A Counterintuitive Guessing Game Analyzed using Markov Processes

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Introduction

The game is very simple. A coin is repeatedly tossed until a pattern of heads or tails occurs. (At least) two people write down a sequence of m heads or tails. Whoevers sequence comes up first wins. Having to match a sequence adds drama to the game-much like having to match 6 numbers in a lottery.

Example: If m=3

Say player 1 writes down HHH and player 2 writes down HTH.

If the coin comes up TH**HTH** player 2 wins on the 5^{th} toss.
If the coin comes up THHTT**HHH** player 1 wins on the 8^{th} toss.

The question is: How should a person bet? Or perhaps a better question is: How should a person not bet?

We shall look at three assumptions people make about this game and show why they are incorrect. The case of having to match a sequence of m=3 coins shall be considered. The general case is basically the same.

Incorrect Intuition I

Given a fair coin, if flipped until we get a particular triple of heads or tails, the 8 possible bets

HHH, HHT, HTH, HTT, THH, THT, TTH, TTT

all have an equal chance of occurring-so it does not matter how one bets.

Let's try a computer simulation of 100,000 of games where player 1 bets on HHH and player 2 bets on HTH and a fair (computer) coin is flipped until either a HHH or a HTH occurs.

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Percent of time player 2 (HTH) wins is 0.599680
Percent of time player 1 (HHH) wins is 0.400320
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Simulation 1 (100,000 trials)

So it appears that HHH only wins about 40% of the time whereas HTH wins about 60% of the time.

Fact: Not all triples are equally likely. We will prove that

P(HHH winning)=.4 and P(HTH winning)=.6

To show this we will need the number of ways to win after n coins are tossed and the probability of each way of winning [1]. Consider the following table which lists the number of ways for each of the 8 possible triples to occur:

Number of ways to win

Last Three Coins ↓	n=# of coins tossed →	3	4	5	6
HHH		1	1	1	2
HHT		1	1	1	2
HTH		1	2	2	3
HTT		1	2	2	3
THH		1	1	2	4
THT		1	1	2	4
TTH		1	2	4	6
TTT		1	2	4	6
Probability of each way of winning	$\frac{1}{2^n}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$

Table 1

<u>Comment 1</u>: The numbers in tables 1 are easily seen as coming from the branches of the corresponding tree diagram. If a branch ends with HHH or HTH that branch is closed. Hence HHH and HTH can be viewed as absorbing states.

<u>Comment 2</u>: Note that the numbers for

HHH & HHT (1^{st} and 2^{nd} row)
HTH & HTT (3^{rd} and 4^{th} row)
THH & THT (5^{th} and 6^{th} row)
TTH & TTT (7^{th} and 8^{th} row)

are the same in table 1. This occurs as each pair HH, HT, TH, and TT splits into two triples **H and **T with equal chance (as we are assuming a fair coin).

We can see from table 1 that HTH has more ways of occurring than HHH and hence has a higher probability of occurring.

Computing a few probabilities:

$$\text{P(HHH wins)=}\begin{cases} 1*\left(\frac{1}{8}\right) = .125 \text{ after 3 tosses} \\ 1*\left(\frac{1}{8}\right)+1*\left(\frac{1}{16}\right) = .1875 \text{ after 4 tosses} \\ 1*\left(\frac{1}{8}\right)+1*\left(\frac{1}{16}\right)+1*\left(\frac{1}{32}\right) = .21875 \text{ after 5 tosses} \end{cases} \tag{1}$$

$$\text{P(HTH wins)=}\begin{cases} 1*\left(\frac{1}{8}\right) = .125 \text{ after 3 tosses} \\ 1*\left(\frac{1}{8}\right)+2*\left(\frac{1}{16}\right) = .25 \text{ after 4 tosses} \\ 1*\left(\frac{1}{8}\right)+2*\left(\frac{1}{16}\right)+2*\left(\frac{1}{32}\right) = .3125 \text{ after 5 tosses} \end{cases} \tag{2}$$

So if the game goes 4 or more tosses HTH has a higher chance of winning than HHH.

Proof that P(HHH)=.4 and P(HTH)=.6.

Define S_n to be the state vector consisting of the number of ways that

$$[\text{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}] \tag{3}$$

can occur, where n coins have been tossed.

From table 1 we see that:

$$S_3=\begin{bmatrix}1\\1\\1\\1\\1\\1\\1\\1\end{bmatrix} \qquad S_4=\begin{bmatrix}1\\1\\2\\2\\1\\1\\2\\2\end{bmatrix} \qquad S_5=\begin{bmatrix}1\\1\\2\\2\\2\\2\\4\\4\end{bmatrix}\ldots \tag{4}$$

Note that the first element of S_n represents the number of ways HHH can occur and the third element of S_n represents the number of ways HTH can occur after n coins have been tossed.

