A GEOGEBRA ILLUSTRATOR FOR MARDEN'S THEOREM

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Abstract: Marden's Theorem shows the relationship between the zeros of a cubic polynomial and the construction of an ellipse, its foci, and its center in the complex plane. We present a GeoGebra applet that illustrates this relationship for cubic polynomials with real coefficients.

In a previous article "Real Polynomials with a Complex Twist," referenced below in the bibliography, we presented a GeoGebra illustrator that showed the points in the complex plane for which a polynomial with real coefficients had real number values. In particular, here is a link to the GeoGebra illustrator for cubic polynomials:

Complex Zeros of a Cubic Polynomial

In this article we will present a GeoGebra applet which provides an interactive illustrator for the construction of the ellipse of Marden's Theorem for cubic polynomials with real coefficients.

Marden's Theorem: Suppose that the zeroes z_1, z_2, and z_3 of a third-degree polynomial $p(z)$ are non-collinear. Then there is a unique ellipse inscribed in the triangle in the complex plane with vertices z_1, z_2, z_3 and tangent to the sides at their midpoints. The foci of that ellipse are the zeroes of the first derivative $p'(z)$ and the center of the ellipse is the zero of the second derivative $p''(z)$.

Dan Kalman has an excellent article about this theorem in the MAA publication Loci/JOMA referenced below.

Here is a link to the GeoGebra applet for Marden's Ellipse: Marden's Theorem Illustrator

Note: the cubic polynomials used in this illustrator have only real coefficients and the center of the ellipse will lie on the x- or y-axis.

We will now consider three situations which can occur with the graph of $p(x)$ where x is a real number:

1. The slope of the tangent line at the inflection point of the graph is negative. In this case there will be two local extrema. The major axis of the ellipse will be along the real *x*-axis and the foci will be real numbers which provide local extremes for the polynomial *p*(*z*). Here [Figure 1] is a picture of this situation for the polynomial

