USING CAS TECHNOLOGY AND COUNTEREXAMPLES TO ALLEVIATE MISCONCEPTIONS IN CALCULUS

Jay L. Schiffman

Rowan University

201 Mullica Hill Road, Glassboro, NJ 08028

schiffman@rowan.edu

Abstract: Calculus students often use rote memorization without conceptual understanding. This paper utilizes CAS technology and counterexamples to help alleviate these pitfalls.

1. Calculus Misconceptions.

In this section, we consider eighteen misconceptions involving calculus ideas dealing with limits, continuity, differentiation, integration and infinite sequences and series. The misconceptions are dealt with in order.

1. If $\lim_{x\to c^-} f(x)$ and $\lim_{x\to c^+} f(x)$ both exist, and moreover, $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x)$, then $f(x)$ is continuous $@\, x = c.$

2. If f is not defined $@\, x = c,$ then $\lim_{x\to c} f(x)$ does not exist.

3. If $\lim_{x\to c} f(x) = L = \lim_{x\to c} g(x)$, then $f(x) \equiv g(x)$ $(f \text{ and } g \text{ are idntical})$.

4. If $f(x) = \dfrac{p(x)}{q(x)}$, where p *and* q are polynomials in the indeterminate $x,$ then it is necessarily the case that $\forall a \in \mathbb{R}$ where $q(a) = 0,$ then $x = a$ is a vertical asymptote to the graph of f.

5. An algebraic function cannot possess more than one horizontal asymptote to its graph.

6. A rational function can never intersect its horizontal or oblique asymptote, but always has one of these types of asymptotes.

7. The derivative of a product of two differentiable functions coincides with the product of their respective derivatives while the derivative of a quotient of two differentiable functions coincides with the quotient of their respective derivatives.

8. The n^{th} derivative of any polynomial function $p(x)$ necessarily vanishes.

9. If the first n derivatives of f (where f is a function of a real variable) exist on $[a,b]$, then it is necessarily the case that the $(n+1)^{st}$ derivative likewise exists on $[a,b]$.

10. A continuous function f over any type interval (open, closed, half-open, or infinite) must necessarily assume both its maximum and minimum values on the interval.

11. Any function f defined over $[a,b]$ necessarily assumes both of its extreme values somewhere over $[a,b]$.

12. A continuous function f defined on any closed interval necessarily assumes both of its extreme values somewhere on the interval.

13. The indefinite integral of an even function is necessarily odd.

14. If $\lim\limits_{n\to\infty} u_n = 0,$ then $\sum\limits_{n=1}^{+\infty} u_n$ converges.

15. Consider the harmonic series $\sum\limits_{n=1}^{+\infty} \frac{1}{n}$ and thin out the series by removing the first one million terms. The resulting series will converge.

16. The series $\sum\limits_{n=1}^{+\infty} \frac{1}{n^2}$ *and the improper* int*egral* $\int_1^{+\infty} \frac{dx}{x^2}$ have the same numerical value.

17. The sum of two divergent series is divergent.

18. Every bounded sequence converges and every monotonic sequence converges.

2. Analysis of the Misconceptions.

Let us examine, in turn, each of these. The examples illustrate a number of important ideas including that hypotheses matter and have consequences. The idea of exploring the why is extremely essential for students to appreciate early in their mathematical development.

1. Consider the function f defined as follows:

$$f(x) = \begin{cases} \dfrac{x^2-1}{x-1}, & x \neq 1 \\ 1, & x = 1 \end{cases}.$$

We note that

$$\lim_{x\to1^-} f(x) = \lim_{x\to1^-}\frac{x^2-1}{x-1} = \lim_{x\to1^-}\frac{(x+1)\cdot(x-1)}{x-1} = \lim_{x\to1^-}(x+1) = \lim_{x\to1^-} x + \lim_{x\to1^-} 1 = 1+1 = 2.$$

Meanwhile

$$\lim_{x\to1^+} f(x) = \lim_{x\to1^+}\frac{x^2-1}{x-1} = \lim_{x\to1^+}\frac{(x+1)\cdot(x-1)}{x-1} = \lim_{x\to1^+}(x+1) = \lim_{x\to1^+} x + \lim_{x\to1^+} 1 = 1+1 = 2.$$

Thus $\lim_{x\to1^-} f(x) = 2 = \lim_{x\to1^+} f(x) = \lim_{x\to1} f(x).$

On the other hand, $f(1) = 1 \neq 2 = \lim_{x\to1} f(x).$ Hence $f(x)$ is discontinuous @ $x = 2.$ Such a discontinuity is called a ***removable discontinuity*** since we can redefine the function so that f would be continuous @ $x = 2$ as follows:

$$f(x) = \left\{\begin{matrix} \frac{x^2-1}{x-1}, & x \neq 1 \\ 2, & x = 1 \end{matrix}\right\}.$$

FIGURES 1-6 display the graphing features for this function and **FIGURE 7** the symbolic capability of the CAS calculator:

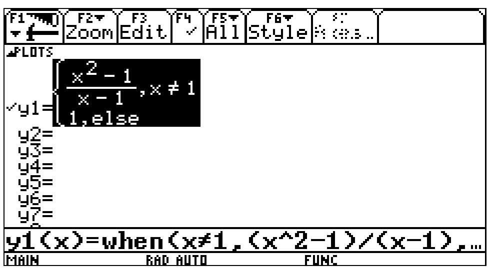


FIGURE 1: The Piecewise Function

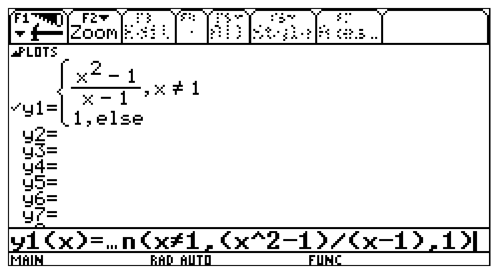


FIGURE 2: The Piecewise Function

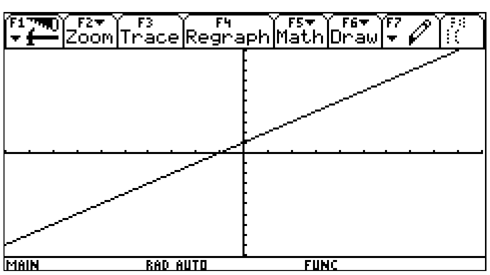


FIGURE 3: Graph of The Function

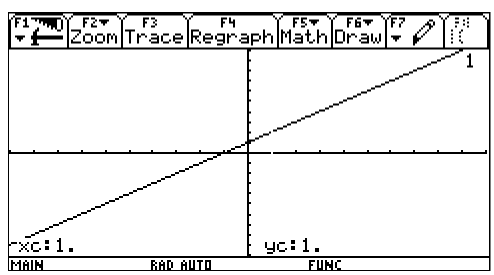


FIGURE 4: A Trace Value

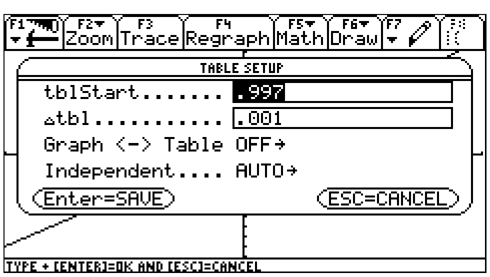


FIGURE 5: The Table Setup

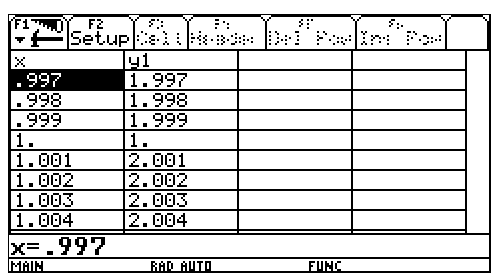


FIGURE 6: The Table

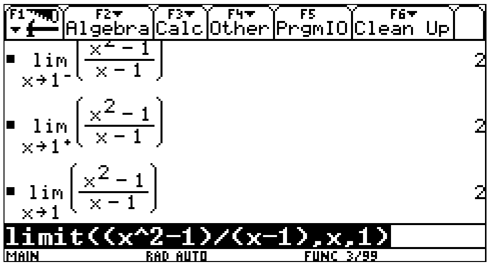


FIGURE 7: Symbolic Representation Using CAS

2. We next consider the function f defined by $f(x)=\dfrac{x^2-1}{x-1}$. Observe that f is not defined @ $x=1$. (Note that $f(1)=\dfrac{1^2-1}{1-1}=\dfrac{1-1}{1-1}=\dfrac{0}{0}$. Nonetheless, $\lim\limits_{x\to 1} f(x)\ \exists\ and\ \lim\limits_{x\to 1} f(x)=2.$)

The limit of a function as the independent variable approaches the point has nothing to do with the value of the function at the point!

