SOLVING CUBIC EQUATIONS BY *EXCEL*

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(*Abstract)* There is a formula to solve a general cubic equation. Here, we will solve examples of cubics: When there is one real solution, when there are two equal real solutions, and when there are three distinct real solutions. We will program in *Excel,* the solutions of cubics for all three cases.

We all know the solution to a quadratic equation, $ax^2 + bx + c = 0$:

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The quantity $\Delta = b^2 - 4ac$ is known as the discriminant:

$$\Delta = \begin{cases} -, & \text{both roots complex;} \\ 0, & \text{one real root;} \\ +, & \text{both roots real and unequal.} \end{cases}$$

There is a similar description to the solutions of a cubic equation,

(1) $$ax^3 + bx^2 + cx + d = 0,$$

published by Cardano (*Ars Magna, 1545).* We derive Cardano's equation.

Define $x = y - \frac{b}{3}$, with $a = 1$. Eq. (1) becomes

(2) $$y^3 + py + q = 0,$$

called the reduced equation,

where $p = c - \frac{b^2}{3}$ and $q = d - \frac{bc}{3} + \frac{2b^3}{27}$.

The discriminant of the cubic is defined as

$$\Delta = (x_1 - x_2)^2 (x_1 - x_3)^2 (x_2 - x_3)^2,$$

where x_1, x_2, x_3 are the roots of Eq. (1). This is also given by

$$\Delta = -4p^3 - 27q^2,$$

where p and q are given in Eq. (2), where

$$\Delta = \begin{array}{ll} -, & \text{two roots complex;} \\ 0, & \text{two equal roots;} \\ +, & \text{all roots real and distinct.} \end{array}$$

Start with the complex cube root of unity:

$$\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2} i,$$

which is one complex root of $z^3 - 1 = 0$.

Let $y = z - p/3z$, then $z^3 - \frac{p^3}{27z^3} + q = 0$, which can be rewritten as

(3) $$(z^3)^2 + q(z^3) - \left(\frac{p}{3}\right)^3 = 0$$

a quadratic in z^3. The roots are

$$z_{1,2}{}^3 = -\frac{q}{2} \pm \sqrt{\left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2}.$$

Choose the two solutions as: $z_1{}^3 z_2{}^3 = -(p/3)^3$, or $z_1 z_2 = -p/3$, i.e., $z_2 = -p/3z_1$, then the other cube roots are $\omega z_1,\ \omega^2 z_1,\ \omega z_2$. And $y_1 = z_1 - p/3z_1 = z_1 + z_2,\ y_2 = \omega z_1 + \omega^2 z_2,\ \ y_3 = \omega^2 z_1 + \omega z_2$ are the roots of Eq. (2). As a check, we can use the identities $\omega^3 = 1$ and $1 + \omega + \omega^2 = 0$.
We can also verify

$$\begin{aligned} y_1 + y_2 + y_3 &= 0, \\ y_1 y_2 + y_2 y_3 + y_3 y_1 &= p, \\ y_1 y_2 y_3 &= -q. \end{aligned}$$

Case (*i*): Two equal roots.
Here, $\Delta = 0$. Then $\frac{p^3}{27} + \frac{q^2}{4} = 0 \Rightarrow z_1 = z_2$. Thus

(4) $$y_1 = z_1 + z_2 = 2\sqrt[3]{-\frac{q}{2}} = \sqrt[3]{-4q}. \text{ And}$$
$$y_2 = -z_1 = \sqrt[3]{q/2}, \quad y_3 = \sqrt[3]{q/2} = y_2.$$

Case (*ii*): Two complex roots.
Here, $\Delta < 0$. Then $\frac{p^3}{27} + \frac{q^2}{4} > 0$ and $\sqrt{\frac{p^3}{27} + \frac{q^2}{4}}$ is a real number. Then

$$z_1 = \sqrt[3]{\frac{-q}{2} + \sqrt{\left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2}}, \quad z_2 = \sqrt[3]{\frac{-q}{2} - \sqrt{\left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2}},$$

And $$y_1 = z_1 + z_2,$$

(5)
$$y_2 = -\frac{1}{2}(z_1 + z_2) + \mathrm{i}\frac{1}{2}(z_1 - z_2)\sqrt{3},$$
$$y_3 = -\frac{1}{2}(z_1 + z_2) - \mathrm{i}\frac{1}{2}(z_1 - z_2)\sqrt{3}.$$

Case (*iii*): All real roots.

