KEPLER'S LAWS FROM NEWTON'S LAWS

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(*Abstract.*) The laws of planetary motion were empirical laws formulated by Kepler in the early 1600s. They remained an enigma until the late 1600s when Newton derived them from his laws of motion. The derivation has great pedagogic value that will be appreciated by intermediate students of physics and the calculus.

1. Introduction

1.1 Kepler's laws of planetary motion (1609, 1619)

1.2 Newton's laws of motion (1687)

2. Derivation by vector calculus

2.1 Kepler's first law

2.2 Second law

2.3 Third law

2.4 Gravitational field of a spherical mass distribution

3. Conclusions

Appendix: Alternate derivation of planetary orbit

References

From the voluminous data of Tycho Brahe (1546-1601), Johann Kepler (1571-1630) deduced the three laws of planetary motion:

I. The orbit of each planet is an ellipse, with the Sun at one focus.
II. The line from Sun to planet sweeps out area at a constant rate.
III. The square of the period of revolution of a planet is proportional to the cube of its mean distance from the Sun.

These laws, the first two published in 1609, the third in 1619, were empirical laws deduced from observations; they had no theoretical basis. They remained so until 1687 when Isaac Newton (1642-1787) formulated his laws of motion in his *Principia Mathematica*, and he derived Kepler's laws from them in his 'System of the World'.

I'. In the absence of a force, a body at rest stays at rest, and a body in motion stays in motion with the same speed in the same direction.

II'. When a net force *f* acts on a body *m*, the body accelerates with acceleration *a* proportional to *f* and inversely to *m;* the II' law is usually stated as, $\vec{f} = m\vec{a}$ (in vector form).

III'. For every action, there is an equal and opposite reaction.

IV'. (Law of gravitation) Between any two bodies *m* and *M* in the Universe, there exists a force of attraction *F* that is proportional to the masses and inversely proportional to the square of their separation *r*:

$$F = G\frac{mM}{r^2}, \quad G = \text{the constant of gravitation.}$$

What follows is an updated derivation of Kepler's laws from Newton's laws using vector calculus.

I. From the second law of motion and the law of gravitation:

$$\vec{a} = -\frac{GM}{r^3}\vec{r},$$

$\Rightarrow \quad \vec{a} \parallel \vec{r},$ i.e., $\vec{r} \times \vec{a} = 0.$ Then,

$$\frac{d}{dt}(\vec{r} \times \vec{v}) = \vec{r}' \times \vec{v} + \vec{r} \times \vec{v}' = \vec{v} \times \vec{v} + \vec{r} \times \vec{a} = 0$$

$\Rightarrow \quad \vec{r} \times \vec{v} = \vec{h},$ a constant vector, in general, $\vec{h} \neq 0,$ i.e., $\vec{r} \nparallel \vec{v};$

$\Rightarrow \quad \vec{r} \perp \vec{h}$ for all *t*, so that the planet always lies in the plane through the origin $\perp \vec{h}$. Thus, the orbit of a planet is confined to a plane, or that the orbit is a plane curve.

To find the equation of the orbit:

$$\vec{h} = \vec{r} \times \vec{v} = \vec{r} \times \vec{r}\,' = r\hat{u} \times (r\hat{u})' = r\hat{u} \times (r'\hat{u} + r\hat{u}') = rr'(\hat{u} \times \hat{u}) + r^2(\widehat{\hat{u} \times u'})$$

$$= r^2(\widehat{\hat{u} \times u'}).$$

Then, $\vec{a} \times \vec{h} = -\frac{GM}{r^2}\hat{u} \times r^2(\widehat{\hat{u} \times u'}) = -GM\hat{u} \times (\hat{u} \times \hat{u}') = -GM[(\hat{u} \cdot \hat{u}')\hat{u} - (\hat{u} \cdot \hat{u})\hat{u}']$

$$= GM\hat{u}', \text{ since } |\hat{u}(t)| = 1 \rightarrow \hat{u} \cdot \hat{u} = |\hat{u}|^2 = 1, \text{ and } \hat{u} \cdot \hat{u}' = 0.$$

And so, $\left(\vec{v} \times \vec{h}\right)' = \vec{v}\,' \times \vec{h} = \vec{a} \times \vec{h} = GM\hat{u}'$.