3. It is not true that if $\lim\limits_{x\to c} f(x)=L=\lim\limits_{x\to c} g(x)$, then $f(x)\equiv g(x)$. Recall that two functions

f *and* g are equal, written $f=g$, if f *and* g have a common domain D and

$f(x)\equiv g(x)\ \forall x\in D.$ To cite an example, let $f(x)=\dfrac{x^3-1}{x-1}$ *and* $g(x)=x^2+x+1.$

Observe that $f(x)=\dfrac{(x-1)\cdot(x^2+x+1)}{x-1}=x^2+x+1=g(x)$ for $x\neq 1$ while $f(1)$ does not exist. On the other hand, $g(1)=3.$ In addition,

$$\lim_{x\to 1} f(x)=\lim_{x\to 1}\frac{x^3-1}{x-1}=\lim_{x\to 1}\frac{(x-1)\cdot(x^2+x+1)}{x-1}=\lim_{x\to 1}(x^2+x+1)=(1)^2+1+1=3=\lim_{x\to 1} g(x).$$

Observe that f *and* g are two functions that agree at all points with the exception of $x=c$, then $\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c} g(x)$. Let us furnish graphical and symbolic analysis for the statements above. See **FIGURES 8-15:**

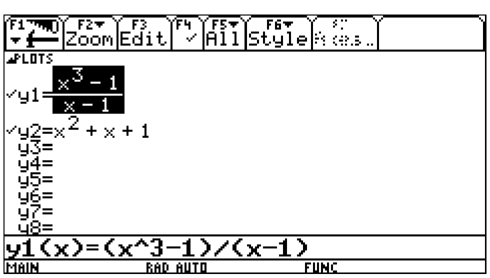


FIGURE 8: The Function Inputs

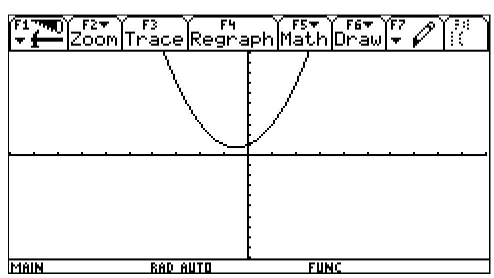


FIGURE 9: The Graph

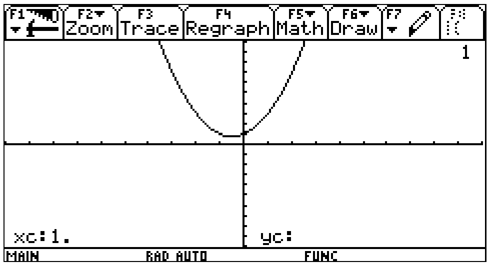


FIGURE 10: A Trace Value for y_1

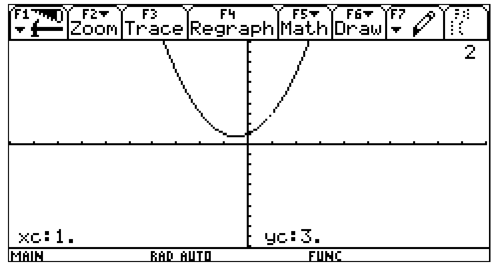


FIGURE 11: A Trace Value for y_2

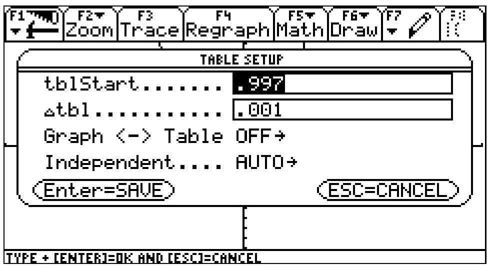


FIGURE 12: The Table Setup

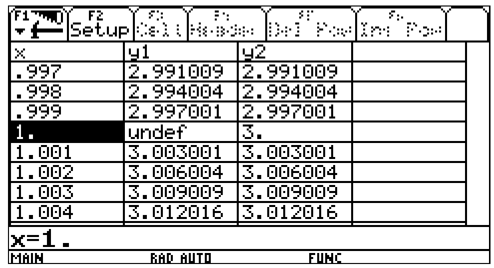


FIGURE 13: The Table

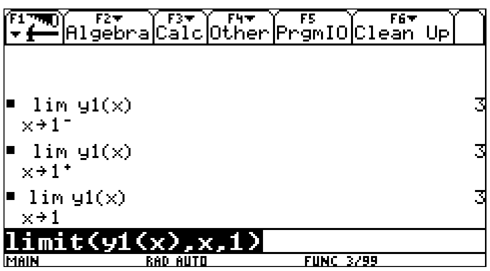


FIGURE 14: CAS Capabilities

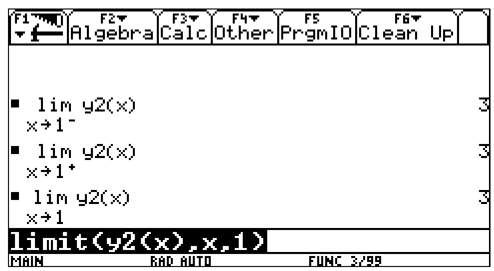


FIGURE 15: CAS Capabilities

Note that the one-sided limits have four arguments following the limit command which is accessed via the keystrokes F3, 3 from the HOME SCREEN: the function, the independent variable, the number the independent variable is approaching, and the direction (any negative number for the left-hand limit and any positive number for the right-hand limit). Limits without regards to direction have the first three arguments only.

4. If $f(x)=\dfrac{p(x)}{q(x)}$, where p *and* q are polynomials in the indeterminate x, then it is not necessarily the case that $\forall a \in D_f$ where $q(a)=0$, then $x=a$ is a vertical asymptote to the graph of f. Let $f(x)=\dfrac{x^3-1}{x-1}$ as in Example 4 above. Here $f(x)=\dfrac{p(x)}{q(x)}$ where $p(x)=x^3-1$ *and* $q(x)=x-1$. Observe that if $x=1$, $p(1)=1^3-1=0=1-1=q(1)$. In order for $x=1$ to be a vertical asymptote of f, $p(1)\neq 0$ *while* $q(1)=0$. Note that $x=1$ is a removable discontinuity of f. If we redefine f as follows, then f will be continuous @ $x=1$:

$$f(x)=\begin{cases}\dfrac{x^3-1}{x-1}, & x\neq 1\\ 3, & x=1\end{cases}.$$

5. While a rational function can possess only one horizontal asymptote to its graph if it has horizontal asymptotes at all, it is indeed the case that an algebraic non-rational function may possess two different horizontal asymptotes, one from the left and one from the right. To cite an example, consider $f(x)=\dfrac{3\cdot x-2}{\sqrt{2\cdot x^2+1}}$. Now

$$\text{if } x>0, \text{ then } \lim_{x\to+\infty} f(x)=\lim_{x\to+\infty}\frac{3\cdot x-2}{\sqrt{2\cdot x^2+1}}=\lim_{x\to+\infty}\frac{(3\cdot x-2)/x}{\left(\sqrt{2\cdot x^2+1}\right)/\sqrt{x^2}}=\lim_{x\to+\infty}\frac{3-\frac{2}{x}}{\sqrt{2+\frac{1}{x^2}}}=$$

$$\frac{\lim\limits_{x\to+\infty}\left(3-\frac{2}{x}\right)}{\lim\limits_{x\to+\infty}\left(\sqrt{2+\frac{1}{x^2}}\right)}=\frac{\lim\limits_{x\to+\infty}3-\lim\limits_{x\to+\infty}\frac{2}{x}}{\sqrt{\lim\limits_{x\to+\infty}\left(2+\frac{1}{x^2}\right)}}=\frac{\lim\limits_{x\to+\infty}3-\lim\limits_{x\to+\infty}\frac{2}{x}}{\sqrt{\lim\limits_{x\to+\infty}2+\lim\limits_{x\to+\infty}\frac{1}{x^2}}}=\frac{\lim\limits_{x\to+\infty}3-2\cdot\lim\limits_{x\to+\infty}\frac{1}{x}}{\sqrt{\lim\limits_{x\to+\infty}2+\lim\limits_{x\to+\infty}\frac{1}{x^2}}}=\frac{\lim\limits_{x\to+\infty}3-2\cdot\lim\limits_{x\to+\infty}\frac{1}{x}}{\sqrt{\lim\limits_{x\to+\infty}2+\lim\limits_{x\to+\infty}\left(\frac{1}{x}\right)^2}}=$$

$$\frac{3-2\cdot 0}{\sqrt{2+0^2}}=\frac{3-0}{\sqrt{2+0}}=\frac{3}{\sqrt{2}}=\frac{3}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}=\frac{3\cdot\sqrt{2}}{2}.$$