$$p(z) = z^3 - 3z - 8.$$

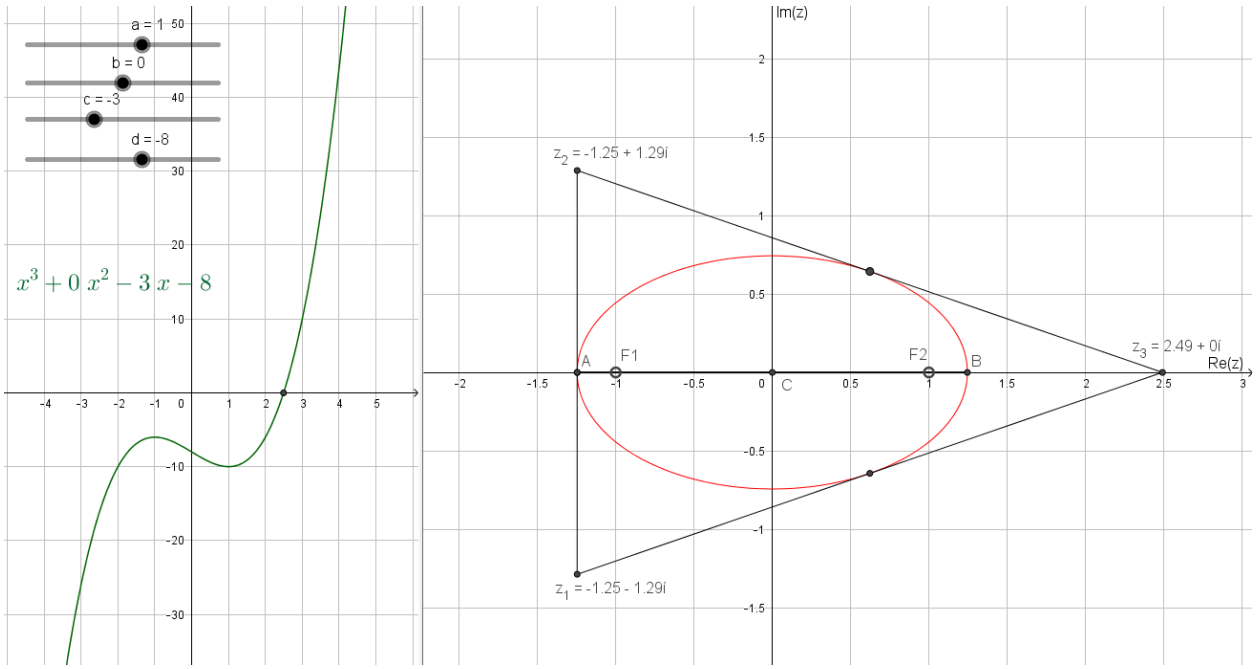

Figure 1 Tangent line at inflection point has negative slope

2. The slope of the tangent line at the inflection point of the graph is 0. In this case the three complex zeros will form and equilateral triangle and Marden's ellipse will be a circle. Here is a picture for

$$p(z) = z^3 - 8$$

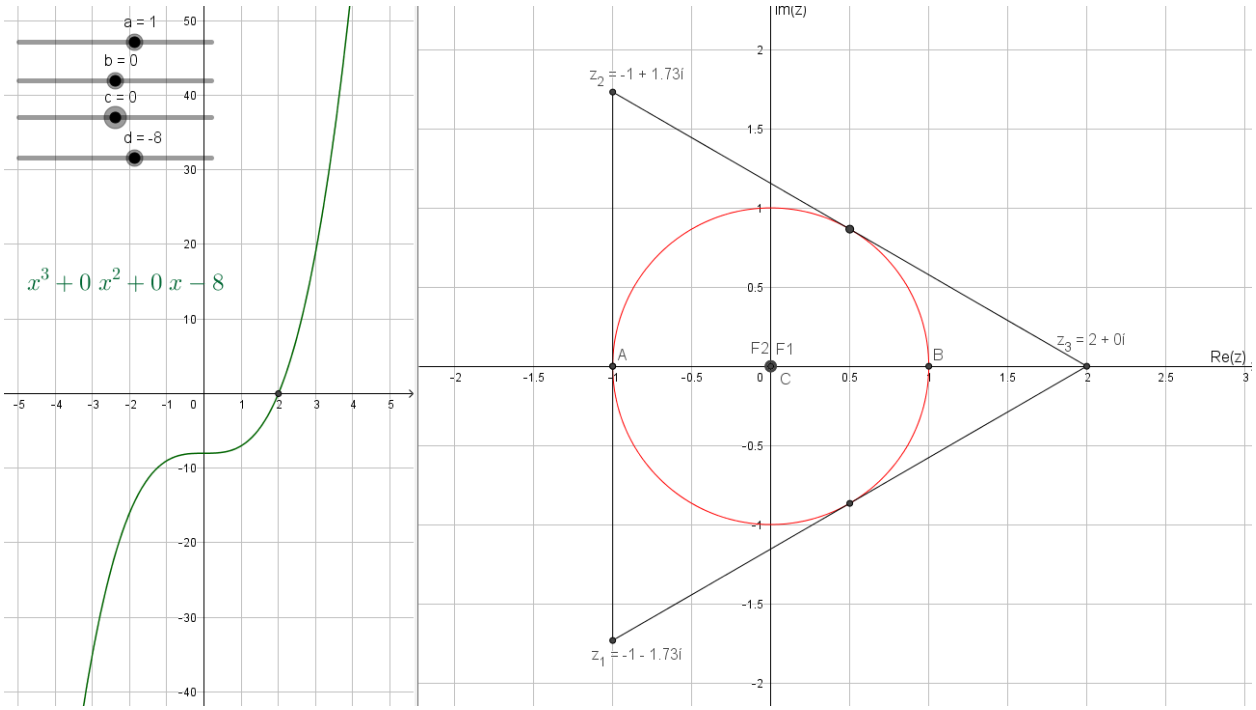

Figure 2 Tangent line at inflection point has slope 0

3. The slope of the tangent line at the inflection point of the graph is positive. In this case there are no extrema and the foci of the ellipse are nonreal complex numbers which lie on the imaginary *y*-axis (their real part is 0). Here is a picture for the polynomial