Right now we consider the state vector Sn with all 8 possible bets as in the general case we would need to consider bets on any of the 8 possible triples (allowing for more than two players and/or multiple bets per player). For our problem we will be able to reduce our state vector to a 4-vector. We will do this shortly.

It is easy to see that the state-transition matrix M which satisfies

$$S_{n+1}=MS_n \text{ for } n\geq 3 \tag{5}$$

is given by

$$M=\begin{bmatrix} 0&0&0&0&1&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&1&0&0&0&1&0&0 \\ 0&1&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&1&0&0&0&1 \\ 0&0&0&1&0&0&0&1 \end{bmatrix} \tag{6}$$

Hence

$$S_{n+3}=M^n S_n \text{ for } n\geq 0 \tag{7}$$

Which, as $S_3=[1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ ]^T$

$$S_{n+3}=M^n\begin{bmatrix}1\\1\\1\\1\\1\\1\\1\\1\end{bmatrix} \quad \text{for } n\geq 0 \tag{7}$$

(7) gives us the number of ways to win after n+3 coins have been flipped. As

$$\text{P(each way n+3 coins can be flipped)}=\frac{1}{2^{n+3}} \tag{8}$$

From (7)-(8) we see

P(HHH winning after n+3 coins tossed) =the first element of

P(HTH winning after n+3 coins tossed) =the third element of

$$M^n \begin{bmatrix}1\\1\\1\\1\\1\\1\\1\\1\end{bmatrix} * \frac{1}{2^{n+3}} \quad \text{for } n\geq 0 \tag{9}$$

Hence the 1st and 3rd elements of the column vector

$$\sum_{n=0}^{\infty} M^n \begin{bmatrix}1\\1\\1\\1\\1\\1\\1\\1\end{bmatrix} * \frac{1}{2^{n+3}} \tag{10}$$

gives the probabilities of HHH and HTH winning respectively.

Rewriting (10) we get

$$\frac{1}{8}\sum_{n=0}^{\infty}(M * \frac{1}{2})^n \begin{bmatrix}1\\1\\1\\1\\1\\1\\1\\1\end{bmatrix} \triangleq \frac{1}{8}\sum_{n=0}^{\infty}(M_1)^n \begin{bmatrix}1\\1\\1\\1\\1\\1\\1\\1\end{bmatrix} \tag{11}$$

where

$$M_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & .5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & .5 & 0 & 0 & 0 \\ 0 & .5 & 0 & 0 & 0 & .5 & 0 & 0 \\ 0 & .5 & 0 & 0 & 0 & .5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & .5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & .5 & 0 \\ 0 & 0 & 0 & .5 & 0 & 0 & 0 & .5 \\ 0 & 0 & 0 & .5 & 0 & 0 & 0 & .5 \end{bmatrix} \tag{12}$$

Noting the row redundancies in the matrix M_1 as stated in Comment 2, as we are only interested in P(HHH) and P(HTH), we can simplify our matrix M_1 to a 4x4 matrix as follows:

P(HHH winning) =the first element of
&
P(HTH winning) =the second element of

$$\frac{1}{8}\sum_{n=0}^{\infty}(M_2)^n\begin{bmatrix}1\\1\\1\\1\end{bmatrix} \qquad (13)$$

where

$$M_2=\begin{bmatrix}0 & 0 & .5 & 0\\ .5 & 0 & .5 & 0\\ 0 & 0 & 0 & .5\\ 0 & .5 & 0 & .5\end{bmatrix} \qquad (14)$$

And our state vector S_n is now the number of ways that the sequences [HHH, HTH, THH, TTH] can occur after the coins have been tossed.

The reason for this follows directly from comment 2 as follows:

We can see from our original state vector S_n=number of ways that [HHH, HHT, HTH, HTT, THH, THT, TTH, TTT] can occur that the number of ways that

HHH & HHT
HTH & HTT
THH & THT
TTH & TTT

can occur are the same. Hence the sequences HHT, HTT, THT, and TTT are redundant to calculating the number of ways HHH and HTH can occur. So we can reduce our state vector S_n to the 4x1 vector [HHH, HTH, THH, TTH] and our state vector S_n to the 4x1 vector [HHH, HTH, THH, TTH] and our state transition matrix to the corresponding 4x4 matrix M_2.