On the other hand,

$$\text{if } x<0, \text{ then } \lim_{x\to-\infty} f(x)=\lim_{x\to-\infty}\frac{3\cdot x-2}{\sqrt{2\cdot x^2+1}}=\lim_{x\to-\infty}\frac{(3\cdot x-2)/x}{\left(\sqrt{2\cdot x^2+1}\right)/-\sqrt{x^2}}=\lim_{x\to-\infty}\frac{3-\frac{2}{x}}{-\sqrt{2+\frac{1}{x^2}}}=$$

$$\frac{\lim\limits_{x\to-\infty}\left(3-\frac{2}{x}\right)}{\lim\limits_{x\to-\infty}\left(-\sqrt{2+\frac{1}{x^2}}\right)}=\frac{\lim\limits_{x\to-\infty}3-\lim\limits_{x\to\infty}\frac{2}{x}}{-\sqrt{\lim\limits_{x\to-\infty}\left(2+\frac{1}{x^2}\right)}}=\frac{\lim\limits_{x\to-\infty}3-\lim\limits_{x\to-\infty}\frac{2}{x}}{-\sqrt{\lim\limits_{x\to-\infty}2+\lim\limits_{x\to-\infty}\frac{1}{x^2}}}=\frac{\lim\limits_{x\to-\infty}3-2\cdot\lim\limits_{x\to-\infty}\frac{1}{x}}{-\sqrt{\lim\limits_{x\to-\infty}2+\lim\limits_{x\to-\infty}\frac{1}{x^2}}}=\frac{\lim\limits_{x\to-\infty}3-2\cdot\lim\limits_{x\to-\infty}\frac{1}{x}}{-\sqrt{\lim\limits_{x\to-\infty}2+\lim\limits_{x\to-\infty}\left(\frac{1}{x}\right)^2}}=$$

$$\frac{3-2\cdot 0}{-\sqrt{2+0^2}}=\frac{3-0}{-\sqrt{2+0}}=\frac{3}{-\sqrt{2}}=\frac{3}{-\sqrt{2}}\cdot\frac{\sqrt{2}}{-\sqrt{2}}=-\frac{3\cdot\sqrt{2}}{2}.$$

One can verify this using technology. See **FIGURES 16-21:**

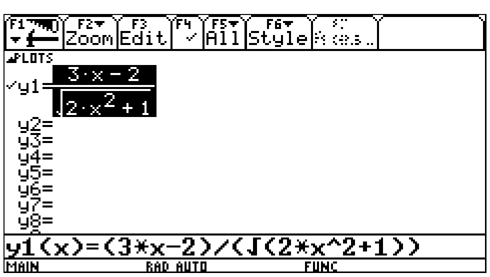


FIGURE 16: The Function Input

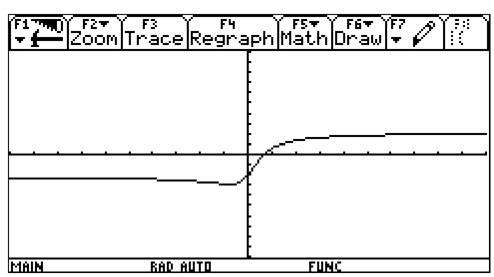


FIGURE 17: The Graph

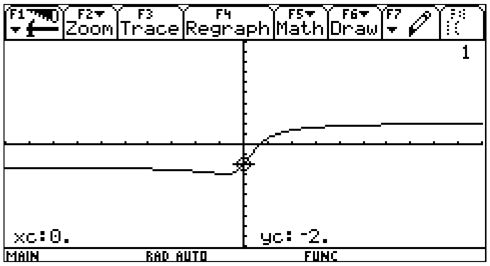


FIGURE 18: A Trace Value

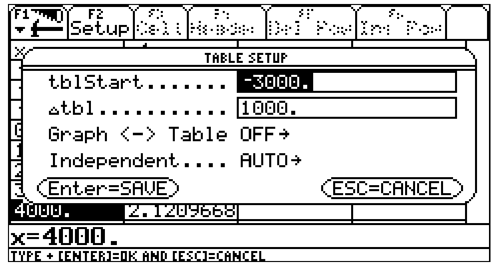


FIGURE 19: The Table Setup

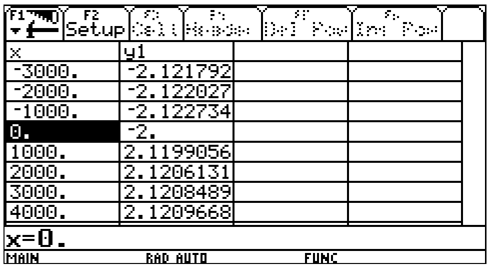


FIGURE 20: The Table

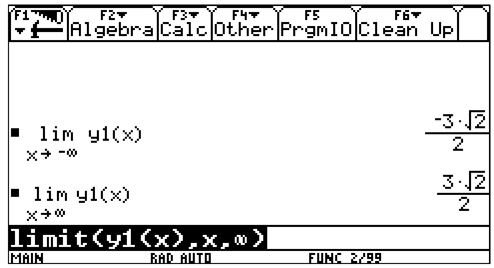


FIGURE 21: CAS Capabilities

In **FIGURE 22,** we view the graph together with its two horizontal asymptotes:

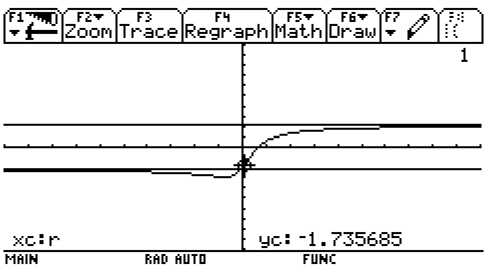


FIGURE 22: The Graph and its two Horizontal Asymptotes

6. While it is indeed the case that a rational function can never intersect a vertical asymptote, it indeed can cross a horizontal or oblique asymptote. We cite an example of each case:

Case 1: The graph of a rational function intersects its horizontal asymptote:

Consider $f(x) = \frac{3 \cdot x^2 + 16 \cdot x + 5}{x^2 + 5 \cdot x + 4}$. One can show that $y = 3$ is the equation of the horizontal asymptote to the graph of f. This is indeed the case by comparing the degrees of the respective polynomials $3 \cdot x^2 + 16 \cdot x + 5$ *and* $x^2 + 5 \cdot x + 4$. Observe that each polynomial is of degree 2. Hence $y = \frac{3}{1} = 3$, the ratio of the leading coefficients to numerator and denominator serves as the equation of the horizontal asymptote to the graph of f. Observe that this rational function intersects its horizontal asymptote; for if one solves the equation $\frac{3 \cdot x^2 + 16 \cdot x + 5}{x^2 + 5 \cdot x + 4} = 3$, then the Solve Command in the Algebra Pull-Down Menu (F2, 1) yields our solution as displayed in **FIGURES 23-31:**