$$p(z) = z^3 + 3z - 8$$

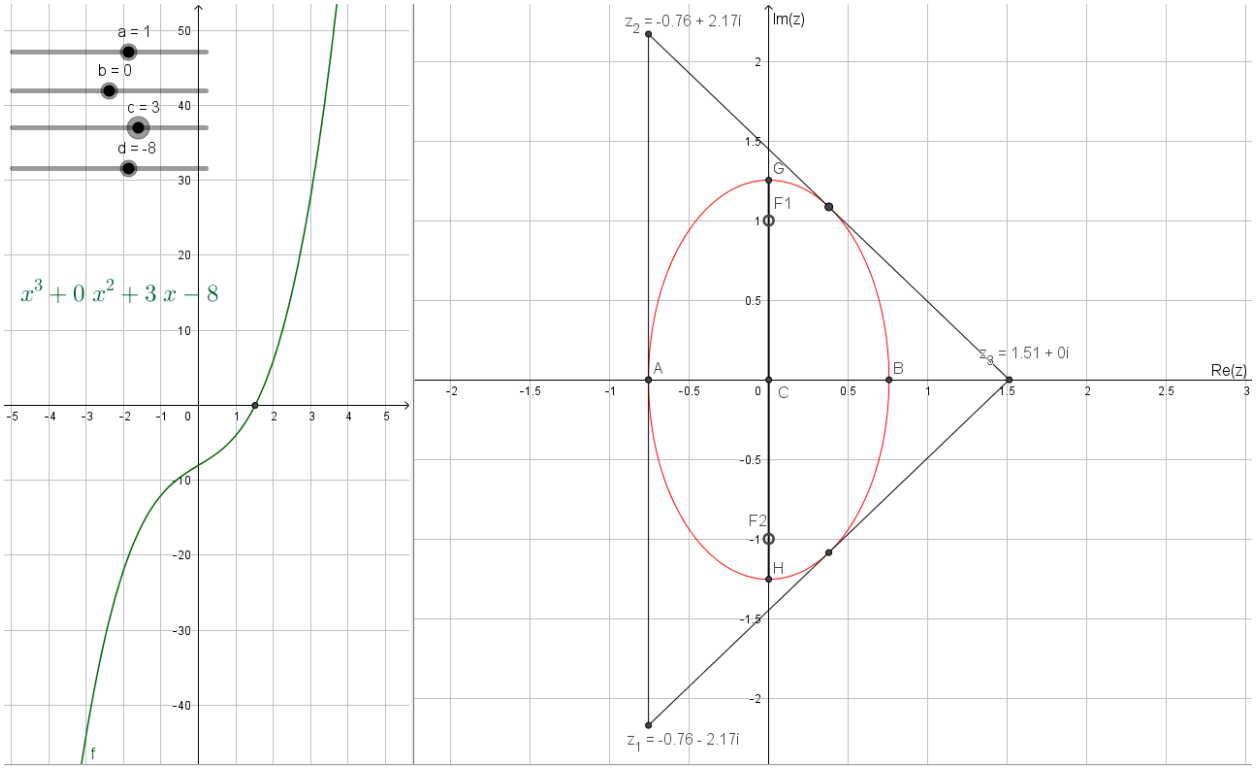

Figure 3 Tangent line at inflection point has positive slope

Graphical representations are valuable when they illustrate the connections between what we know from symbolic mathematics and the underlying concepts. Marden's Theorem reveals relationships between the zeros of a cubic polynomial and a certain ellipse in the complex plane. We hope the reader will be able to use this applet to investigate further geometric relationships between cubic polynomials and the Marden Ellipse.

References:

- Kalman, Dan (2008a), "An Elementary Proof of Marden's Theorem", The American Mathematical Monthly, 115 (4): 330–338, doi:10.1080/00029890.2008.11920532
- Kalman, Dan (2008b), "The Most Marvelous Theorem in Mathematics", Journal of Online Mathematics and Its Applications, Volume 8. March 2008. Article ID 1663
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