Integrate, $\quad \vec{v} \times \vec{h} = GM\hat{u} + \vec{c}, \;\; \vec{c}$ a constant vector.

Choose the coordinates so that $\hat{k}$ is along $\vec{h}$, and the planet moves in the *xy*-plane. Since both $\vec{v} \times \vec{h}$ and $\hat{u}$ are $\perp \vec{h}, \Rightarrow \;\; \vec{c}$ lies in the *xy*-plane; choose $\vec{c}$ to lie along $\hat{\imath}$ (Figure 1).

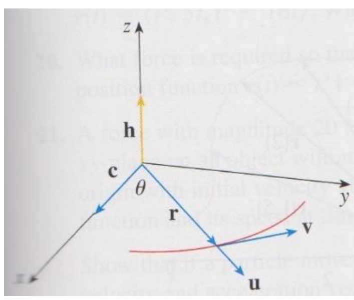

Figure 1. Plane of orbit

Then, (r, θ) are the polar coordinates of the planet, and:

$$\vec{r} \cdot \left(\vec{v} \times \vec{h}\right) = \vec{r} \cdot (GM\hat{u} + \vec{c}) = GM\vec{r} \cdot \hat{u} + \vec{r} \cdot \vec{c}$$

$$= GMr\hat{u} \cdot \hat{u} + rc\cos\theta$$

$$= GMr + rc\cos\theta.$$

Then, $\quad r = \frac{\vec{r} \cdot (\vec{v} \times \vec{h})}{GM + rc\cos\theta} = \frac{1}{GM}\frac{\vec{r} \cdot (\vec{v} \times \vec{h})}{1 + e\cos\theta}, \;\; e = \frac{c}{GM}.$

But, $\quad \vec{r} \cdot \left(\vec{v} \times \vec{h}\right) = (\vec{r} \times \vec{v}) \cdot \vec{h} = \vec{h} \cdot \vec{h} = h^2.$

So, $\quad r = \frac{h^2/GM}{1 + e\cos\theta} = \frac{eh^2/c}{1 + e\cos\theta} = \frac{ed}{1 + e\cos\theta}, \;\; d = \frac{h^2}{c}.$

This is the polar equation of a conic section with focus at the origin and eccentricity e. (An alternate way to obtain the trajectory of a planet is to solve the equation of motion in polar coordinates, cf. Appendix.)

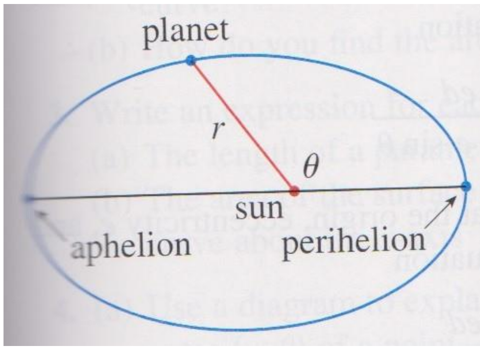

Figure 2. Kepler's I law

II. In polar coordinates:

$$\vec{r} = [(r\cos\theta)\hat{\imath} + (r\sin\theta)\hat{\jmath}], \quad \vec{h} = h\hat{k}$$

$$\vec{h} = \vec{r} \times \vec{v} = \vec{r} \times \vec{r}' = [(r\cos\theta)\hat{\imath} + (r\sin\theta)\hat{\jmath}] \times \left[\left(r'\cos\theta - r\sin\theta\frac{d\theta}{dt}\right)\hat{\imath} + \left(r'\sin\theta + r\cos\theta\frac{d\theta}{dt}\right)\hat{\jmath}\right] = r^2\frac{d\theta}{dt}\hat{k}$$

$$\Rightarrow \quad h = |\vec{h}| = r^2\frac{d\theta}{dt}.$$

The element of area: $dA = \frac{1}{2}r^2 d\theta$, and $\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{h}{2}$, constant since h is constant. We may restate the II Law: The line from Sun to planet sweeps out equal areas in equal times.