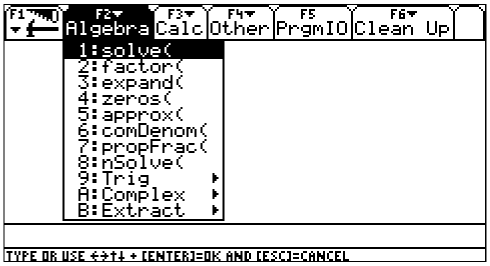


FIGURE 23: The Solve Command

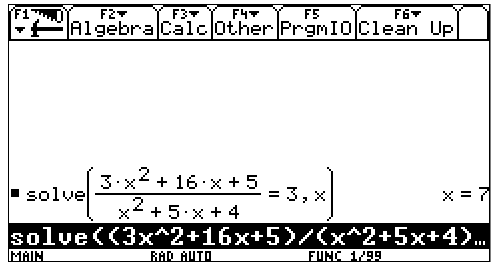


FIGURE 24: The Point of Intersection

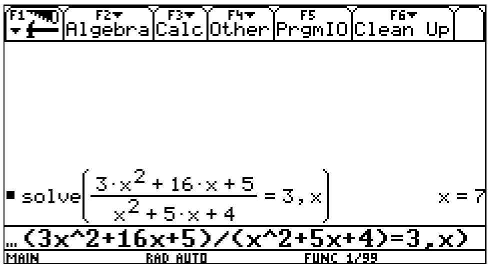


FIGURE 25: The Point of Intersection

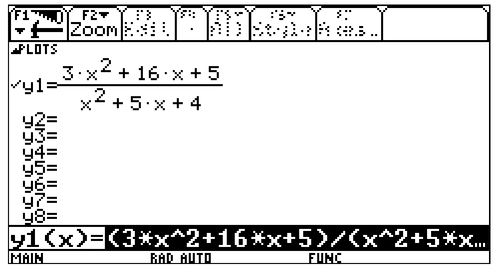


FIGURE 26: The Function Input

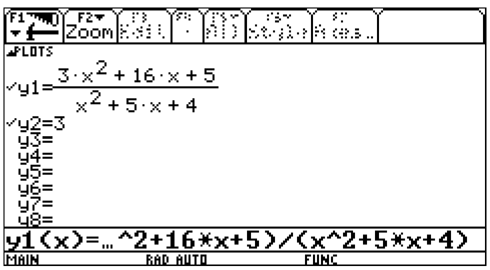


FIGURE 27: The Function Input

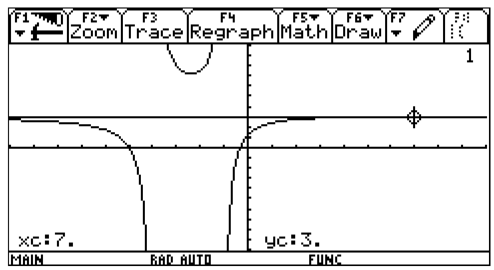


FIGURE 28: The Graph

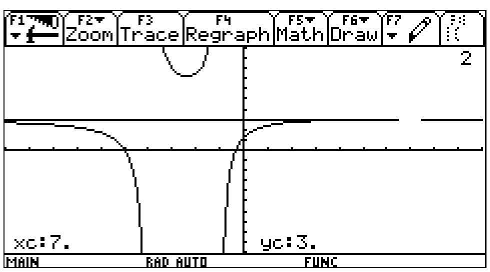


FIGURE 29: A Trace Value

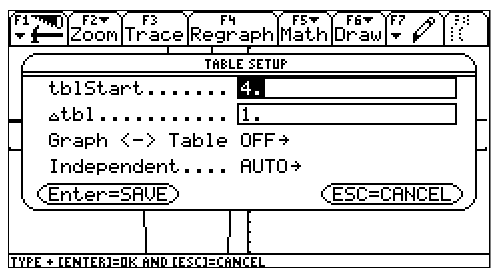


FIGURE 30: The Table Setup

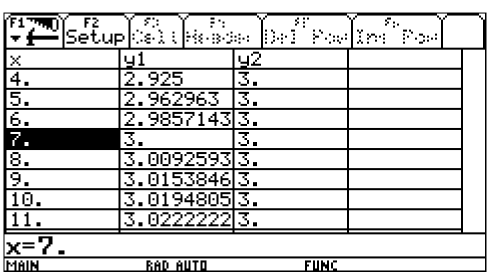


FIGURE 31: The Table

Case 2: The graph of a rational function intersects its oblique asymptote:

Consider $f(x)=\dfrac{x^3}{2\cdot x^2-8}$. Since the degree of the numerator polynomial (3) is one higher than the degree of the denominator polynomial (2), there is an oblique asymptote to the graph of f. To find the equation of this oblique asymptote, we use long division with the proper fraction command from the Algebra Menu (F2, 7). See **FIGURE 32:**

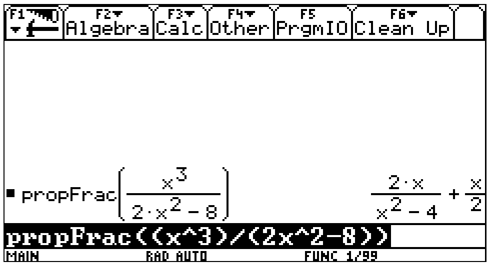


FIGURE 32: The Proper Rational Function

Hence the horizontal asymptote to the graph of $f(x)=\dfrac{x^3}{2\cdot x^2-8}$ is $y=\dfrac{x}{2}$. Note that

$$\lim_{x\to\pm\infty}\left|\frac{2\cdot x}{x^2-4}\right|=0.$$

Moreover, f intersects its horizontal asymptote @ $x=0$ as displayed in **FIGURE 33:**

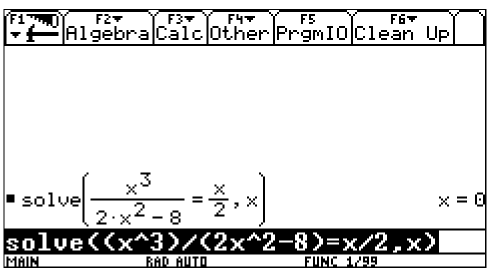


FIGURE 33: The Point of Intersection with the Oblique Asymptote

In FIGURES 34-38, we view the function and its oblique asymptote with all of the necessary particulars.

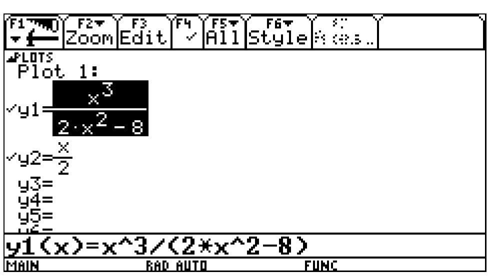


FIGURE 34: The Function Inputs

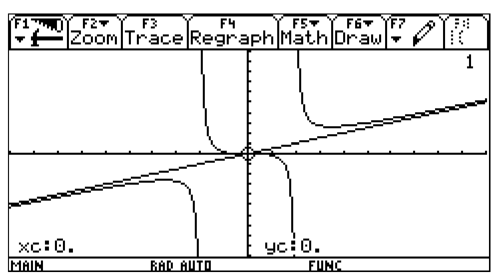


FIGURE 35: The Point of Intersection

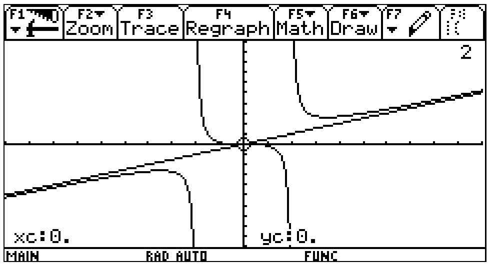


FIGURE 36: The Point of Intersection

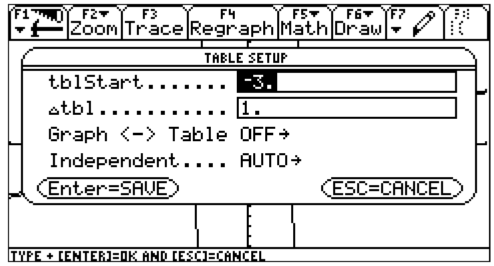
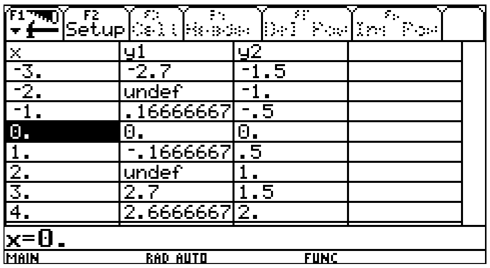


FIGURE 37: The Table Setup



x	y1	y2
-3.	-2.7	-1.5
-2.	undef	-1.
-1.	.16666667	-.5
0.	0.	0.
1.	-.1666667	.5
2.	undef	1.
3.	2.7	1.5
4.	2.6666667	2.

x=0.

FIGURE 38: The Table

7. The derivative of a product is not equal to the product of the derivatives and the derivative of a quotient is not equal to the quotient of the derivatives. (Even though it is true that the companion theorems for limits hold and the derivative of a sum is equal to the sum of the derivatives and the derivative of a difference is the difference between the derivatives.) To furnish counterexamples, let

$f(x) = (4\cdot x+3)\cdot(2\cdot x^2+5)$ *and* $g(x) = \dfrac{2\cdot x+8}{x+4}.$

Observe that

$$\frac{d}{dx}\left[f(x)\cdot g(x)\right] = \frac{d}{dx}\left[(4\cdot x+3)\cdot\left(2\cdot x^2+5\right)\right] = \frac{d}{dx}\left[8\cdot x^3+6\cdot x^2+20\cdot x+15\right] =$$

$$\frac{d}{dx}\left[8\cdot x^3\right]+\frac{d}{dx}\left[6\cdot x^2\right]+\frac{d}{dx}[20\cdot x]+\frac{d}{dx}[15] = 8\cdot\frac{d}{dx}\left[x^3\right]+6\cdot\frac{d}{dx}\left[x^2\right]+20\cdot\frac{d}{dx}[x]+\frac{d}{dx}[5] =$$

$$8\cdot 3\cdot x^{3-1}+6\cdot 2\cdot x^{2-1}+20\cdot 1+0 = 24\cdot x^2+12\cdot x+20.$$

On the other hand, $\dfrac{d}{dx}\left[(4\cdot x+3)\right]\cdot\dfrac{d}{dx}\left[\left(2\cdot x^2+5\right)\right] = 4\cdot 4\cdot x = 16\cdot x.$

Meanwhile, $\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{d}{dx}\left[\dfrac{2\cdot x+8}{x+4}\right] = \dfrac{d}{dx}\left[\dfrac{2\cdot(x+4)}{x+4}\right] = \dfrac{d}{dx}[2] = 0.$

$$\frac{d}{dx}\left[f(x)\cdot g(x)\right] = f(x)\cdot g'(x)+g(x)\cdot f'(x) \text{ and } \frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\cdot f'(x)-f(x)\cdot g'(x)}{\left[g(x)\right]^2}.$$

On the other hand, $\dfrac{\frac{d}{dx}[2\cdot x+8]}{\frac{d}{dx}[x+4]} = \dfrac{2}{1} = 2.$

