divide each term in the numerator and denominator by F_n. Then

$$r = \frac{\left[F_n \cdot x + F_{n+1}\cdot y\right]/F_n}{\left[F_{n-1}\cdot x + F_n \cdot y\right]/F_n} = \frac{F_n \cdot x/F_n + F_{n+1}\cdot y/F_n}{F_{n-1}\cdot x/F_n + F_n\cdot y/F_n} = \frac{x + \frac{F_{n+1}}{F_n}\cdot y}{\frac{F_{n-1}}{F_n}\cdot x + y}.$$

Now $\lim\limits_{n\to+\infty} \frac{F_{n+1}}{F_n} = \phi \ and \ \lim\limits_{n\to+\infty} \frac{F_{n-1}}{F_n} = \frac{1}{\phi}$, where ϕ is the Golden Ratio.

$$\lim_{n\to+\infty} r = \lim_{n\to+\infty}\left[\frac{x + \frac{F_{n+1}}{F_n}\cdot y}{\frac{F_{n-1}}{F_n}\cdot x + y}\right] = \frac{x+\phi\cdot y}{\left(\frac{x}{\phi}+y\right)} = \frac{x+\phi\cdot y}{\frac{x+\phi\cdot y}{\phi}} = \frac{1}{\frac{1}{\phi}} = \phi.$$

Hence the ratio of consecutive terms tends to The Golden Ratio, completing the proof.

4. Concluding Remarks and Further Directions.

In this paper, we explored divisibility patterns and primes in the reverse Lucas sequence which included securing the complete factorizations for the initial one hundred terms in the sequence. I have been able to secure the prime factorizations of the first four hundred seventy-five terms of the sequence. Determining whether an integer is prime can be achieved in polynomial time. In contrast, the actual factorization of large composite integers is an NP Hard problem that cannot be achieved in this manner; for things are growing exponentially. Factoring a large integer is contingent upon the second largest prime factor and if this factor contains more than thirty digits, it becomes a challenge for even the best machines to achieve a complete factorization. Hence breaking a problem into a smaller sub problem is the manner in which one needs to proceed to obtain additional fruitful outcomes in this endeavor and at least obtain partial factorizations generating small prime factors.

In addition, in the Fibonacci sequence 1, 1, 2, 3, 5, 8, …, by the n^2 term, n appears as a factor of some term in the sequence. In contrast, one-third of the primes and 746 of the first one thousand integers never appear as factors of any term in the Lucas sequence. For the Reverse Lucas sequence, of the 168 primes less than 1000, 72 do not appear as factors of any term in the sequence and 721 of the first one thousand counting integers do not either. All three sequences enjoy the Fibonacci number tricks discussed in this paper and many others.

5. References.

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