ROUTINES FOR ENGAGING STUDENTS AND EDUCATORS IN A VIRTUAL SETTING

Jay L. Schiffman

Retired, Rowan University

604-36 South Washington Square # 815, Philadelphia, PA 19106

schiffmanjay911@gmail.com

Abstract: Our mindset has certainly been altered as we transitioned to a virtual platform in March 2020. This hands-on workshop focuses on rich problems that incorporate several routines to promote engagement and assist learners to become empowered as well as enjoy mathematics. These routines include Which One Does Not Belong? Always, Sometimes or Never, What if?, manipulatives, puzzles and posing purposeful questions. Please join us and see how these routines demonstrate the power and beauty that is mathematics rendering the discipline a fun and pleasurable enterprise.

We initiate our discussion by considering several discussion activities. At the conclusion of the activities, I pose possible solutions. Since these routines are open-ended, multiple correct solutions are possible rendering such activities dynamic and purposeful.

SOME PROBLEMS AND DISCUSSION ACTIVITIES:

I. In each puzzle consisting of four entries, furnish reasons why each entry is possible as not belonging relative to the others. Multiple solutions are possible.

9	16
25	43

Number 1 from Pam Wilson

17	26
44	65

Number 3 from Mary Borassa

121	16
9	73

Number 5 from Isabelle and Noah Bourassa

$\frac{1}{20}$	$\frac{20}{25}$
$\frac{2}{3}$	$\frac{5}{4}$

Number 10 from Helene Matte

1, 1, 2, 3, 5, 8, …	2, 4, 8, 16, …
-1, 1, 3, 5, 7, …	100, 99, 98, 97, …

Number 23 from Stephanie Orosz

33%	$\frac{1}{3}$
$\frac{5}{3}$	$.\overline{6}$

Number 33 from Erick Lee

$2^x = 4$	$4^x = 2$
$x^2 = 4$	$2^x = -4$

Number 44 from Chris Bolognese

$(-2)^4$	$(-2)^4$
$(-2)^4$	$4^{1/2}$

Number 45 from Chris Bolognese

$\frac{19\pi}{6}$	1110°
$\frac{-11\pi}{6}$	$\frac{7\pi}{3}$

Number 46 from Chris Bolognese

3.14	$\frac{22}{7}$
π	$3.\overline{14}$

Number 50 from Jennifer Wilson

Possible solutions follow.

II. ALWAYS, SOMETIMES OR NEVER ACTIVITY

Determine if each of the following statements is true or false. If it is true, construct a formal proof. For all false statements, determine an appropriate counterexample and explain what needs to be altered to make the false statements true.

(a). The product of any three integers is divisible by 2.

(b). The sum of four consecutive integers is divisible by 4.

(c). The product of four consecutive integers plus one is always a perfect square.

(d). The sum of two prime numbers is never a prime number.

(e). The sum of five consecutive integers is divisible by 5.

(f). If the sides of a right triangle are tripled, then both the perimeter and area of the right triangle are tripled.

III. Consider the calendar for the month of MARCH 2017:

MARCH 2017

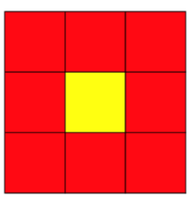
S	**M**	**T**	**W**	**R**	**F**	**S**
			1	**2**	**3**	**4**
5	**6**	**7**	**8**	**9**	**10**	**11**
12	**13**	**14**	**15**	**16**	**17**	**18**
19	**20**	**21**	**22**	**23**	**24**	**25**
26	**27**	**29**	**30**	**30**	**31**	

a. Select several 3×3 groups of numbers and find the sum of these numbers. Determine how the obtained sums related to the middle number.

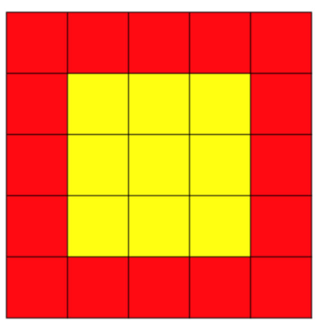
b. Prove that the sum of any 9 integers in any 3×3 set of numbers selected from a monthly calendar will always be equal to 9 times the middle number. Use algebra and technology to furnish a convincing argument.

IV. Use both inductive reasoning (five cases) and then deductive reasoning to solve the following number puzzle employing the given directives:
a. Pick any number.
b. Add 221 to the given selected number.
c. Multiply the sum by 2652.
d. Subtract 1326 from your product.
e. Divide your difference by 663.
f. Subtract 870 from your quotient.
g. Divide your difference by 4.
h. Subtract the original number from your quotient.
i. Observe your result.