It should be noted that

$$\frac{d}{dx}\left[f(x)\cdot g(x)\right]=f(x)\cdot g'(x)+g(x)\cdot f'(x) \text{ and } \frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)\cdot f'(x)-f(x)\cdot g'(x)}{\left[g(x)\right]^2}.$$

These rules are referred to as the product and quotient rules respectively. One can utilize technology on The VOYAGE 200 to view these rules as well and perform the calculations involving the previous examples. See **FIGURES 39-40** for the product and **FIGURES 41-42** for the quotient:

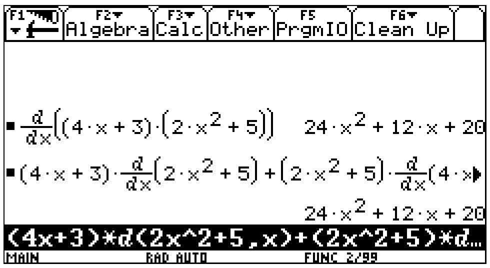


FIGURE 39: The Product Rule

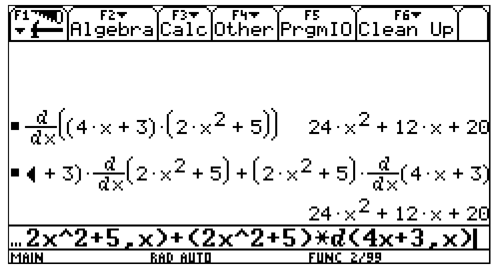


FIGURE 40: The Product Rule

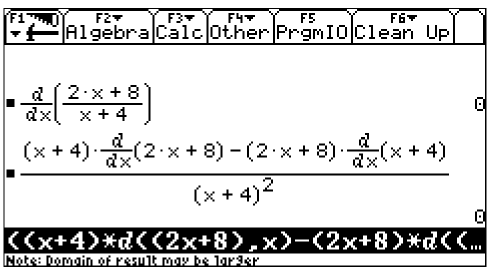


FIGURE 41: The Quotient Rule

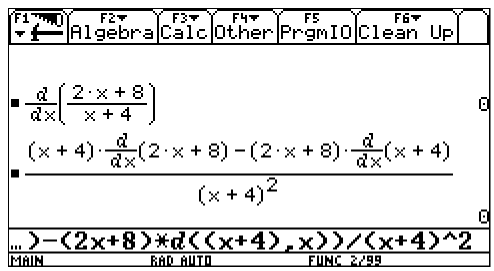


FIGURE 42: The Quotient Rule

The calculator actually generates the product and quotient rules. See **FIGURES 43-45:**

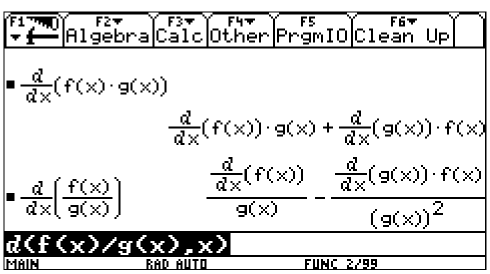


FIGURE 43: Symbolic Calculations

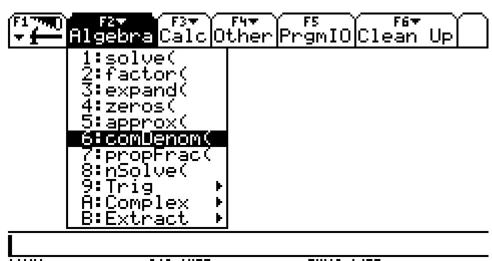


FIGURE 44: The Common Denominator

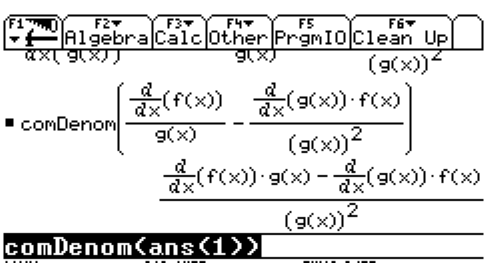


FIGURE 45: The Quotient Rule in Textbooks

8. The n^{th} derivative of a polynomial function of degree n is constant and the $n+1^{st}$ derivative vanishes. To cite an example, let $p(x)=x^3$. *Then*:

$$p'(x)=3\cdot x^2$$
$$p''(x)=6\cdot x$$
$$p'''(x)=6$$
$$p^{(iv)}(x)=0 \text{ and } p^{(n)}(x)=0 \text{ for } n\geq 4.$$

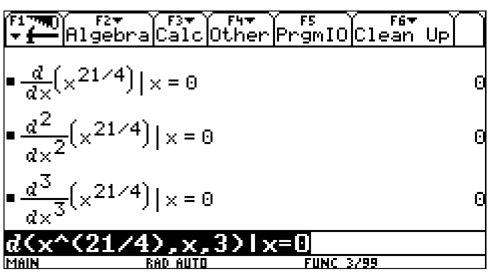
9. If the first n derivatives of f where f is a function of a real variable exist over $[a,b]$, then it is not necessarily the case that the $(n+1)^{st}$ derivative likewise exists over $[a,b]$.

For example, let $f(x)=x^{\frac{4}{3}}$ over $[-1,1]$. Then we observe that:-

$$f'(x)=\frac{4}{3}\cdot x^{\frac{4}{3}-1}=\frac{4}{3}\cdot x^{\frac{1}{3}} \text{ and } f'(0)=\frac{4}{3}\cdot(0)^{\frac{1}{3}}=\frac{4}{3}\cdot 0=0.$$

$$f''(x)=\frac{1}{3}\cdot\frac{4}{3}\cdot x^{\frac{1}{3}-1}=\frac{4}{9}\cdot x^{-\frac{2}{3}} \text{ and } f''(0)=\frac{4}{9}\cdot(0)^{-\frac{2}{3}} \nexists.$$

Hence $f''(0)$ fails to exist. One can similarly show that if we let $f(x)=x^{\frac{21}{4}}$, then the first five derivatives exist @ $x=0$, but the sixth derivative does not. The VOYAGE 200 displays this in **FIGURES 46-47:**



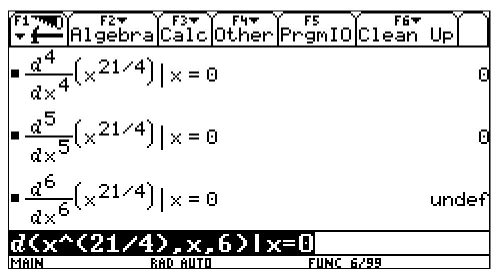


FIGURE 46: The First Three Derivatives FIGURE 47: The Next Three Derivatives

It should be noted that the above behavior is in contrast to analytic function theory in the theory of functions of a complex variable in the sense that if a function f has a first derivative (where f is analytic), then f has derivatives of all higher orders.
As an exercise for further discovery, it is neat to explore and then conjecture the following: Suppose $f(x)=x^{\frac{m}{n}}$ *where* $m,n\in\mathbb{N}$ and determine the number of derivatives one can take at zero before derivatives after that order fail to exist at the origin.

10. It is not necessarily the case that a continuous function f defined over any type of interval will necessarily secure its absolute extrema somewhere on the interval.

Consider the function f defined by $f(x)=x^2+1$ *over* $[-1,2)$. Now f is continuous, being a polynomial function. Moreover, f attains its minimum value 1 @ $x=0$ and $0\in[-1,2)$. Unfortunately, f does not attain a maximum value anywhere over $[-1,2)$. The point $(2,5)$ is NOT the maximum value of f on this interval; for $x=2$ is not a point in the half-open interval $[-1,2)$. For, suppose I claim that $y=4.61$ is the maximum value of f on $[-1,2)$. This vertical value corresponds to $1.9.$ $\left(f(1.9)=1.9^2+1=3.61+1=4.61.\right)$ It turns out that I can find a larger y value; say $y=4.8025$ corresponding to $x=1.95.$ Thus $y=4.61$ is not the maximum value of f attained on $[-1,2)$. One can similarly demonstrate (by taking $x>1.95,$ *but* $x<2$) that $y=4.8025$ is not the largest y value attained. We conclude that there is no largest y value for attained by $f(x)=x^2+1$ *over* $[-1,2)$. The problem is that the interval $[-1,2)$ is not a closed bounded interval. In contrast to the above situation, The Extreme Value Theorem asserts that if f is continuous over a closed bounded interval $[a,b]$, then f attains both its absolute minimum value m and its absolute maximum value M somewhere on $[a,b]$. Hence the hypothesis that the interval must be closed and bounded cannot be relaxed in the Extreme Value Theorem.

11. It is not the case that a function f defined over a closed bounded interval $[a,b]$ necessary assumes its absolute extrema somewhere over $[a,b]$. Hence the hypothesis of continuity cannot be relaxed from the Extreme Value Theorem.
The following counterexample serves to be illustrative:

Consider the function f defined by $f(x)=\left\{\begin{matrix} x^2+1, & x\neq 0 \\ 2, & x=0\end{matrix}\right\}$ over $[-1,2]$.