Here, $\Delta > 0$. Then $\frac{p^3}{27} + \frac{q^2}{4} < 0$ and $\sqrt{\frac{p^3}{27} + \frac{q^2}{4}}$ is not a real number. From Eq. (5), then z_1 and z_2 are cube roots of complex numbers and it is not easy to do. So, to avoid doing this, we use another approach, by trigonometry.
We can easily show that $\cos 3\theta = 4cos^3\theta - 3cos\theta$, so that $z = \cos\theta$ is a root of the equation $4z^3 - 3z = k$ if $\cos 3\theta = k$, i.e., $\theta = (arccosk)/3$. In this case $\cos(\theta + 2\pi/3)$ and $\cos(\theta + 4\pi/3)$ are also solutions, provided $|k| \leq 1$. In the reduced Eq. (2), use

$$y = 2\sqrt{-p/3}\ z.$$

Then, we get back $4z^3 - 3z = k$, with

$$k = \frac{-q}{2}\sqrt{-\left(\frac{3}{p}\right)^3} = \frac{-q}{2}\left(\frac{3}{-p}\right)^{3/2}.$$

And the roots are given by,

$$\theta = \frac{1}{3}arccos\left(\frac{-q}{2}\left(\frac{3}{-p}\right)^{3/2}\right),$$

(6)
$$y_k = 2\sqrt{\frac{-p}{3}}\ cos\left(\theta + \frac{2n\pi}{3}\right), n = 0,\ 1,\ 2.$$

Examples:

Example 1: Two equal real roots.
To find the roots of the cubic: $x^3 - 7x^2 + 15x - 9 = 0$.
We can check that the discriminant $\Delta = -4p^3 - 27q^2 = 0$
From Eq. (2) $p = \ c - \frac{b^2}{3}$ and $q = d - \frac{bc}{3} + \frac{2b^3}{27}$
Here, the reduced Eq. (2) is $y^3 - \frac{4}{3}y + \frac{16}{27} = 0$, with $p = -\frac{4}{3}, q = \frac{16}{27}$.
$\Rightarrow \ \Delta = 0$.
Using Eq. (4), the roots are: $y_1 = -\frac{4}{3}, y_2 = y_3 = \frac{2}{3}$, and

$$x_1 = 1,\ x_2 = x_3 = 3.$$

Example 1 Solution using *Excel*

cubic equation inputs	
a	1
b	-7
c	15
d	-9

From the reduced equation	
p	-1.33
q	0.59
Discriminant delta	
Del	0
Therefore, roots are equal	
y1	-1.33
y2	0.67
y3	0.67
x1	1
x2	3
x3	3

Example 2: Two complex roots
To find the roots of the cubic: $x^3 - x^2 + x - 1 = 0$.
From Eq. (2) $p = \frac{2}{3}$ and $q = -\frac{20}{27}$.
We can check that the discriminant $\Delta = -4p^3 - 27q^2 = -\frac{496}{27} < 0$.
Here, the reduced Eq. is $y^3 + \frac{2}{3}y + \frac{2}{9} = 0$, with $p = \frac{2}{3}$, $q = -\frac{20}{27}$.
Using Eq. (4), the roots are: $y_1 = \frac{2}{3}$, $y_2 = -\frac{1}{3} + i$, $y_3 = -\frac{1}{3} - i$, and

$$x_1 = 1,\ x_2 = i, x_3 = -i.$$

Example 2 Solution using *Excel*

cubic equation inputs	
a	1
b	-1
c	1
d	-1
From the reduced equation	
p	0.67
q	-0.74
Discriminant delta	
Del	-16
Therefore, one real root and two complex roots.	
c2	0.333333333333333
s	0.384900179
z1	0.910683603

z2	-0.244016936
y1	-0.333333333
y	1
y2	-0.333333333333333 + i
y3	-0.333333333333333 - i
x1	1
x2	i
x3	-i

Example 3: Three distinct real roots

To find the roots of the cubic: $x^3 - x^2 - 32x + 60 = 0$.

$$a = 1,\ b = -1,\ c = -32,\ d = 60.$$

From Eq. (2), $p = -\frac{97}{3}$ and $q = \frac{1330}{27}$.

We can check that the discriminant $\Delta = -4p^3 - 27q^2 = 69696 > 0$.

Here, the reduced Eq. is $y^3 - \frac{97}{3}y + \frac{1330}{27} = 0$, with $p = -\frac{97}{3}$, $q = \frac{1330}{27}$.

Here, $k = -\frac{q}{2}\left(\frac{3}{-p}\right)^{3/2} = -0.696088$, and $|k| \leq 1$.

Then, $\theta = \frac{1}{3} arccos\left(\frac{-q}{2}\left(\frac{3}{-p}\right)^{3/2}\right) = 0.780243\, rad.$

Using Eq. (6), the roots are: $y_0 = 2\sqrt{\frac{-p}{3}}\, cos\theta$=4.666667, $y_1 = 2\sqrt{\frac{-p}{3}}\cos\left(\theta + \frac{2\pi}{3}\right) = -6.3333333$, $y_2 = 2\sqrt{\frac{-p}{3}}\cos\left(\theta + \frac{4\pi}{3}\right)$=1.666667, and

$$x_0 = 5,\ x_1 = -6, x_2 = 2.$$

Example 3 Solution using *Excel*

cubic equation inputs

a	1
b	-1
c	-32
d	60

From the reduced equation

p	-32.33

q	49.26
Discriminant delta	
Del	69696
Therefore, three distinct real roots	
k	-0.696087835
t	0.780243433
y1	4.666666667
y2	-6.333333333
y3	1.666666667
x1	5
x2	-6
x3	2

Example 4: Two complex roots

To find the roots of the cubic: $x^3 - 4x^2 + 13x = 0$.

