ARROWGRAMS: SECRET MESSAGE PUZZLES BASED ON TRANSITIVITY

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1 Introduction

Mathematical puzzles provide enjoyable and accessible ways to engage learners in the broader discourse of mathematics. An arrowgram is a puzzle built on vertices connected by arrows, that is, a puzzle built on a directed graph. The term, "Arrowgram" was suggested by one of the author's coworker's, Joan Hart. The name is a play on words. Letters sent over air mail are called, "aerograms." Messages contained in arrowgrams are contained in directed graphs, which are built on vertices and arrows. We usually denote a directed graph by D. The directed graphs we study are finite, weakly connected, and contain no loops and no repeated arrows. See [5] for more precise definitions and notation concerning directed graphs.

The author invented arrowgrams in order to explain some aspects of graded ring theory to his students. He has incorporated Arrowgrams into several classes including liberal arts math, linear algebra, and abstract algebra courses on groups and rings. One of his coauthored puzzles appeared in a 2011 issue of Focus, a publication of the Mathematical Association of America (see [6]). A fun thing about arrowgrams is they can contain secret messages. The words are formed by pairs of letters which stand for the arrows.

Definition 1 *The label of an arrow is called its* ***grade****. The grade of an arrow is an element of a group, which is called the* ***grading group****.*

This terminology comes from graded ring theory. The inspiration for arrowgrams came from a method to induce gradings on matrix rings using directed graphs, which was developed in [2] and [3] and studied further in [1] and [5]. Grades for some of the arrows are given. To solve the puzzle you must grade the rest of the arrows using a rule which is based on transitivity. This is covered in section 2.

Throughout, we use additive notation for the group operation. In practice, the grading group is a cyclic group such as the integers or the group of integers modulo a positive integer n. For example, the arrowgram in [6] uses integer addition mod 20. Interesting counting questions arise when finite cyclic groups are used. Methods for constructing gradings are described in section 3. A computer program created by undergraduate student Matthew Zahm for creating arrowgrams is described in section 4. Combinatorics and linear algebra arise naturally in this context and are considered in section 5. Two arrowgrams are offered for your entertainment in section 6.

2 How to Solve and Arrowgram

2.1 Transitivity Rule

We steal language from right triangles to describe the rule for solving an arrowgram.

Definition 2 *In a directed graph, vertices x, y, and z form a* ***transitive triple*** *if there are arrows from x to y, from y to z, and from x to z.*

1. *The arrow from x to z is called the* ***hypotenuse****.*
2. *The arrows from x to y and from y to z are called* ***legs****.*
3. *The* ***transitivity rule*** *requires the sum of the grades of the legs to equal the grade of the hypotenuse in every transitive triple.*

The transitivity rule for grading arrows is based on preserving transitivity in addition. We do not square the legs or the hypotenuse because the numbers on the arrows are not distances. This is a puzzle and the transitivity rule is just the rule you use to solve the puzzle.

The sum described in the transitivity rule is the operation of the grading group. If no grading group is specified, you can assume it is the integers with ordinary addition. In this case we can drop the term "grade" and replace it with "number." We solve the arrowgram by assigning numbers to the arrows so that the sum of the numbers on the legs equals the number on the hypotenuse in every transitive triple. We form words using pairs of letters that represent arrows.

2.2 Secret Message Arrowgram

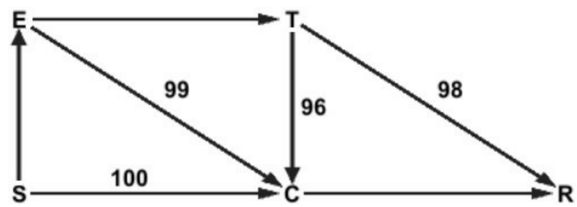


Figure 1: Secret Message Arrowgram

The puzzle in figure 1 has seven arrows CR, EC, ET, SC, SE, TC, and TR. The secret message is formed by the pairs of letters for the arrows numbered by 1, 2, and 3.

2.2.1 Solution

In the transitive triple formed by S, E, and C the number on leg EC is 99 and the number on the hypotenuse SC is 100 so the missing number on leg SE must be 1 since $99 + 1 = 100$. Similarly, in the transitive triple formed by E, C, and T the missing number on leg ET must be 3 since $96+3 = 99$. Finally, in the transitive triple formed by T, C, and R the missing number on leg CR must be 2 since $96 + 2 = 98$. The secret message determined by arrows 1, 2, and 3 is SE CR ET. Now that we've solved it, the secret message is no longer a secret.

3 How to Construct an Arrowgram

These are only guidelines. Be creative!