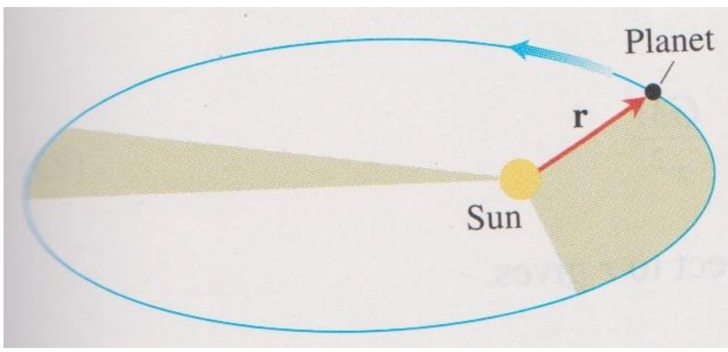

Figure 3. Kepler's II law

From a physics point of view, the area swept out by the planet is proportional to the angular momentum, L, of the planet:

$$dA = \frac{1}{2}|\vec{r} \times \vec{v}dt| = \frac{1}{2m}|\vec{r} \times m\vec{v}|dt = \frac{1}{2m}Ldt = \text{constant.}$$

We note that this result is true for any *central force*, $\vec{f} = f\vec{r}$.

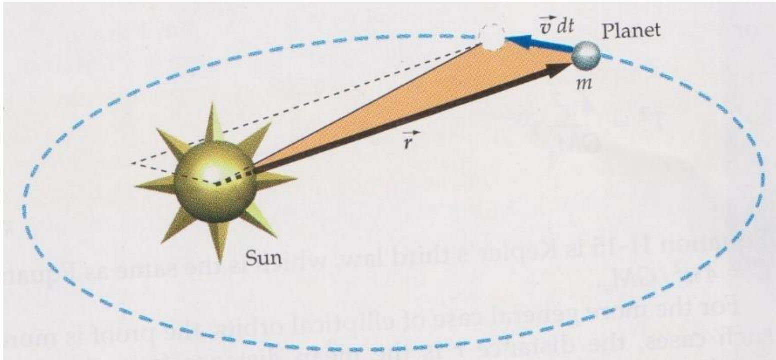

Figure 4. The area swept out in terms of angular momentum

III. Kepler's third law

From the II law:

$$\frac{dA}{dt} = \frac{1}{2}h, \text{ a constant,}$$

$$A(t) = \frac{1}{2}ht + C_1. \;\; A(0) = \pi 0 \quad \rightarrow \quad A(t) = \frac{1}{2}ht.$$

$A =$ area of ellipse, $T =$ period of orbit

$$A(T) = \frac{1}{2}hT \quad \rightarrow \quad T = \frac{2A}{h} = \frac{2\pi ab}{h}.$$

From the I law:

$$\frac{h^2}{GM} = ed, \;\; a = \frac{ed}{1-e^2} \quad \rightarrow \quad ed = a(1-e^2). \text{ Also, } 1 - e^2 = \frac{b^2}{a^2}. \text{ Hence,}$$

$$\frac{h^2}{GM} = ed = \frac{b^2}{a^2}.$$

$$\therefore \quad T^2 = \frac{4\pi^2 a^2 b^2}{h^2} = 4\pi^2 a^2 b^2 \frac{a}{GMb^2} = \frac{4\pi^2}{GM}a^3.$$

We need to show that the mean distance from Sun to planet over one period is equal to the semimajor axis a of the ellipse.