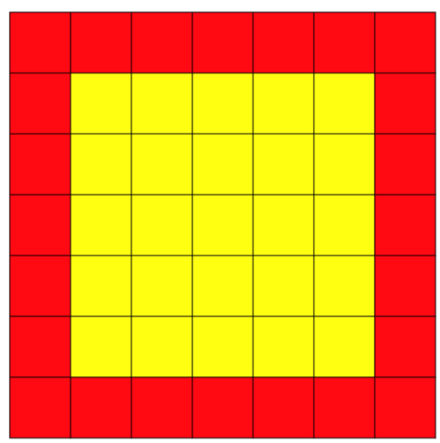
V. Here are the first three figures of a sequence formed by color tiles



1^{st}



2^{nd}



3^{rd}

(a). Find a pattern and describe the next two figures in the sequence.

(b). Describe the 100^{th} figure. Include the number of each color of tile and the total number of tiles in the figure.

(c). Write algebraic expressions for the n^{th} figure for (1) the number of yellow tiles, (2) the number of red tiles and (3) the total number of tiles.

POSSIBLE SOLUTIONS TO DISCUSSION PROBLEMS AND ACTIVITIES:

I. Number 1: One can argue that each of the entries does not belong for a variety of reasons. For example, 9 does not belong; for it is a single digit integer and the sum of the digits of all the other entries is 7. 16 is the only even integer so it does not belong as the others are odd integers. 25 is the only entry that is divisible by 5 and 43 is the only prime integer.

Number 3: 17 is the only prime integer while the other entries are composite. 44 is the only integer that is divisible by 4 and 65 is the only integer that is divisible by 5. Finally, 26 is the only integer that is between a perfect square and a perfect cube.

Number 5: 73 is the only prime integer while 16 is the only even integer. 9 is the only integer that is divisible by 3 and 121 is the only integer that is divisible by 11.

Number 33: 33% is the only entry that is written in percent form, $\frac{5}{3}$ is the only entry larger than one. Meanwhile $\frac{1}{3} = .\overline{3}$ is the only entry that represents an infinitely repeating decimal less than one-half. Finally, $.\overline{6} = \frac{2}{3}$ represents the sole entry expressed as an infinitely repeating decimal in the puzzle.

Number 10: $\frac{1}{20}$ is the sole rational number less than one-half while $\frac{20}{25}$ is the only rational number not expressed in simplest form and $\frac{5}{4}$ is the only rational number entry that is greater than one. Finally, $\frac{2}{3}$ is the sole rational number that can be represented as an infinitely repeating decimal while all the others are represented as terminating decimals.

Number 23: The famous Fibonacci sequence 1, 1, 2, 3, 5, 8, … does not belong as one requires one to know the previous two terms to obtain the next term which is not the case for arithmetic and geometric progressions. 2, 4, 8, 16, … is the sole geometric progression in the puzzle while the arithmetic sequence -1, 1, 3, 5, 7, … is the only sequence which has an initial term which is a negative integer and the arithmetic sequence 100, 99, 98, 97, … is the only decreasing sequence in the puzzle. All the other sequences are increasing.

Number 44: $2^x = 4$ is the only equation that has a unique positive integer as a solution while $x^2 = 4$ is not an exponential equation and has two solutions. $2^x = -4$ is the only equation that has no real number as a solution while $4^x = 2$ is the only equation whose solution is not an integer.

Number 45: This one drives me crazy when I am grading papers. The lack of precision and the use of proper notation is a problem in both traditional and online formats. $(-2)^4$ is the sole entry where one has a negative base. The problem -2^4 represents the only one in the puzzle that has a solution that is a negative integer. 2^{-4} is the sole case where the solution is not an integer. Finally, $4^{1/2}$ is the only problem where the exponent is not an integer.

Number 46: 1110° is the only entry in degree mode while all the others are in radian mode. $\frac{-11\pi}{6}$ is the only member of the puzzle that is negative and $\frac{7\pi}{3}$ is the only entry that is not a multiple of $\frac{\pi}{6}$. $\frac{19\pi}{6}$ is the only positive entry expressed in radian measure that is a multiple of $\frac{\pi}{6}$.

Number 50: π is the only irrational number in the set while 3.14 is the only terminating decimal in the set and $3.\overline{14}$ is an infinitely repeating decimal with the length of the period being two. $\frac{22}{7}$ is the only element that is expressed in rational number form. One can show that the period of the repetend for the fraction $\frac{22}{7}$ is six. Moreover, $\frac{22}{7} = 3.\overline{142857}$.

<u>Note</u>: These are not the only possible solutions, and this is what renders such an activity extremely useful.

II. We determine if each of the following statements is true or false. If it is true, construct a formal proof. For all false statements, determine an appropriate counterexample and explain what needs to be altered to make the false statements true.