Before computing $\sum_{n=0}^{\infty}(M_2)^n$ and deriving our chances of winning, let's use (13)-(14) to compute the probability of winning with HHH and HTH in 3, 4,…, 10 tosses

Coin Tosses	P(HHH)	P(HTH)
3	.125	.125
4	.0625	.125
5	.03125	.0625
6	.03125	.046875
7	.03125	.046875
8	.0234375	.0390625

9	.017578125	.029296875
10	.0146484375	.0234375

Table 2

Now we can clearly see that P(HTH)>P(HHH) once more than three coins are tossed.

To show P(HHH)=.4 and P(HTH)=.6 we need to compute

$$\sum_{n=0}^{\infty}(M_2)^n \tag{15}$$

A standard result from matrix theory [2] is that

$$\sum_{n=0}^{\infty}(M_2)^n = (I - M_2)^{-1} = \begin{bmatrix} 1.2 & 0.4 & 0.8 & 0.8 \\ 0.8 & 1.6 & 1.2 & 1.2 \\ 0.4 & 0.8 & 1.6 & 1.6 \\ 0.8 & 1.6 & 1.2 & 3.2 \end{bmatrix} \tag{16}$$

(The sum must converge as we are dealing with probabilities).

By (13) and (16)

P(HHH winning) =the first element of
&
P(HTH winning) =the second element of

$$\frac{1}{8}\begin{bmatrix} 1.2 & 0.4 & 0.8 & 0.8 \\ 0.8 & 1.6 & 1.2 & 1.2 \\ 0.4 & 0.8 & 1.6 & 1.6 \\ 0.8 & 1.6 & 1.2 & 3.2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} .4 \\ .6 \\ .55 \\ .85 \end{bmatrix} \tag{17}$$

Thus we have proved that P(HHH)=.4 is a far worse bet than P(HTH)=.6 and hence the assumption that each triple has the same probability is incorrect.

We shall now consider the mean length of time it takes to win. This leads to

Incorrect Intuition II

As P(HTH)>P(HHH) the mean time to victory for HTH must be shorter.

To compute the mean number of coin flips until victory we need to compute

$$\frac{1}{8}\sum_{n=0}^{\infty}(M_2)^n * (n+3)\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 5.12 & 3.84 & 5.28 & 6.88 \\ 5.28 & 8.96 & 8.32 & 10.72 \\ 3.84 & 6.88 & 8.96 & 12.16 \\ 6.88 & 12.16 & 10.72 & 21.12 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.64 \\ 4.16 \\ 3.98 \\ 6.36 \end{bmatrix} \tag{18}$$

The n+3 term comes from the fact that our first chance of winning occurs after 3 tosses. So we cannot have the number of tosses being 0, 1, or 2. Again, simplifying the lhs of (18) is a standard result from linear algebra [2].

The mean number of flips until HHH and HTH occur are the first and second elements of (18) divided by their corresponding probabilities, hence:

Mean number of flips until

HHH occurs=2.64/P(HHH)=2.64/.4=6.6 tosses

(19)

HTH occurs=4.16/P(HTH)=4.16/.6=6.93333 tosses

Mean length of game =6.6*(.4)+(6.9333333)*(.6)=6.8 tosses

Hence, although HHH has a smaller chance of winning, if it wins it wins, on average, faster.

A simulation of 100,000 runs of the game backs up our result.

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Percent of time player 2 (HTH) wins is 0.599490
Mean time till HTH wins is 6.942001
Percent of time player 1 (HHH) wins is 0.400510
Mean time till HHH wins is 6.602382
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Simulation 2 (100,000 trials)

It can be shown in a similar fashion that the standard deviation of a typical game is approximately 4.6 tosses.

All the above work was based on the assumption that we were dealing with a fair coin. But what if P(H)>.5? This leads to

Incorrect Intuition III

Surely if P(H)>.5 then P(HHH)>P(HTH). However, as there are more ways for HTH to occur than HHH, even if P(H)>.5 we can still get P(HTH)>P(HHH).

Example: Using similar analysis as above, if we let P(H)=.55 we get P(HTH)=.559 vs P(HHH)=.441. Still a sizeable advantage for HTH-over a 10% advantage.

By experimentation we would need P(H) to be about .62 to have P(HHH)=P(HTH).

Summary

The above results can be extended to the case of sequences of length more than 3 to show results such as P(HTHT)>P(HHHH). Another way to consider this problem would be as a runs test. After all, betting on HTHT seems to be betting on the fact that the coin is fair whereas betting on HHHH seems to betting on an assumption that the coin is not fair.

References

[1] Mendenhall, William & Scheaffer, Richard. *Mathematical Statistics with Applications*. North Scituate, Massachusetts: Duxbury Press, 1973.

[2] Strang, Gilbert. *Linear Algebra and its Applications*, 3rd Ed. Fort Worth, TX: Saunders HBJ, 1988.