To this end, take an arbitrary ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \quad c^2=a^2-b^2.$$

The distance from Sun to planet, FP, is:

$$d(x,y)=FP=\sqrt{(x-c)^2+(y-0)^2}=\sqrt{x^2-2cx+c^2+b^2-\frac{b^2}{a^2}x^2}$$

$$=\sqrt{\left(1-\frac{b^2}{a^2}\right)x^2-2cx+a^2}=\frac{1}{a}\sqrt{c^2x^2-2ca^2x+a^4}$$

$$=\frac{1}{a}\sqrt{(cx-a^2)^2}. \quad d=\frac{a^2-cx}{a}, \text{ since } a^2>cx.$$

Then, $d=\frac{1}{2a}\int_{-a}^{a}\frac{a^2-cx}{a}dx=\frac{1}{2a^2}\left[a^2x-\frac{cx^2}{2}\right]_{-a}^{a}=\frac{1}{2a^2}\left[a^3-\frac{ca^2}{2}-\left(-a^3-\frac{ca^2}{2}\right)\right]=a,$
i.e., the mean distance from one focus F to a point P on the ellipse is the semimajor axis a.

From the physics point of view, the gravitational force provides the centripetal force on the planet in a circular orbit with mean radius, $r=a$:

$$\frac{GmM}{r^2}=m\frac{v^2}{r} \quad \rightarrow \quad v^2=\frac{GM}{r}.$$

Also, in a circular orbit:

$$v=\frac{2\pi r}{T} \quad \rightarrow \quad v^2=\frac{4\pi^2r^2}{T^2}.$$

And, $$T^2=\frac{4\pi^2}{GM}r^3=\frac{4\pi^2}{GM}a^3.$$

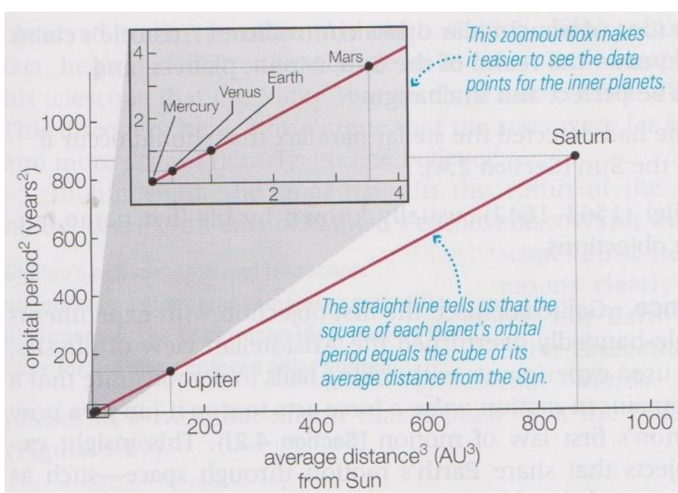

Figure 5. Kepler's III Law, $T^2\ vs.\ a^3$

Gravitational field due to a spherically distributed mass. One of Newton's motivations for developing calculus was to prove that the gravitational field outside a solid sphere is the same as if the mass of the sphere were concentrated at its center. We will show that the gravitational field at a distance *r* from the center of a uniform spherical shell of mass M and radius *R* is:

$$\vec{g} = -\frac{GM}{r^2}\hat{u}, \quad r > R$$

and $$\vec{g} = 0, \qquad r < R.$$

For a solid sphere, we consider it to consist of a continuous set of spherical shells. Since the field due to each shell is the same as if its mass was concentrated at the center of the shell, the field due to the entire solid is as if the entire mass were concentrated at its center:

$$\vec{g} = -\frac{GM}{r^2}\hat{u}, \quad r > R.$$

Inside, only the mass inside the radius *r* contributes to the field:

$$M' = \frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R^3}M = \frac{r^3}{R^3}M \quad \rightarrow$$

$$\vec{g} = -\frac{GM'}{r^2}\hat{u} = -\frac{\frac{GMr^3}{R^3}}{r^2}\hat{u} = -\frac{GM}{R^3}\vec{r}, \quad r < R.$$

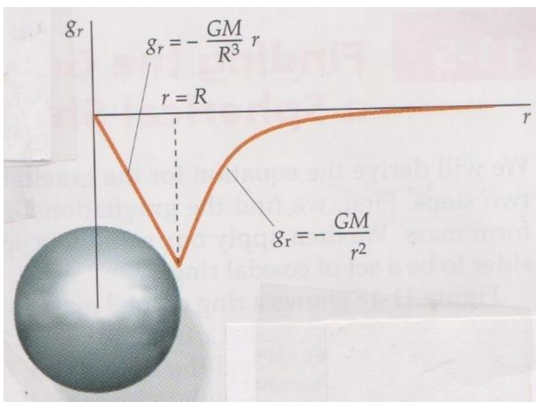

Figure 6. Gravitational field due to a sphere

The results above hold whether or not the sphere has constant density ρ, so long as $\rho = \rho(r)$ for spherical symmetry.