Observe that f is not continuous @ $x=0$ and $0\in[-1,2]$. We note that

$$\lim_{x\to 0} f(x)=\lim_{x\to 0}\left(x^2+1\right)=\lim_{x\to 0}x^2+\lim_{x\to 0}1=\left(\lim_{x\to 0}x\right)^2+\lim_{x\to 0}1=(0)^2+1=0+1=1\neq 2=f(0).$$

Now f attains its maximum value of 5 @ $x=2$; for $f(2)=2^2+1=4+1=5$ and each y value is no greater than 5 for any x value on $[-1,2]$. In contrast, f does not achieve its minimum value anywhere on $[-1,2]$. Clearly $y=1$ is NOT the minimum value of f on $[-1,2]$ as $f(0)\neq 1$ since we defined $f(0)=2.$ Since we defined $f(0)=2,$ f does not have $y=1$ as a vertical value for ANY $x\in[-1,2]$. There are y values very close to 1 (but very slightly above 1) as $x\to 0^-$ and as $x\to 0^+$. None of these values is the minimum value either. (If one supposes that $y=\frac{17}{16}=1.0625$ is the minimum y value assumed by f, corresponding to $x=\pm\frac{1}{4}$. One can secure a smaller y value; say $y=1.015625$

assumed by f, corresponding to $x = \pm\frac{1}{8}$, contradicting the assumption that $y = \frac{17}{16}$ was the least possible vertical value assumed by f. In a similar manner, it can be shown that NO vertical value is the least value assumed by f demonstrating that f does not attain its absolute minimum on $[-1, 2]$.)

We provide a graphing analysis of the above in **FIGURES 48-54:**

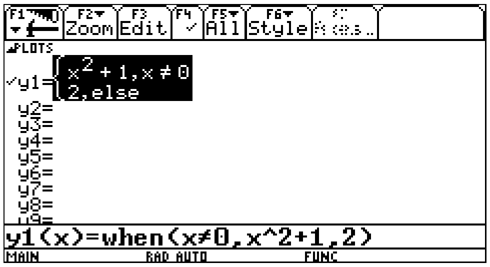


FIGURE 48: The Piecewise Function

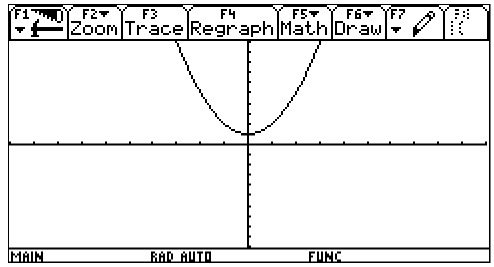


FIGURE 49: The Graph

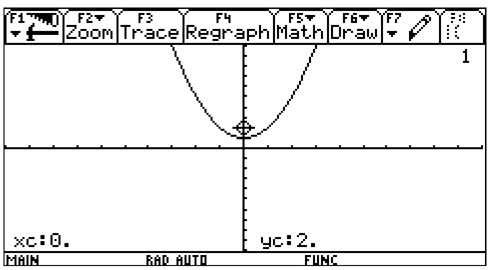


FIGURE 50: The Value @ $x = 0$

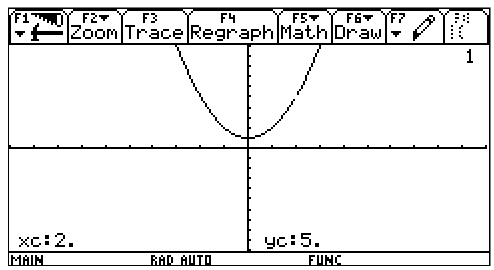


FIGURE 51: The Value @ $x = 2$

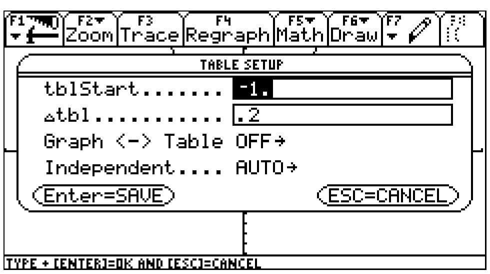
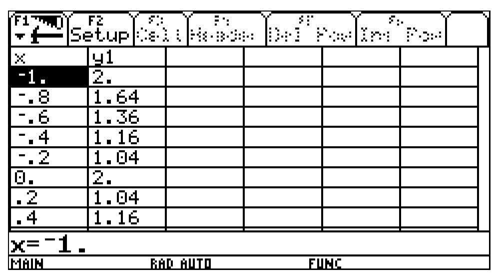


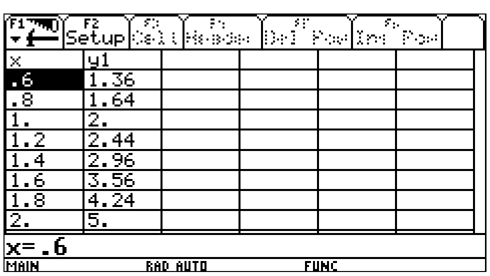
FIGURE 52: The Table Setup



x	y1
-1.	2.
-.8	1.64
-.6	1.36
-.4	1.16
-.2	1.04
0.	2.
.2	1.04
.4	1.16

x=-1.

FIGURE 53: The Table



x	y1
.6	1.36
.8	1.64
1.	2.
1.2	2.44
1.4	2.96
1.6	3.56
1.8	4.24
2.	5.

x=.6

FIGURE 54: The Table

12. It is not the case that a continuous function f defined on any closed interval necessarily assumes both of its extreme values somewhere on the interval. To cite a

counterexample, let f be defined on the closed but unbounded interval $[0,+\infty)$ by $f(x)=\dfrac{1}{x+1}$. Observe that the greatest value of f, namely $y=1$, is achieved @ $x=0$. This decreasing function $\left(f'(x)=-\dfrac{1}{(x+1)^2}<0\ \forall x\in[0,+\infty)\right)$ tends to zero as $x\to+\infty$. Nonetheless, the value $y=0$ is never attained by the function. (Note that 0 is the g.l.b. – greatest lower bound of f for its range.) (If we assumed, on the contrary, that $y=0.001$ is the least value attained by f, corresponding to $x=999$, then for the value $x=99999$, the corresponding y value will be $y=0.00001$. This shows that $y=0.001$ is not the least vertical value assumed by f. A similar argument will demonstrate that f will never achieve its absolute minimum on $[0,+\infty)$. A complete graphical analysis of this function is illustrated **in FIGURES 55-60:**

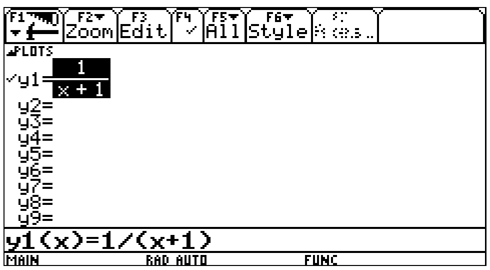


FIGURE 55: The Function Input

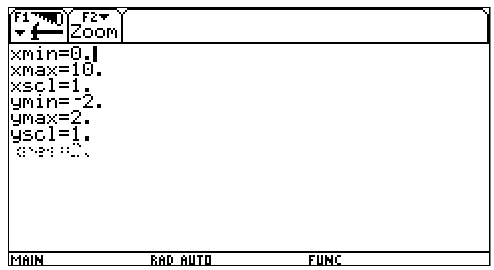


FIGURE 56: The Viewing Window

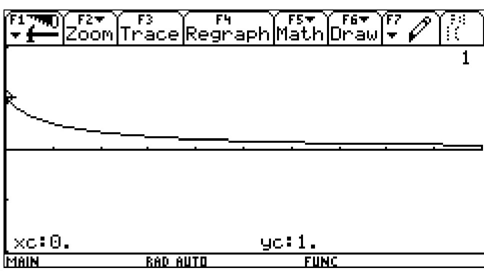


FIGURE 57: The Graph

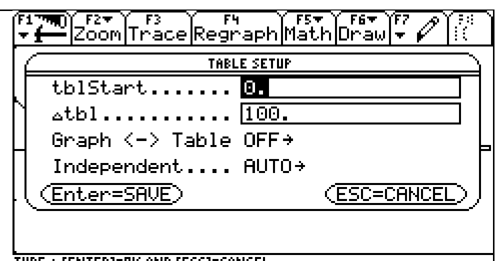


FIGURE 58: The Table Setup

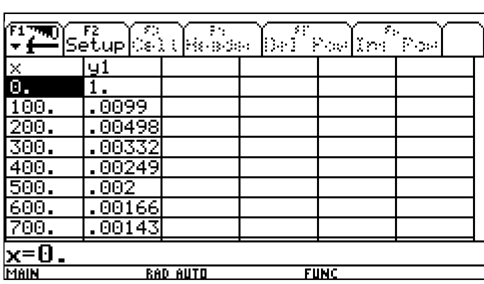


FIGURE 59: The Table

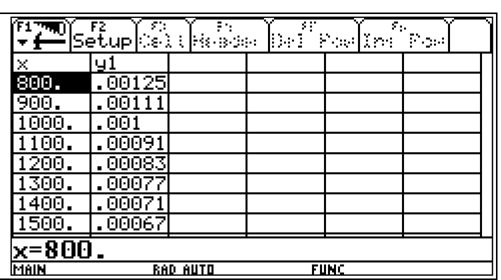


FIGURE 60: The Table

13. It is not necessarily the case that the indefinite integral of an even function is odd. We consider the function f on $[a,b]$ defined by $f(x)=x^4$. Note that f is an even function: $\forall x\in D_f=\mathbb{R},\ f(-x)=(-x)^4=x^4=f(x)$. Observe that

