IMPOSSIBLE CONSTRUCTIONS IN GEOMETRY AND WHY

Jay Villanueva

Miami Dade College

11011 SW 104 St.

Miami, FL 33176

jvillanu@mdc.edu

(*Abstract*.) Some ancient problems in geometry came down to us as impossible constructions by straightedge and compass: the doubling of the cube, the trisection of an angle, the squaring of the circle, and the construction of certain regular polygons. They have only been solved in the 19th Century by the invention of new mathematics. We can state: A geometric construction is possible iff the process occurs in a field whose dimension over the rational field is a power of 2, i.e., 2^k. And, a regular polygon of *n* sides is constructible iff

$$n = 2^k \cdot p_1 p_2 \cdots p_m,$$

where *k* and *m* are non-negative integers, and the p_ms are of the form

$$p_m = 2^{2^m} + 1.$$

Since ancient times some problems in geometry have come down to us from the Greeks as impossible constructions: the doubling of the cube, the trisection of an angle, the squaring of the circle, and the construction of certain regular polygons. But they remained enigmas until the invention of new mathematics in the 19th Century. Now, we have solved these problems completely by merging geometry with other areas of study, such as algebra, trigonometry, and analysis, under the unifying thread of number theory.

A favorite demonstration by Euclid in *The Elements,* ~300 BC, was to construct figures just using an unmarked straightedge and compass. His first problems were to construct an equilateral triangle of given side and how to bisect a given length.

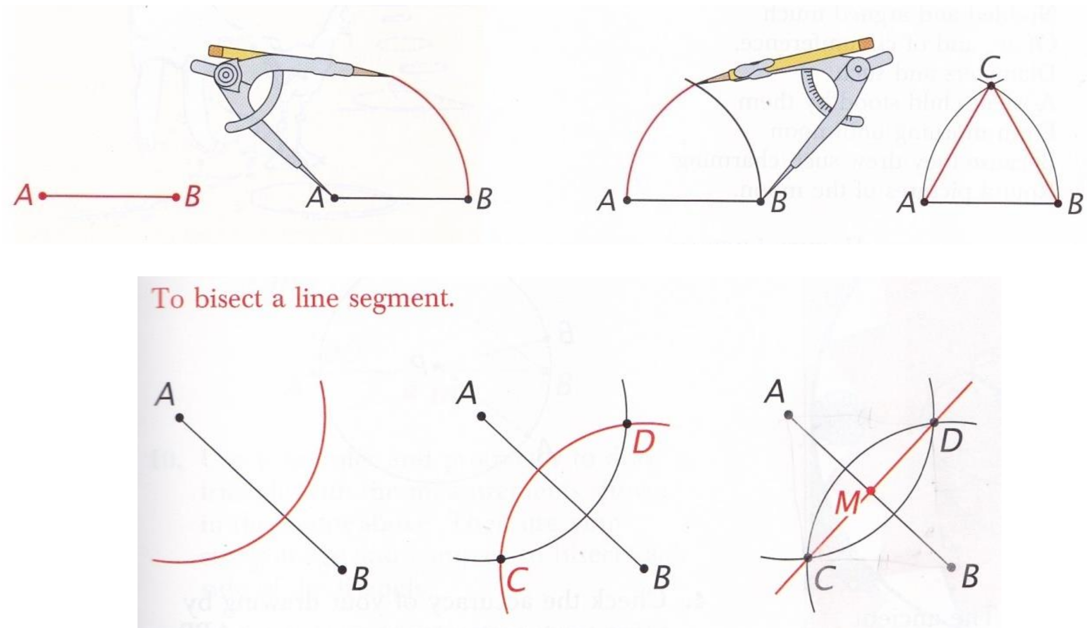


Figure 1. Equilateral triangle and bisect a line segment

Other familiar problems were:

Bisect an angle

Construct an isosceles triangle of given base and side

Construct a 30-60-90 degree triangle

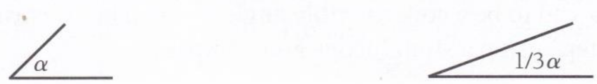
Construct a parallel to a given line

Construct a perpendicular to a given line through a point on or off the line.

And then there are the famous problems of antiquity, first considered by the Greeks, which are important, not only for their practical value, but also for the creative and challenging ideas they presented.

The Three Famous Problems of Antiquity

The Problem of Trisecting an Angle. Is it always possible to trisect a given angle?



The Problem of Doubling a Cube. Is it possible to construct a cube whose volume is twice that of a given cube?

The Problem of Squaring a Circle. Can one construct a square whose area is equal to the area of a given circle?

Figure 2. Three famous problems of antiquity

A fourth famous problem is a regular polygon of n sides. For what values of n is this constructible?

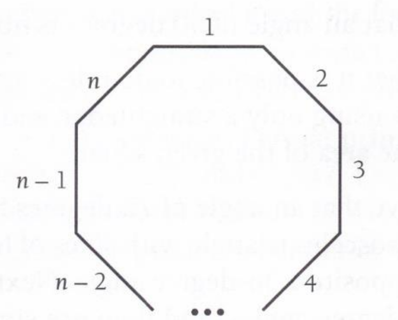


Figure 3. The regular n – gon

These problems date from 500 BC.