Think of a secret message. The secret message is formed by arrows, which are pairs of vertices. It should have an even number of letters.

Choose a directed graph D**.** There must be as many vertices as there are different letters in the secret message. If a letter appears more than once in the secret message, we can use the same vertex repeatedly as an endpoint for different arrows. It is usually a good idea to throw in some extra letters to make the secret message harder to guess.

Choose a big enough group G**.** Each arrow in the secret message should have a different grade from all of the other arrows in the directed graph. This uniquely determines the secret message. The integers under addition is certainly big enough, but modular arithmetic or any other group operation can be used.

Determine which arrows to give grades at the start. The puzzle should be somewhat hard to solve. We want to make sure there are enough ungraded arrows that the solver has to work at it. We should also make sure there is enough information to solve for the secret message uniquely.

Definition 3 *A G-**grading set** is a set, S, of arrows such that for any assignment of grades to all of the arrows of S there is a uniquely determined grading of D satisfying the transitivity rule. For example, a grading set in figure 1 is $\{SC, EC, TC, TR\}$.*

Remark 4 *If there is a G-grading set, then no matter how we assign grades to its arrows there is a unique solution to the arrowgram. We can use linear algebra to see if a G-grading set exists.*

3.1 Linear Algebra for Arrowgrams

We form a linear system in standard form, so the unknowns should all be on one side of the equation. The transitivity rule gives a system of equations of the form, leg $-$ hypotenuse $+$ leg $= 0$. (We adopt linear algebra notation and terminology from [4].)

Notation 5 *We denote a transitive triple by (x, y, z), where x, y, and z are vertices, xy and yz are the legs, and xz is the hypotenuse.*

Definition 6 *Suppose G is a finitely written group and D is a finite and weakly connected directed graph, which contains no loops and no repeated arrows. For a grading of D, we denote the grade of the arrow from a to b by X_{ab}.*

1. *A **transitivity equation** is a linear equation $X_{ab} - X_{ac} + X_{bc} = 0$ such that (a, b, c) is a transitive triple in D .*
2. *The **transitive triple equations** is the list all transitivity equations for D.*
3. *We say A is a **transitivity matrix for** D if $\begin{bmatrix} A & \mathbf{0} \end{bmatrix}$ is an augmented matrix for the transitive triple equations.*

Remark 7 *A transitivity matrix A is an $m \times n$ integer matrix, where m is the number of transitive triples and n is the number of arrows. If G is abelian, we can write the transitivity equations as $AX = \mathbf{0}$, where X is a variable vector determined by the arrows of D and the solution space is a subgroup of G^n. Every entry of A is either 0, 1, or -1.*

4 Software to Create Arrowgrams

The math needed to determine the arrow grades is basic linear algebra, which is easy to program. While he was an undergraduate student and under supervision of the author, Matthew Zahm developed software that can create Arrowgrams. It runs on the game development platform Unity, which easily handles our graphical needs because computer games also use a lot of graphics. This software constructs directed graph diagrams quickly and easily and automatically determines grades for all arrows, which helps to create Arrowgrams much faster than doing the calculations by hand.

We explain how to use the software to create the secret message arrowgram provided in figure 1.

Step 1: Create the Directed Graph.

In the first step, SETCR was entered in the box shown on the upper left-hand corner of figure 2. A vertex for each letter was generated as well as arrows SE, ET, TC, and CR. The other three arrows were added using the mouse.

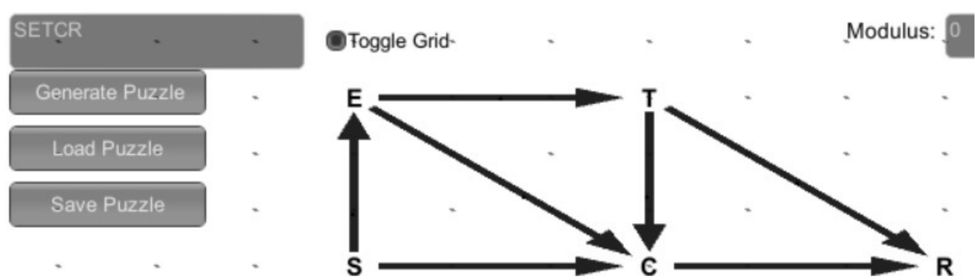


Figure 2: Create Secret Message Arrowgram

Several features of the software are shown in figure 2. Grid points let us line up the vertices horizontally and vertically. They can be turned on and off with a toggle button. The modulus setting is for modular arithmetic. Set it to zero for ordinary addition.

The software determines the transitivity equations directly from the directed graph. Figure 3 shows the transitivity equations on the left side of the screen. They are listed next to their transitive triples.