A spherical shell of mass *M* and radius *R* consists of rings a distance *x* from the field point *P*. (i) Outside, $r > R$. The field due to *dm* has magnitude $dg = \frac{Gdm}{s^2}$ along *s;* over the ring,

$$g_x = -\int \frac{Gdm}{s^2} \cos\alpha = -\frac{Gm}{s^2}\cos\alpha.$$

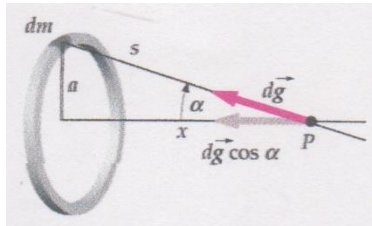

Figure 7. Ring element of mass *dm*

The shell consists of ringstrips of mass *dM* and area *dA:*

$$dM = M\frac{dA}{A} = M\frac{(2\pi R\sin\theta)(Rd\theta)}{4\pi R^2} = \frac{M}{2}\sin\theta d\theta$$

$$\rightarrow \quad dg_r = -\frac{GdM}{s^2}\cos\alpha = -\frac{GM\sin\theta d\theta}{2s^2}\cos\alpha$$

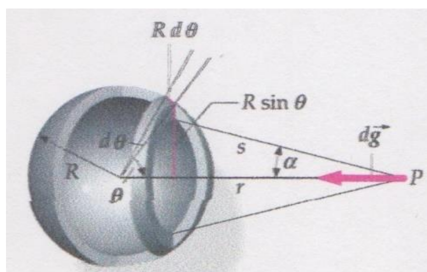

Figure 8. Relation between the variables

The three variables s, θ, α are related by:

$$s^2 = r^2 + R^2 - 2rR\cos\theta \quad \rightarrow \quad 2sds = 2rR\sin\theta d\theta;$$

$$R^2 = s^2 + r^2 - 2sr\cos\alpha \quad \rightarrow \quad \cos\alpha = \frac{s^2+r^2-R^2}{2sr}\ .$$

$$\rightarrow \quad dg_r = -\frac{GM}{2s^2}\frac{sds}{rR}\frac{s^2+r^2-R^2}{2sr} = -\frac{GM}{4r^2R}\left(1+\frac{r^2-R^2}{s^2}\right)ds.$$

The field due to the entire shell is found by integrating from $s = r - R\ (\theta = 0)$ to $s = r + R\ (\theta = 180^0)$,

$$g_r = -\frac{GM}{4r^2R}\int_{r-R}^{r+R}\left(1+\frac{r^2-R^2}{s^2}\right)ds = -\frac{GM}{4r^2R}\left[s-\frac{r^2-R^2}{s}\right]_{r-R}^{r+R} = -\frac{GM}{r^2}.$$

(ii) If the field point P is inside the shell ($r < R$), the calculation is identical except for a change in limits:

$$g_r = -\frac{GM}{4r^2R}\left[s - \frac{r^2-R^2}{s}\right]_{R-r}^{R+r} = 0.$$

Conclusions:

- The laws of planetary motion (Kepler) are derivable from the laws of motion (Newton);
- The derivation is a beautiful elegant powerful demonstration of planetary motion from the laws of motion using a synthesis of physics, geometry, vector analysis, polar coordinates, calculus, and differential equations at a level well appreciated by the beginning student;
- Historically, Newton formulated his laws of motion with Kepler's laws as guides;
- Nowadays, we assign this problem as a weekend homework in our classes in intermediate physics.

Appendix: Alternate derivation of planetary orbit.