From Eq. (2) $p = \frac{23}{3}$ and $q = \frac{12580}{999}$.

We can check that the discriminant $\Delta = -4p^3 - 27q^2 = -6084 < 0$.

The reduced Eq. is $y^3 - \frac{23}{3}y + \frac{12580}{999} = 0$, with $p = -\frac{23}{3}$, $q = \frac{12580}{999}$.

Using Eq. (4), the roots are: $y_1 = -\frac{4}{3}$, $y_2 = \frac{2}{3} + 3i$, $y_3 = \frac{2}{3} - 3i$, and

$$x_1 = 0,\ x_2 = 2 + 3i, x_3 = 2 - 3i.$$

Example 5: Three distinct real roots

To find the roots of the cubic: $4x^3 + 4x^2 - x - 1 = 0$.

Which can be re-written as $x^3 + x^2 - \frac{1}{4}x - \frac{1}{4} = 0$.

$$a = 1,\ b = 1,\ c = -\frac{1}{4},\ d = -\frac{1}{4}.$$

From Eq. (2), $p = -\frac{7}{12}$ and $q = -\frac{5}{54}$.

We can check that the discriminant $\Delta = -4p^3 - 27q^2 = \frac{9}{16} > 0$.

Here, the reduced Eq. is $y^3 - \frac{7}{12}y - \frac{11}{54} = 0$, with $p = -\frac{7}{12}$, $q = -\frac{5}{54}$.

Here, $k = -\frac{q}{2}\left(\frac{3}{-p}\right)^{3/2} = 0.539949$, and $|k| \leq 1$.

Then, $\theta = \frac{1}{3}arccos\left(\frac{-q}{2}\left(\frac{3}{-p}\right)^{3/2}\right) = 0.333473\, rad.$

Using Eq. (6), the roots are: $y_0 = 2\sqrt{\frac{-p}{3}}\ cos\theta$=0.833333, $y_1 = 2\sqrt{\frac{-p}{3}}\cos\left(\theta + \frac{2\pi}{3}\right) = -0.666667$, $y_2 = 2\sqrt{\frac{-p}{3}}\cos\left(\theta + \frac{4\pi}{3}\right)$= -0.166667, and

$$x_0 = \frac{1}{2},\ x_1 = -1, x_2 = -\frac{1}{2}.$$

Conclusions:

A cubic is a polynomial equation with real coefficients of the form:

(1) $\qquad ax^3 + bx^2 + cx + d = 0, \quad a,\ b,\ c\ d$ are real.

There is a solution when we transform the equation by $x = y - \frac{b}{3a}$ and we get a reduced form:

(2) $\qquad y^3 + py + q = 0$, where $p = \ c - \frac{b^2}{3},\ \ q = d - \frac{bc}{3} + \frac{2b^3}{27}$.

And we get Cardan’s solutions for three cases.

Defining the discriminant $\Delta = -4p^3 - 27q^2$, we have:

(i) $\Delta < 0$, there are two complex roots;

$$z_1 = \sqrt[3]{\frac{-q}{2} + \sqrt{\left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2}},\ \ z_2 = \sqrt[3]{\frac{-q}{2} - \sqrt{\left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2}},$$

And

$$y_1 = z_1 + z_2,$$
$$y_2 = -\frac{1}{2}(z_1 + z_2) + \mathrm{i}\frac{1}{2}(z_1 - z_2)\sqrt{3},$$
$$y_3 = -\frac{1}{2}(z_1 + z_2) - \mathrm{i}\frac{1}{2}(z_1 - z_2)\sqrt{3}.$$

(ii) $\Delta = 0$, two equal roots;

$$y_1 = z_1 + z_2 = 2\sqrt[3]{-\frac{q}{2}} = \sqrt[3]{-4q}.\text{ And}$$
$$y_2 = -z_1 = \sqrt[3]{q/2}, \quad y_3 = \sqrt[3]{q/2} = y_2.$$

(iii) $\Delta > 0$, there are three distinct roots given by;

$$\theta = \frac{1}{3}arccos\left(\frac{-q}{2}\left(\frac{3}{-p}\right)^{3/2}\right),$$
$$y_k = 2\sqrt{\frac{-p}{3}}\ cos\left(\theta + \frac{2n\pi}{3}\right), n = 0,\ 1,\ 2.$$

We have programmed the solutions to a cubic equation in *Excel* and given examples to illustrate them. We find that any cubic equation can now be solved very readily using *Excel*.

We can see from the formulae of the roots that the computations involve square roots inside cube roots, and oftentimes imaginary numbers are involved. This might explain why the cubic formula is not as popular as the quadratic formula.
But with an algorithm as Cardan's formulae, written in *Excel,* any cubic equation can now be solved fast. In fact, from the formula itself, we can tell at once whether the roots are all distinct real, two are equal, or two are complex, even before we get the actual roots.

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