(a). The product of any three integers is divisible by 2 is false. Consider the odd integers 3, 5 and 7. Then $3 \cdot 5 \cdot 7 = 105$ *and* $2 \nmid 105$. On the other hand, the product of three consecutive integers is divisible by 2. In addition, the product of three integers, two of which are of even parity is even and hence is divisible by 2. Likewise, the product of three integers such that two are of odd parity is even and hence is divisible by 2.

(b). The sum of four consecutive integers is divisible by 4 is false. Consider the consecutive integers 5, 6, 7 and 8. Then $5 + 6 + 7 + 8 = 26$ *and* $4 \nmid 26$. Moreover, the sum of four consecutive integers is NEVER divisible by 4. To see this, we appeal to the algebra of remainders. In any string of four consecutive integers, if the first is evenly divisible by four, the second will have a remainder of one upon division by four, the third will have a remainder of two upon division by four and the fourth will have a remainder of three upon division by four. A similar proof can be constructed in the respective cases where the first integer in the sequence has a remainder of one, two and three upon division by 4. Just shift accordingly. Hence the sum of the remainders will be six which when reduced modulo four is two. It is true that the sum of any four consecutive integers is always even and thus divisible by two.

(c). The product of four consecutive integers plus one is always a perfect square is indeed true. Let the consecutive integers be denoted respectively by $n,\ n+1,\ n+2\ \textit{and}\ n+3.$ Then $n\cdot(n+1)\cdot(n+2)\cdot(n+3)+1=n^4+6\cdot n^3+11\cdot n^2+6\cdot n+1=\left(n^2+3\cdot n+1\right)^2$. This can be verified using the TI-89/VOYAGE 200. See **FIGURES 1-2:**

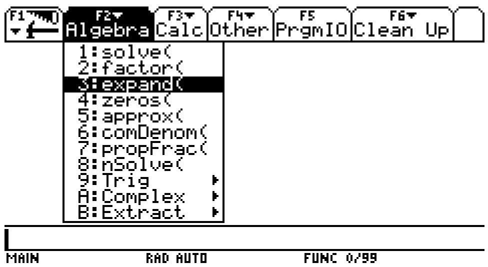


FIGURE 1: The Expand Command

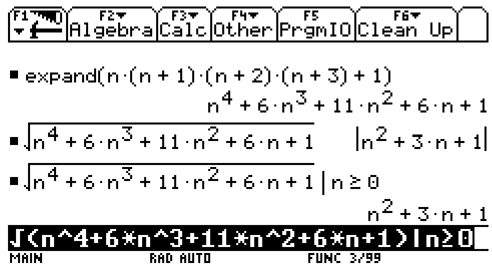


FIGURE 2: Illustrating the Proof

(d). The sum of two prime numbers is never a prime number is false! Consider the prime integers 2 and 5. Then $2+5=7$ which is also a prime number. What is true is that the sum of two odd prime numbers is never a prime; for such a sum yields an even integer which is greater than two and hence has two as a divisor and thus is not prime.

(e). The sum of five consecutive integers is divisible by 5 is indeed true; for if we denote the five consecutive integers by $n,\ n+1,\ n+2,\ n+3\ \textit{and}\ n+4,$ then $n+(n+1)+(n+2)+(n+3)+(n+4)=5\cdot n+10=5\cdot(n+2).$ Notice that the sum of the five consecutive integers is five times the median which is $n+2.$ One can verify this via our graphing calculator. **See FIGURE 3:**

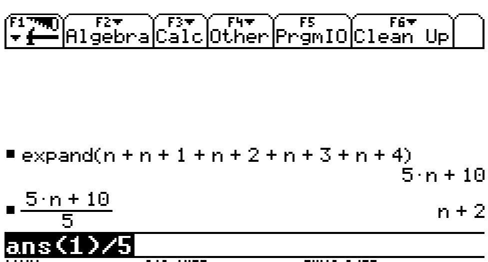


FIGURE 3: Illustrating the Proof

(f). If the sides of a right triangle are tripled, then both the perimeter and area of the right triangle are tripled is false. While the perimeter is indeed tripled, the area is increased nine-fold. To cite an example, consider the 3-4-5 right triangle. Note that $3^2+4^2=9+16=25=5^2.$ This triangle has a perimeter of 12 $(P=a+b+c=3+4+5=12)$ and an area of 6

$\left(A = \frac{1}{2} \cdot a \cdot b = \frac{1}{2} \cdot 3 \cdot 4 = \frac{1}{2} \cdot 12 = 6\right)$. Tripling the sides of the 3-4-5 right triangle yields the similar right triangle 9-12-15. The perimeter is 36 and the area is 54. The new perimeter is three times the original perimeter while the new area is nine times the original area. **FIGURE 4** utilizing the graphing calculator can furnish a formal proof where the area of any right triangle is equal to the product of its legs while the perimeter is the sum of its sides. Let the legs be denoted by a *and* b and the hypotenuse by c.