$F\left(x\right)=\int f\left(x\right)dx=\int x^4dx=\frac{x^{4+1}}{4+1}+C=\frac{x^5}{5}+C,$ where C is an arbitrary constant. If $C=1$, then $F\left(x\right)=\frac{x^5}{5}+1$ and F is not odd:

$\forall x\in D_f=\mathbb{R},\ F\left(-x\right)=\frac{\left(-x\right)^5}{5}+1=-\frac{x^5}{5}+1\neq -F\left(x\right)=-\frac{x^5}{5}-1=-\left(\frac{x^5}{5}+1\right).$ On the other hand, we note that the derivative of an odd function is indeed necessarily even: Let f be an odd function. Then $\forall x\in D_f,\ f\left(-x\right)=-f\left(x\right).$ Now

$\frac{d}{dx}\left[f\left(-x\right)\right]=f'\left(-x\right)\cdot\frac{d}{dx}\left(-x\right)=-f'\left(-x\right)$ and $\frac{d}{dx}\left[-f\left(x\right)\right]=-\frac{d}{dx}\left[f\left(x\right)\right]=-f'\left(x\right).$
Now $-f'\left(-x\right)=-f'\left(x\right)\Leftrightarrow f'\left(-x\right)=f'\left(x\right)\Rightarrow f$ is even.

14. If $\lim_{n\to+\infty}u_n=0,$ then $\sum_{n=1}^{+\infty}u_n$ converges. This statement is false. While the $n-th$ term approaching zero is necessary for the truth of the conclusion, it is not sufficient. The counterexample is the divergent harmonic series $\sum_{n=1}^{+\infty}\frac{1}{n}$. The problem with this series is despite $\lim_{n\to\infty}\left(\frac{1}{n}\right)=0,$ the sequence of partial sums becomes does not possess an upper bound if one selects enough terms and any unbounded sequence is divergent. It can be shown that the minimum number of terms required for the sum of the harmonic series to exceed the integers 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10 is respectively 2, 4, 11, 31, 83, 227, 616, 1674, 4550, and 12367. We verify with the VOYAGE 200 in **FIGURES 61-70:**

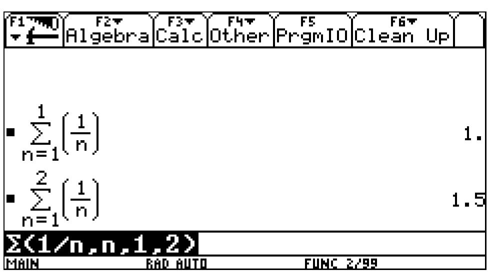


FIGURE 61: The Sum Exceeds One

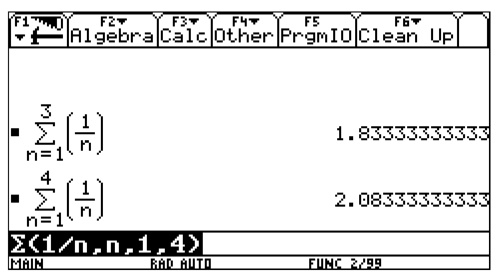


FIGURE 62: The Sum Exceeds Two

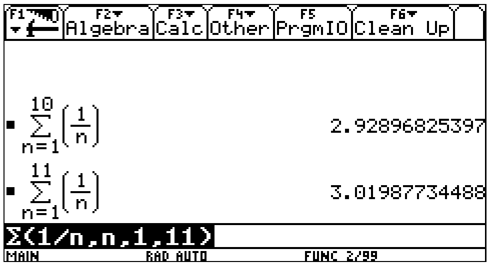


FIGURE 63: The Sum Exceeds Three

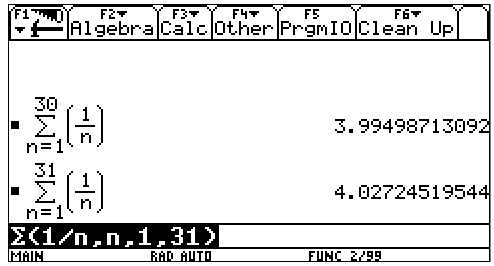


FIGURE 64: The Sum Exceeds Four

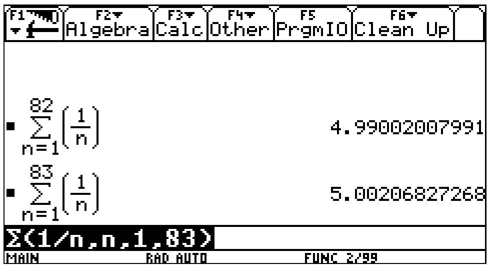


FIGURE 65: The Sum Exceeds Five

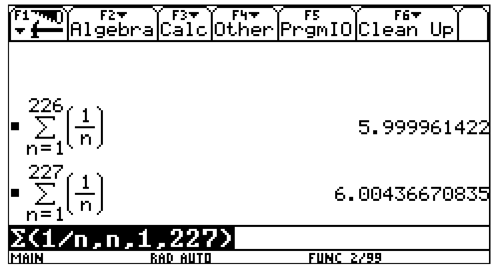


FIGURE 66: The Sum Exceeds Six

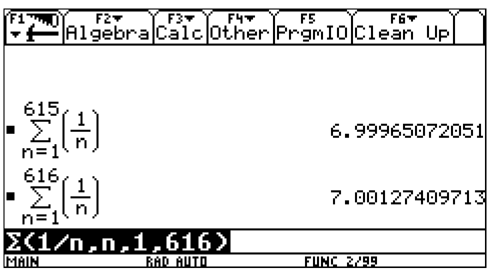


FIGURE 67: The Sum Exceeds Seven

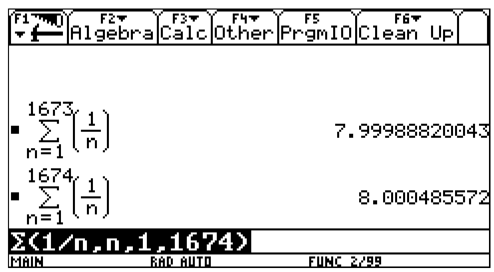


FIGURE 68: The Sum Exceeds Eight

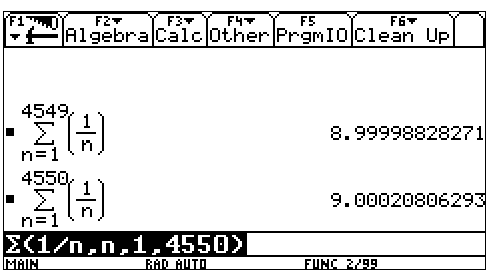


FIGURE 69: The Sum Exceeds Nine

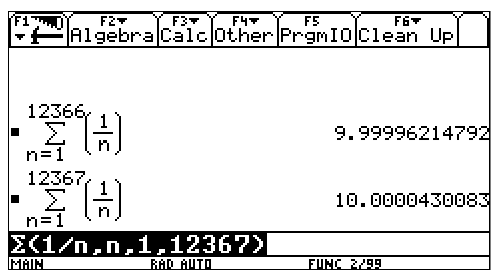


FIGURE 70: The Sum Exceeds Ten

15. If one thins out the harmonic series $\sum_{n=1}^{+\infty} \frac{1}{n}$ by removing the first one million terms, the resulting series will still diverge; for convergence or divergence is not affected by the removal of finitely many terms from the series. The removal of infinitely many terms from the series is a different matter. Euler proved that the infinite series of primes

$\sum_{n=1}^{+\infty} \frac{1}{p_n} = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + ... + \frac{1}{p_n} + ...$ diverges although far more slowly than the harmonic series. In fact, it takes 3 terms for the sum to exceed one, 59 terms for the sum to exceed two, and an incredible 361139 terms for the sum to exceed three. MATHEMATICA 6.0 was employed to verify these assertions.