(1) Legend has it that Athens, beset by a serious plague, sent a delegation to the oracle of Apollo at Delos for advice. The delegation was told to double the cubical altar to Apollo. Unfortunately, they doubled each side, thereby increased the volume by a factor of 8; and the plague got worse. It was called the Delian problem.

(2) We know less about the other problems. How they arose may have been something like this. It is easy to divide a line segment to any number of parts, and just as easy to bisect an angle. It is natural to try to trisect an angle.

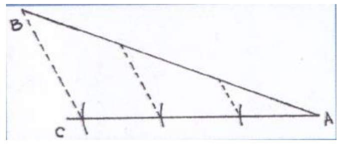
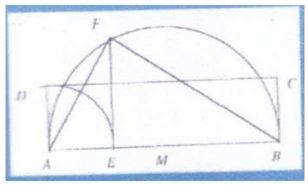


Figure 4. Trisecting a line segment

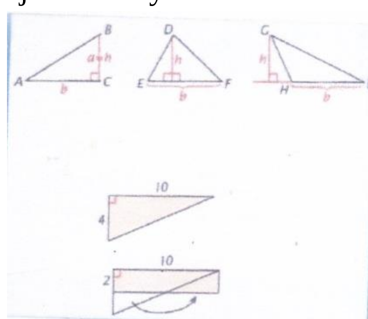
(3) Squaring a rectangle is straightforward.



ABCD rectangle, $AD < AB$. Circle *DE*, center *A*. Circle *AFB*, center *M*. $EF \perp AB$.

$$\triangle AFB \sim \triangle AEF \;\; \Rightarrow \frac{AE}{AF} = \frac{AF}{AB} \;\; \Rightarrow AF^2 = AE \cdot AB = AD \cdot AB.$$

Squaring a triangle is just as easy.



From the formula $Area = \frac{1}{2}bh$, all 3 triangles have the same area. The area of a triangle is half that of a rectangle with the same base and altitude. So we can transform a triangle into a rectangle, then into a square.

Figure 5. Squaring a rectangle and a triangle

(4) Since the 5th C BC the only regular polygons known constructible were for $n = 3, 4, 5, 15,$ and their double sides; it was long thought that there were no others. It took 2200 years before solutions to these 4 problems were discovered. Problem 4 was solved by Gauss in 1801; 1 and 2 by Wantzel in 1837; 3 by Lindeman in 1882.

To answer the question: “Which constructions are possible with unmarked straightedge and compass?”, there is an established analytic criterion for constructability. Every construction problem presents certain given elements *a, b, c, …* and requires to find certain other elements $x, y, z\ldots$, satisfying certain conditions. Any construction consists of a sequence of steps, and each step is one of the following:

1. drawing a straight line between two points;

2. constructing a circle with a given center and radius;
3. finding the intersection points of two straight lines, two circles, or a straight line and a circle.

Given a set of coordinate axes with a unit length and that all the given elements can be represented by rational numbers, we know that the sum, difference, product, and quotient (excluding division by 0) of two rational numbers is a rational number. In addition, finding the intersection points of two lines, two cicles, or a line and a circle involves only the extraction of square roots. To illustrate

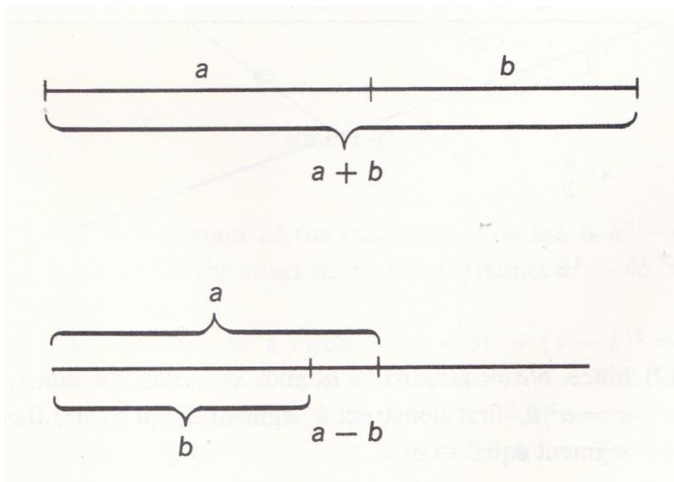


Figure 6. Sums and differences of lengths are constructible

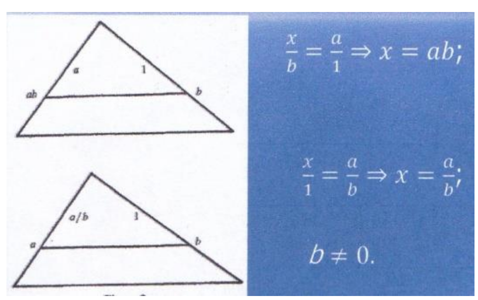


Figure 7. A product and ratio of lengths are constructible

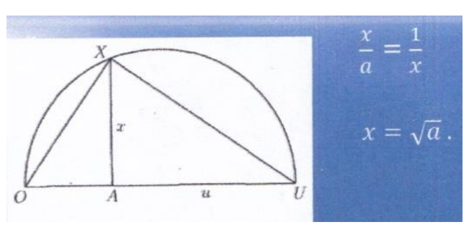
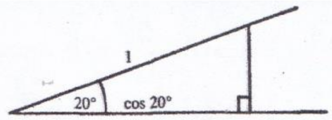


Figure 8. A square root of a length is constructible



Mark off a unit length on one side and drop a perpendicular to the other.