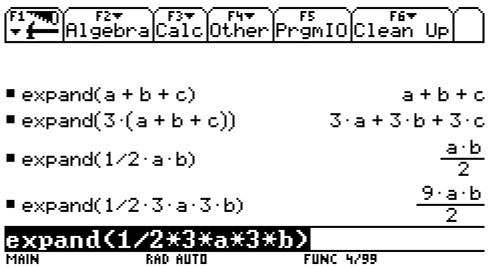


FIGURE 4: The Expand Command

III. Consider the calendar for the month of MARCH 2017 below:

MARCH 2017

S	M	T	W	R	F	S
			1	**2**	**3**	**4**
5	**6**	**7**	**8**	**9**	**10**	**11**
12	**13**	**14**	**15**	**16**	**17**	**18**
19	**20**	**21**	**22**	**23**	**24**	**25**
26	**27**	**28**	**29**	**30**	**31**	

a. We select several 3×3 groups of numbers and find the sum of these numbers and determine how the obtained sums are related to the middle number.

We first consider the group highlighted in **blue** above. The integers are 6, 7, 8, 13, 14, 15, 20, 21 and 22 with the middle number being 14. Adding these nine numbers, we obtain $6+7+8+13+14+15+20+21+22=126=9 \cdot 14$.

Let us next select a second 3×3 group of numbers highlighted in **green** below. These numbers are 9, 10, 11, 16, 17, 18, 23, 24 and 25 with 17 serving as the middle number. We note that $9+10+11+16+17+18+23+24+25=153=9 \cdot 17$.

MARCH 2017

S	M	T	W	R	F	S
			1	**2**	**3**	**4**
5	**6**	**7**	**8**	**9**	**10**	**11**
12	**13**	**14**	**15**	**16**	**17**	**18**
19	**20**	**21**	**22**	**23**	**24**	**25**
26	**27**	**28**	**29**	**30**	**31**	

We finally select a third 3×3 group of numbers highlighted in red below. These numbers are 10, 11, 12, 17, 18, 19, 24, 25 and 26 where 18 is the middle number. Note that $10+11+12+17+18+19+24+25+26=162=9\cdot18.$

MARCH 2017

S	M	T	W	R	F	S
			1	**2**	**3**	**4**
5	**6**	**7**	**8**	**9**	**10**	**11**
12	**13**	**14**	**15**	**16**	**17**	**18**
19	**20**	**21**	**22**	**23**	**24**	**25**
26	**27**	**28**	**29**	**30**	**31**	

b. We next prove that the sum of any 9 digits in any 3×3 set of numbers selected from a monthly calendar will always be equal to 9 times the middle number using both algebra and technology to furnish a convincing argument.

Based on our analysis of three specific cases, it seems plausible to conjecture that the sum of any nine elements in the 3×3 group is always nine times the middle number. To show that this is always true, one can employ algebraic reasoning. We let $x=$ the first number in the array. The next numbers are thus $x+1,\ x+2,\ x+7,\ x+8,\ x+9,\ x+14,\ x+15$ *and* $x+16.$ Next note that $x+x+1+x+2+x+7+x+8+x+9+x+14+x+15+x+16=9x+72=9\cdot(x+8).$ Observe that $x+8$ is the median (middle number) in the array completing our proof.

Technology can play a role as well. Let us use a graphing calculator (TI-89) to furnish the specific cases as well as a formal proof. See **FIGURES 5-8:**