To find $r = f(\theta, t) \ \rightarrow \ f(\theta)$. On the *x*,*y*-plane, in terms of the moving unit vectors $\widehat{u_r}, \widehat{u_\theta}$:

(1) $\widehat{u_r} = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}, \qquad \widehat{u_\theta} = -\sin\theta\hat{\imath} + \cos\theta\,\hat{\jmath}$

(2) $\frac{d\widehat{u_r}}{d\theta} = \widehat{u_\theta}\ , \qquad \frac{d\widehat{u_\theta}}{d\theta} = -\widehat{u_r}$

(3) $\frac{d\widehat{u_r}}{dt} = \frac{d\widehat{u_r}}{d\theta}\theta' = \theta'\widehat{u_\theta}\ , \qquad \frac{d\widehat{u_\theta}}{dt} = \frac{d\widehat{u_\theta}}{d\theta}\theta' = -\theta'\widehat{u_r}$

$\vec{r} = r\widehat{u_r}$

(4) $\vec{v} = \vec{r}' = r'\widehat{u_r} + r\widehat{u_r}' = r'\widehat{u_r} + r\theta'\widehat{u_\theta}$

$\vec{a} = \vec{v}' = (r''\widehat{u_r} + r'\widehat{u_r}') + \left(r'\theta'\widehat{u_\theta} + r\theta''\widehat{u_\theta} + r\theta'\widehat{u_\theta'}\right)$

$= (r'' - r\theta'^2)\widehat{u_r} + (r\theta'' + 2r'\theta')\widehat{u_\theta}$

[In 3-D space (though we won't be needing these):

$\vec{r} = r\widehat{u_r} + z\hat{k}$

(5) $\vec{v} = \vec{r}' = r'\widehat{u_r} + r\theta'\widehat{u_\theta} + z'\hat{k}$

$\vec{a} = (r'' - r\theta'^2)\widehat{u_r} + (r\theta'' + 2r'\theta')\widehat{u_\theta} + z''\hat{k},$

where $(\widehat{u_r}, \widehat{u_\theta}, \hat{k})$ make a triad: $\widehat{u_r} \times \widehat{u_\theta} = \hat{k}$, $\widehat{u_\theta} \times \hat{k} = \widehat{u_r}$, $\hat{k} \times \widehat{u_r} = \widehat{u_\theta}$.]

The equation of motion is given by:

(6) $$\vec{F} = m\vec{r}'' = -\frac{GmM}{r^3}\vec{r} \quad \rightarrow \quad \vec{r}'' = -\frac{Gm}{r^3}\vec{r},$$

i.e., the planet is accelerated toward the Sun at all times. This $\Rightarrow$ $\vec{r} \times \vec{r}'' = 0$. But,

$$\frac{d}{dt}(\vec{r} \times \vec{r}') = \vec{r}' \times \vec{r}' + \vec{r} \times \vec{r}'' = \vec{r} \times \vec{r}'' = 0.$$

(7) $\therefore$ $\vec{r} \times \vec{r}' = \vec{h}$, constant $\Rightarrow$ $\vec{r}$ & $\vec{v}$ always lie in a plane $\perp$ to a fixed vector $\vec{h}$, i.e., the orbit is planar. Choose coordinates so that at $t = 0$:

(i) $r = r_0$,

(ii) $r' = 0, minimum$,

(iii) $\theta = 0$,

(iv) $v_0 = r\theta'$ because $v_0 = |\vec{v}|_{t=0} = |r'\widehat{u_r} + r\theta'\widehat{u_\theta}|_{t=0} = r\theta'$.

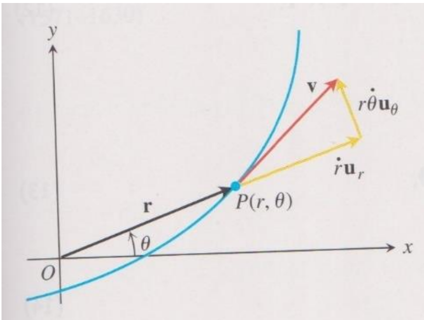

Figure 9. Orbit of planet in a plane

From (5), (6):

(8) $$r'' - r\theta'^2 = -\frac{GM}{r^2}$$

(7) $\rightarrow$ $\vec{h} = \vec{r} \times \vec{v} = r\widehat{u_r} \times (r'\widehat{u_r} + r\theta'\widehat{u_\theta}) = r(r\theta')\hat{k}$.