16. It is false that the convergent $p-series$ $\sum_{n=1}^{+\infty} \frac{1}{n^p}$ and the convergent improper integral $\int_1^{+\infty} \frac{dx}{x^p}$ have the same value. One can show that $\sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ while $\int_1^{+\infty} \frac{dx}{x^2} = 1.$
These assertions can be verified using the TI-89 / VOYAGE 200 or MATHEMATICA CAS. See **FIGURES 71-72:**

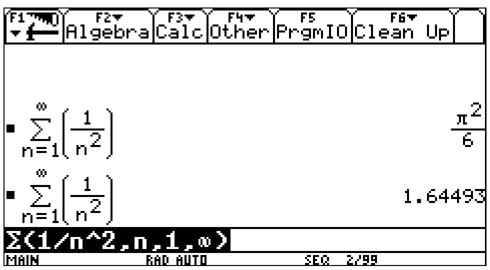


FIGURE 71: The Infinite Sum

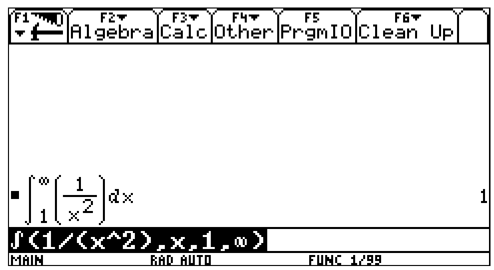


FIGURE 72: The Improper Integral

17. The sum of two divergent series can converge. Consider the divergent harmonic series and its negation. The sum of these two series is the zero series which converges to 0. What is indeed true is that the sum of two convergent series is necessarily convergent while the sum of a convergent series with a divergent series must necessarily diverge.

18. Our final example illustrates that not all bounded sequences are convergent although the converse is true (every convergent sequence is bounded). To cite an example, consider the sequence $u_n = (-1)^{n+1}$.This sequence oscillates between -1 and 1 with value 1 for each of the odd numbered term and value -1 for each even numbered term. The

sequence of partial sums is 1, 0, 1, 0, 1, 0,… does not have a limit and hence the sequence diverges. See **FIGURES 73-78** for a graphical analysis and table:

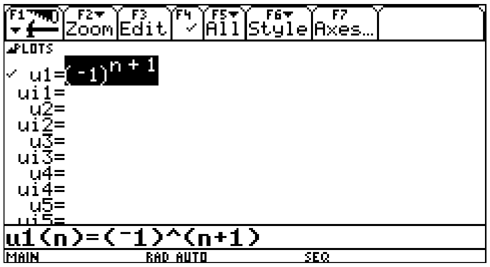


FIGURE 73: The Oscillating Sequence

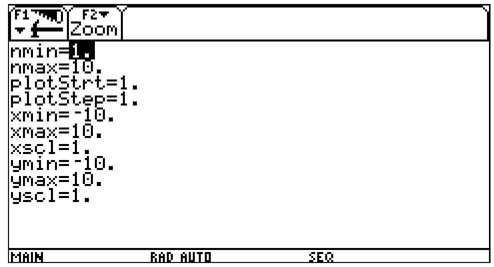


FIGURE 74: The Viewing Window

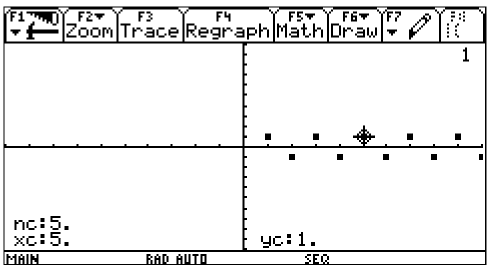


FIGURE 75: A Trace Value

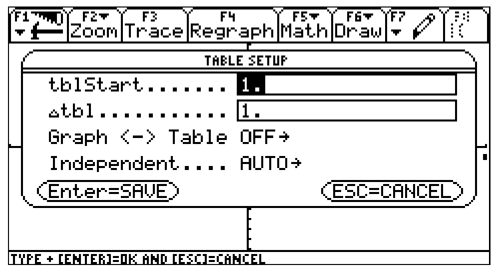


FIGURE 76: The Table Setup

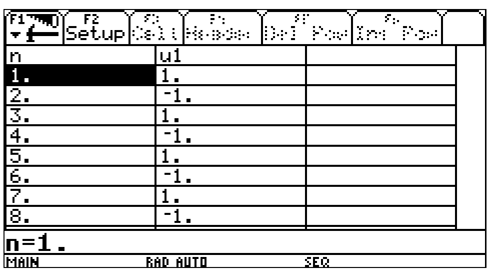


FIGURE 77: The Table

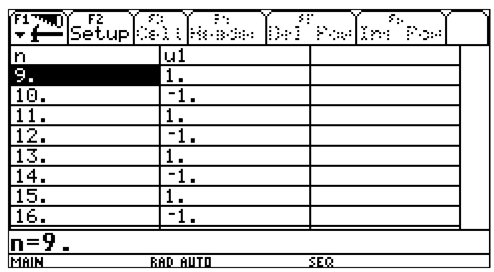


FIGURE 78: The Table

It is also not the case that every monotonic sequence converges. Consider the increasing sequence $u_n = 2^n$. This sequence can be made larger than any positive number by taking enough terms. For example, if one believes that the sequence never exceeds one million, then take 20 terms. Note that $u_{20} = 2^{20} = 1048576 > 1000000.$ A graph and TABLE for this sequence is depicted in **FIGURES 79-85:**

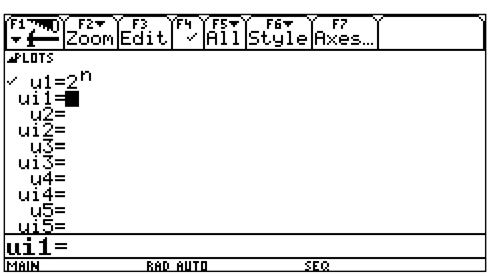


FIGURE 79: The Sequence Input

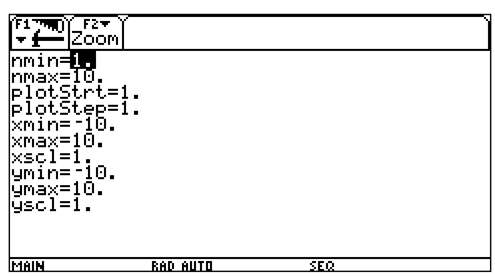


FIGURE 80: The Viewing Window

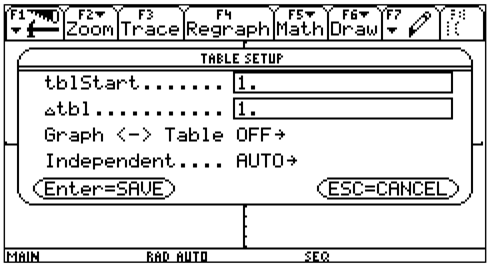


FIGURE 81: The Table Setup

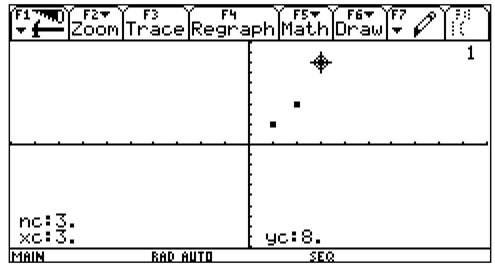


FIGURE 82: A Trace Value

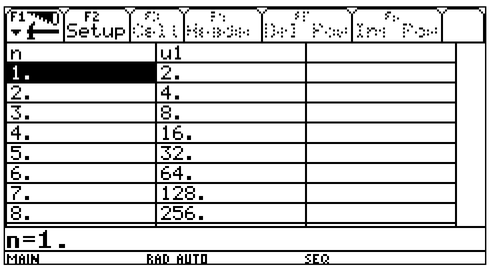


FIGURE 83: The Table

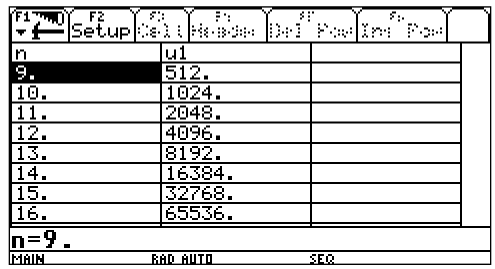


FIGURE 84: The Table

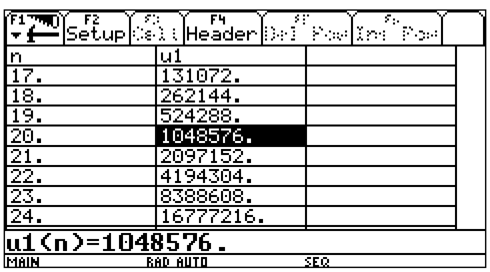


FIGURE 85: The Table

3. Conclusion.

The ideas embodied in this paper constitute only a small segment of rich calculus ideas that are often misconstrued. The use of counterexamples and CAS technology can serve as a vehicle to overcome these pitfalls. In addition, exploring the hypotheses in a theorem and discovering false conclusions when the hypotheses are relaxed is often helpful in breaking new ground and promoting mathematical maturity and aids in transitioning the student to obtain successful outcomes in theoretical mathematics courses including analysis, algebra and topology.