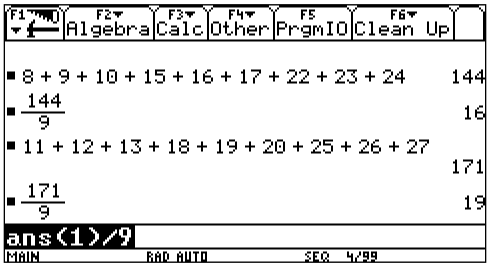


FIGURE 5: A Numerical Example

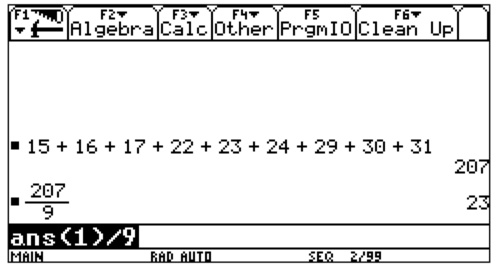


FIGURE 6: A Numerical Example

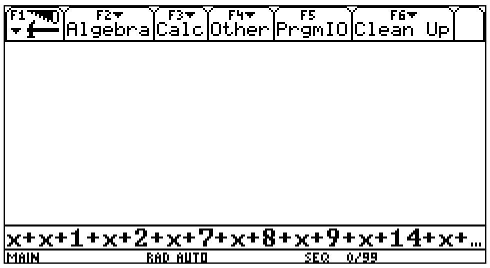


FIGURE 7: The Deductive Proof

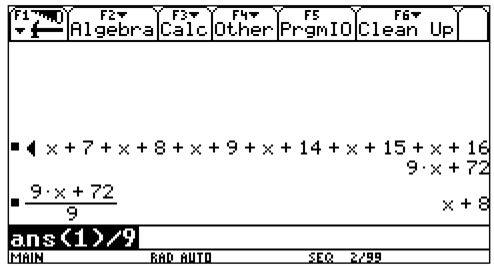


FIGURE 8: The Deductive Proof

IV. In inductive reasoning, we reason to a general conclusion via the observations of specific cases. The conclusions obtained via inductive reasoning are only probable but not certain. In contrast, deductive reasoning is the method of reasoning to a specific conclusion using general observations. The conclusions obtained using deductive reasoning are certain. In the following number puzzle, we employ the five specific numbers 5, 23, 12, 10, and 85 to illustrate inductive reasoning and then employ algebra to furnish a deductive proof. The puzzle and the solutions are provided below:

Pick any Number.	**5**	**23**	**12**	**10**	**85**	n
Add 221 to the given selected number.	**226**	**244**	**233**	**231**	**306**	$n+221$
Multiply the sum by 2652.	**599352**	**647088**	**617916**	**612612**	**811512**	$2652\cdot(n+221)=$ $2652\cdot n+586092$
Subtract 1326 from your product.	**598026**	**645762**	**616590**	**611286**	**810186**	$2652\cdot n+584766$
Divide your difference by 663.	**902**	**974**	**930**	**922**	**1222**	$4\cdot n+882$
Subtract 870 from your quotient.	**32**	**104**	**60**	**52**	**352**	$4\cdot n+12$

Divide your difference by 4.	**8**	**26**	**15**	**13**	**88**	$n+3$
Subtract the original number from your quotient.	**3**	**3**	**3**	**3**	**3**	**3**

The answer we obtain is always 3. We next deploy the calculator to show the inductive cases in **FIGURES 9-18** and the deductive case in **FIGURES 19-20:**