Figure 9. An angle is constructible if its cosine is constructible

In the last figure, if $cos\theta$ is constructible, then the angle θ is constructible. To summarize, a construction with a sytaightedge and compass is possible iff the numbers which define the desired elements can be derived from the given elements by a finite number of rational operations and the extraction of square roots. A corollary of this is that if a is a constructible number, then a is algebraic over the rationals Q (i.e., it satisfies a polynomial equation in Q), and the degree of its minimal polynomial is a power of 2.

We now present the impossibility proofs for the first three problems. How is it possible to prove that certain problems cannot be solved? When the solution violates the constraints of the problem, or some other condition known to be true, when the solution is unacceptable, or that there is no solution.

(1) It is impossible to construct a cube double a given cube. Let cube 1 have a side of length 1. Then

$$V_1 = s^3 = 1. \ \ V_2 = x^3 = 2V_1 = 2. \ \ \Rightarrow \ x^3 = 2.$$

Or, the polynomial $f(x) = x^3 - 2$ has root $\sqrt[3]{2}$. Now, $f(x)$ has no rational root; it is irreducible; it is the minimal polynomial of $\sqrt[3]{2}$. But the degree of this polynomial is 3, not a power of 2. Therefore, $\sqrt[3]{2}$ is not constructible. Thus, to double a cube is impossible.

(2) It is impossible to trisect a given angle. For any θ, $cos3\theta = 4cos^4\theta - 3cos\theta$. Let $\theta = 20^0$. Then, $cos60^0 = \frac{1}{2} = 4x^3 - 3x, \ \Rightarrow cos20^0 = x$ is a root of $f(x) = 8x^3 - 6x - 1$. $f(x)$ has no rational roots. The minimal polynomial of x has degree 3, not a power of 2. Therefore, $x = cos20^0$ is not constructible, and that 20^0 is not constructible.

(3) It is impossible to square a circle. A circle with radius 1 has area $A = \pi r^2 = \pi$. A square of equal area $A = s^2 = \pi$, has side $s = \sqrt{\pi}$.

But Lindeman showed in 1882 that π is transcendental, i.e., not algebraic (no polynomial over Q where it is a root). Therefore, π is not constructible. Thus, a square cannot be constructed with equal area as a circle.

(4) With regards to the regular polygons, the question was something else: Which regular polygons are possible? For over 2000 years, the only known regular polygons are the triangle, square, pentagon, and their doubles. Gauss in 1801 showed that the regular polygon of 17 sides can be constructed using a straightedge and compass. In fact, he showed more, for *n* a prime of the form

$$n = 2^{2^m} + 1, \quad m \text{ a nonnegative integer,}$$

an n-gon was possible. For $m = 0, n = 3$ and for $m = 1, n = 5$; we know these are constructible. For $m = 2, n = 17$. Gauss showed specifically that this was constructible. In 1822, Magnus Paucher and independently Friedrich Richelot in 1832 explicitly constructed $n = 257$ $(m = 3)$; in 1894, Johann Hermes constructed $n = 65537$ $(m = 4)$, working for 10 years on his 200-page manuscript.

The number $p = 2^{2^m} + 1$ is known as a Fermat prime; it is a prime number for $m = 0,1,2,3,4$. For $m = 5, p = 4{,}294{,}967{,}297 = (641)(6{,}700{,}417)$ and for $m = 6, p = 18{,}446{,}744{,}073{,}709{,}551{,}617 = 274{,}177 \cdot 67{,}280{,}421{,}310{,}721$. It is not known whether p is prime for higher values of *m*. In summary, a regular *n* – gon is constructible iff *n* has the values given by

$$n = 2^k p_1 p_2 \ldots p_m, \quad k, m \text{ non-negative integers,}$$

to wit, for $n < 100$:

$$n = 3,4,5,6,8,10,12,15,16,17,20,24,30,32,34,40,48,51,60,64,68,80,85,96.$$

Conclusions. Using a straightedge and compass alone, it is:

1. impossible to double a cube
2. impossible to trisect an angle
3. impossible to square a circle
4. impossible to construct certain regular polygons.

- In #2, some special angles can be trisected, but not in general.
- In #4, in antiquity, the only constructible regular polygons were the triangle, square, pentagon, quindecagon, and their double sides. Now, all *n* – gons of the form $n = 2^k p_1 p_2 \ldots p_m$, are constructible, where *k*, *m* are nonnegative integers, and $p_m = 2^{2^m} + 1$. In particular, $n = 17, 257,$ and 65537 are constructible.
- If other instruments are allowed, one or another of these problems can be solved.

- Today’s computer can solve all of these problems.

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