(9) At $t = 0$: $r_0 v_0 = r^2\theta'$ $\rightarrow$ $\theta' = \frac{r_0 v_0}{r^2}$, (8) becomes $r'' = \frac{{r_0}^2{v_0}^2}{r^3} - \frac{GM}{r^2}$ $\rightarrow$

(10) $$p\frac{dp}{dr} = \frac{{r_0}^2{v_0}^2}{r^3} - \frac{GM}{r^2} \quad \rightarrow \quad p^2 = -\frac{{r_0}^2{v_0}^2}{r^2} + \frac{GM}{r} + C.$$

At $t = 0$: $C = {v_0}^2 - \frac{2GM}{r_0}$; (10) becomes

(11) $r'^2 = v_0{}^2\left(1-\frac{r_0{}^2}{r^2}\right) + 2GM\left(\frac{1}{r}-\frac{1}{r_0}\right).$

But, $\frac{r'}{\theta'} = \frac{dr/dt}{d\theta/dt} = \frac{dr}{d\theta}$, and from (9),

(12) $\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \frac{1}{r_0{}^2} - \frac{1}{r^2} + \frac{2GM}{r_0{}^2 v_0{}^2}\left(\frac{1}{r}-\frac{1}{r_0}\right)$

$$= \frac{1}{r_0{}^2} - \frac{1}{r^2} + 2h\left(\frac{1}{r}-\frac{1}{r_0}\right),\ \ h = \frac{GM}{r_0{}^2 v_0{}^2}.$$

Let $u = \frac{1}{r}$, $\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}$, $\left(\frac{du}{d\theta}\right)^2 = \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$;

(12) $\rightarrow$ $\left(\frac{du}{d\theta}\right)^2 = u_0{}^2 - u^2 + 2hu - 2hu_0 = (u_0-h)^2 - (u-h)^2 \quad \rightarrow$

(13) $\frac{du}{d\theta} = \pm\sqrt{(u_0-h)^2-(u-h)^2}\,.$

We use $(-)$: $\theta' > 0, r' > 0$, $\frac{dr}{d\theta} = \frac{r'}{\theta'} > 0$, and $\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta} \quad \rightarrow$

(14) $\frac{-1}{(u_0-h)^2-(u-h)^2}\frac{du}{d\theta} = 1$

$$cos^{-1}\left(\frac{u-h}{u_0-h}\right) = \theta + C';$$

when $\theta = 0$: $u = u_0, cos^{-1}(1) = 0 \quad \rightarrow \quad C' = 0.$

The solution is:

$$\frac{u-h}{u_0-h} = \cos\theta$$

and, $\frac{1}{r} = u = h - (u_0-h)\cos\theta$

(15) or, $r = \frac{(1+e)r_0}{1+e\cos\theta}$, $e = \frac{1}{r_0 h} - 1 = \frac{r_0 v_0{}^2}{GM} - 1,$

a conic section with the Sun at one focus and with eccentricity *e*.

References

1. I Asimov, 1982. *Asimov's Biographical Encyclopedia of Science & Technology.* NY: Doubleday & Co., Inc.
2. ET Bell, 1965. *Men of Mathematics.* NY: Simon & Schuster, Inc.

3. J Bennett, M Donahue, N Schneider, M Voit, 2018. *Cosmic Perspective.* NY: Pearson.
4. S Chandrasekhar, 1995. *Newton's Principia for the Common Reader.* NY: Oxford University Press.
5. J Stewart, 2008. *Calculus Early Transcendentals.* CA: Thomson Learning, Inc.
6. J Stillwell, 2010. *Mathematics and its History*. NY: Springer.
7. G Thomas, 2005. *Calculus.* NY: Pearson Education, Inc.
8. PA Tipler, 1999. *Physics for Scientists & Engineers*. NY: WH Freeman & Co.