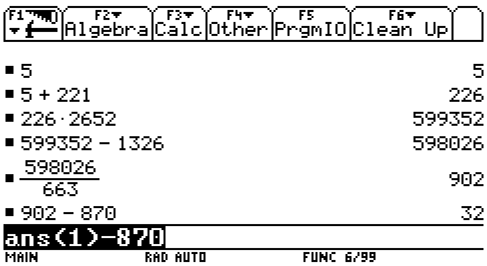


FIGURE 9: A First Example

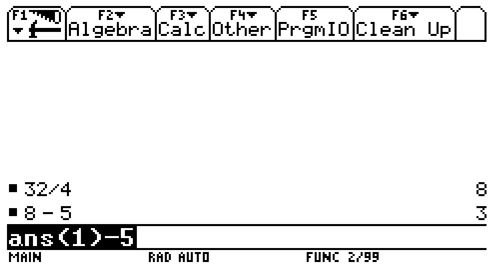


FIGURE 10: A First Example

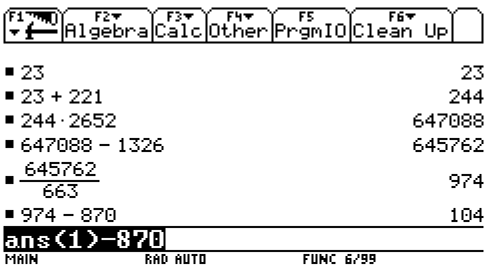


FIGURE 11: A Second Example

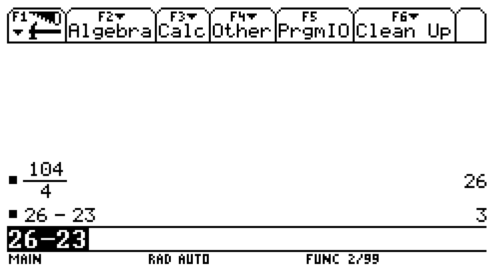


FIGURE 12: A Second Example

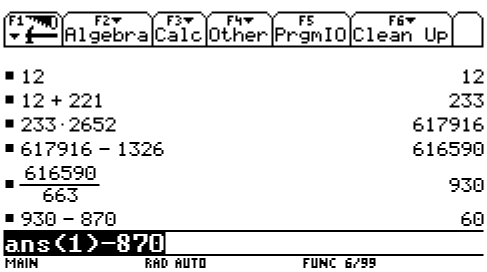


FIGURE 13: A Third Example

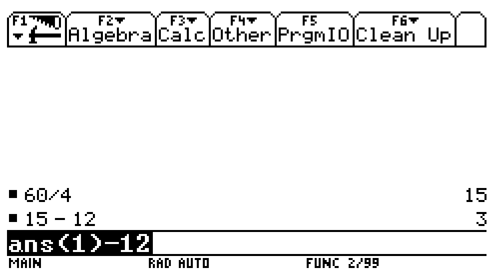


FIGURE 14: A Third Example

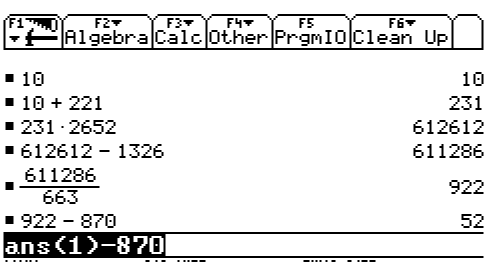


FIGURE 15: A Fourth Example

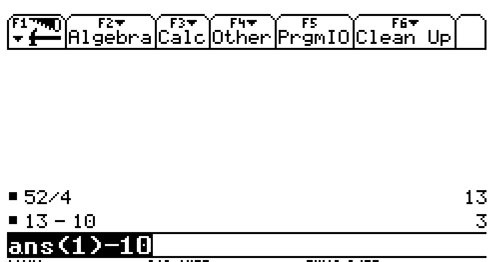


FIGURE 16: A Fourth Example

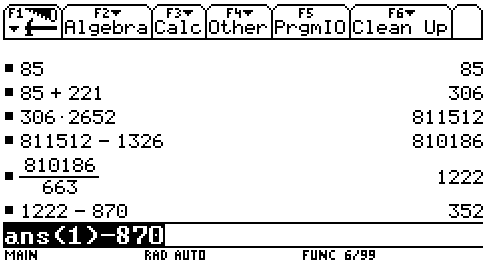


FIGURE 17: A Fifth Example

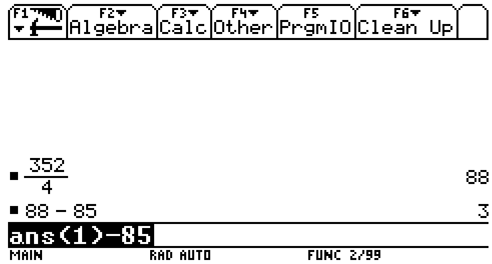


FIGURE 18: A Fifth Example

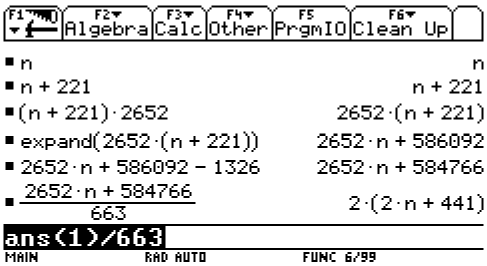


FIGURE 19: The Deductive Proof

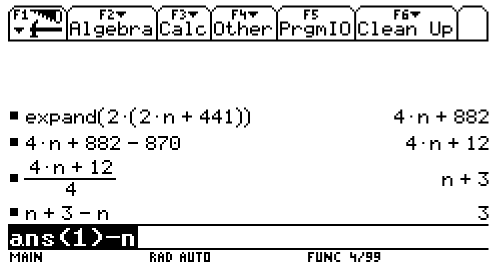
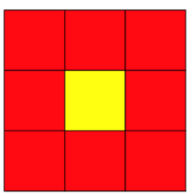
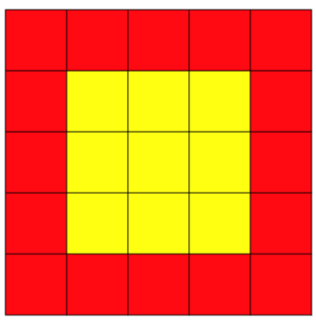


FIGURE 20: The Deductive Proof

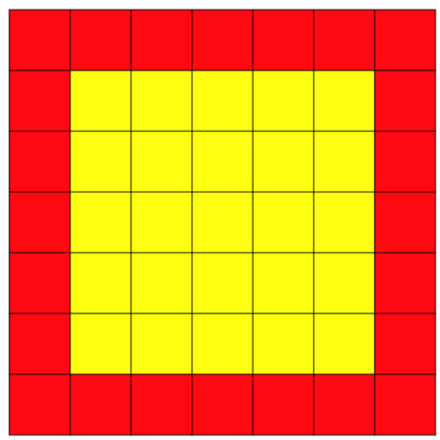
V. Here are the first three figures of a sequence formed by color tiles:



1st

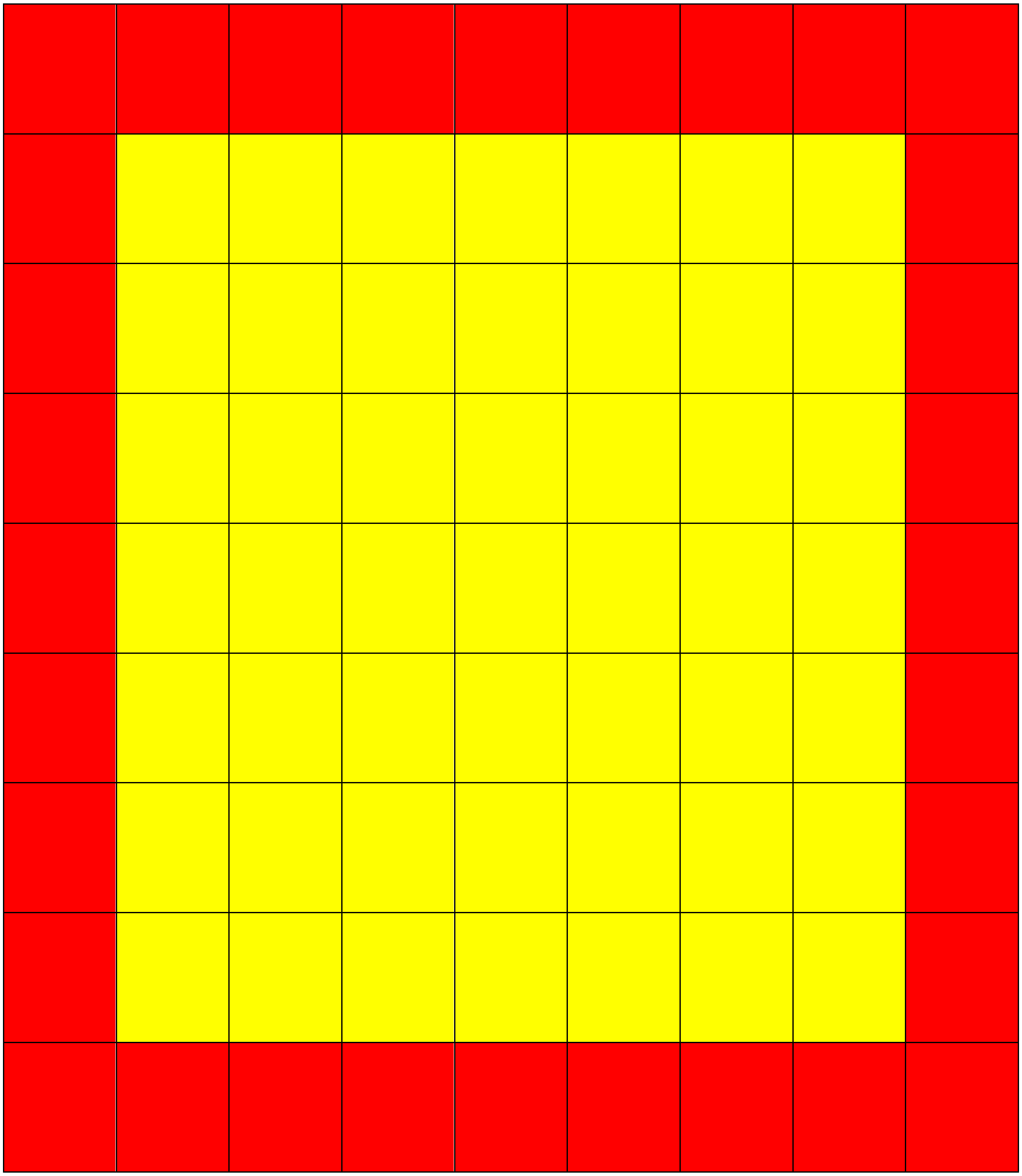


2nd

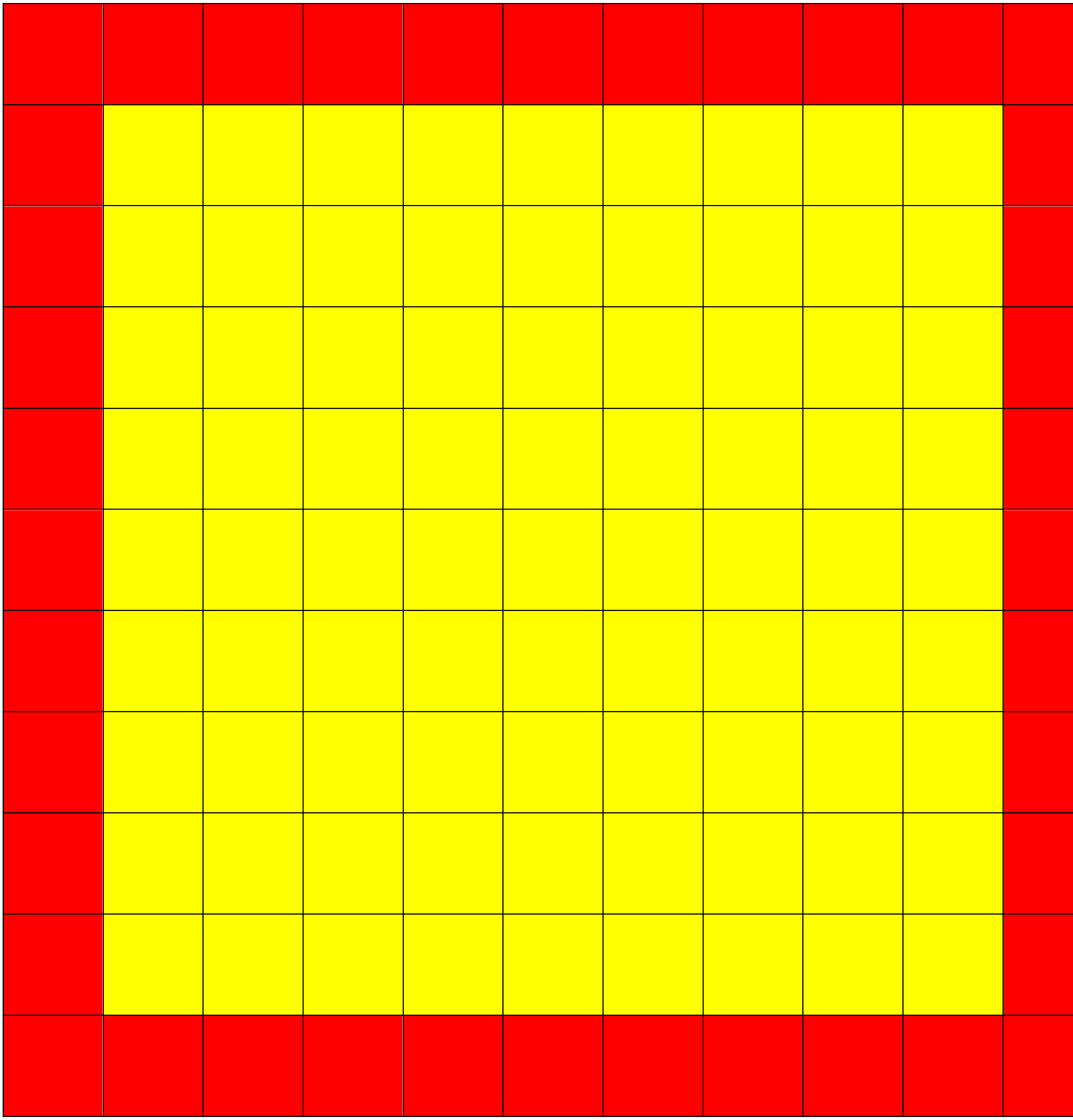


3rd

(a). The first figure consists of nine squares, eight of which are shaded red along the border and one that is shaded yellow in the center. The second consists of twenty-five squares, sixteen are shaded red along the border and nine are shaded yellow. The third is composed of forty-nine squares, twenty-four are shaded red along the border and twenty-five that are shaded yellow. The next two figures respectively contain eighty-one squares, thirty-two which are shaded red along the border and forty-nine shaded yellow and one hundred twenty-one squares, forty of which are red along the border and eighty-one that are shaded yellow. See the figures below:



4^{th}



5th

(b). Following the above pattern, the 100^{th} figure would consist of forty thousand, four hundred one squares of which eight hundred squares are red on the border and thirty-nine thousand six hundred one are yellow. Let us create a table and show the first ten iterations with the number of squares of each color as well as the total number of squares to detect a pattern:

Iteration	Number of Red Squares	Number of Yellow Squares	Total Number of Squares
1^{st}	**8**	**1**	**9**
2^{nd}	**16**	**9**	**25**
3^{rd}	**24**	**25**	**49**
4^{th}	**32**	**49**	**81**
5^{th}	**40**	**81**	**121**
6^{th}	**48**	**121**	**169**
7^{th}	**56**	**169**	**225**
8^{th}	**64**	**225**	**289**
9^{th}	**72**	**289**	**361**
10^{th}	**80**	**361**	**441**
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100^{th}	**800**	**39601**	**40401**
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$n-th$	$8\cdot n$	$(2\cdot n-1)^2$	$(2\cdot n+1)^2$

Observe that the number of red squares coincides with eight times the iteration number while the number of yellow squares is the square of one less than twice the iteration number and the total number of squares is the square of one more than twice the iteration number. Note that
$8\cdot n+(2\cdot n-1)^2=8\cdot n+4\cdot n^2-4\cdot n+1=4\cdot n^2+4\cdot n+1=(2\cdot n+1)^2$.

CONCLUSION:

The routines alluded to in this paper are just a few that have been successfully utilized in both in person and virtual settings. The resiliency of academic professionals during these unprecedented times has been nothing short of amazing as we hopefully transition back to more normal times. Let us keep what has proven successful and refigure those practices that can be improved.