Step 2: Construct a Transitivity Matrix.

Notation 8 *We depart from the convention set in definition 6 and write ab instead of X_{ab}. We drop the X's in order to focus on the connection with the arrows.*

There is a unique solution for each assignment of grades to the free variables. We want to solve for the secret message, so SE, CR, and ET should be free variables. We can choose the ordering of the arrows in the columns of a transitivity matrix. Putting SE in column 5, CR in column 6, and ET in column 7 increases the likelihood that these become free variables. These choices are shown in the boxes on the upper right side of figure 3. The software finds the transitivity matrix for the homogeneous linear system automatically. It also finds the reduced echelon form and gives a parametric description of the solutions.

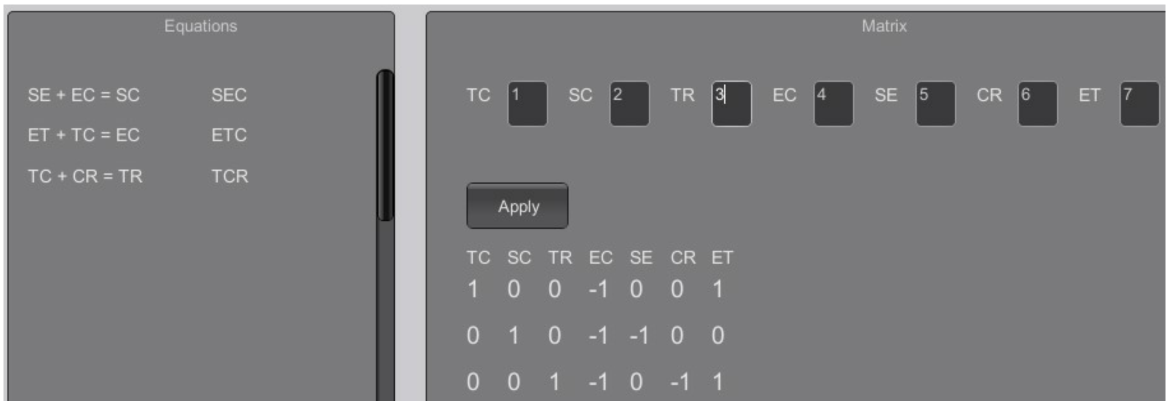


Figure 3: Transitivity Matrix for SECRET Message

Step 3: Find G-Grading Set.

The "Apply" button shown in figure 3 opens up a new page, which displays the free variables on the left and the basic variables on the right.

Figure 4: Variables for Secret Message

As shown in figure 4, we can enter whatever we like for the free variables. Values for basic variables are determined by formulas, which the software obtains from the reduced echelon form of the matrix. The desired solution uses 1, 2, and 3 for the grades of SE, CR, and ET. Setting the fourth free variable, EC, equal to 99 ensures that none of the basic variables are equal to 1, 2, or 3. The software automatically inserts the grades on the arrows in the next step.

Step 4: Remove Unwanted Grades.

Figure 5 shows the directed graph after we removed the unwanted grades. We also curved the arrows to make a more visually interesting graph. It looks very different from the one in figure 1. The control points used to curve the arrows can be toggled on and off so that they don't appear in the final image.

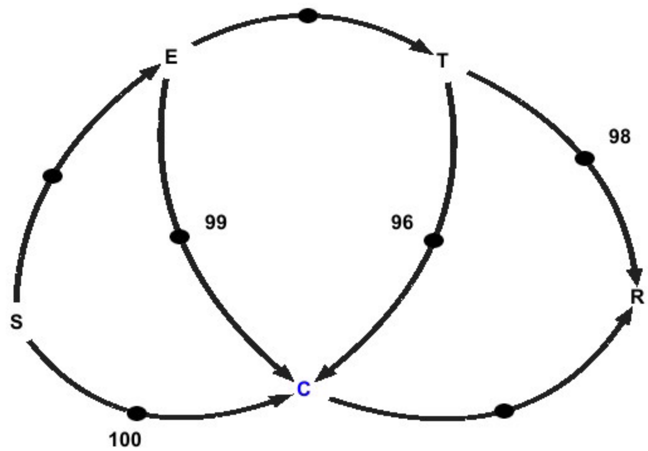


Figure 5: Alternative Directed Graph

Using this software to build Arrowgrams is faster, but it can still take a long time to determine the grading and have a nice-looking diagram.

5 Counting Gradings

Definition 9 *A vertex in a directed graph is called **isolated** if it is neither the start nor the end of some arrow.*

Isolated vertices are not useful for creating arrowgrams. We will often assume there are none.

5.1 Elementary Gradings

To create an arrowgram, we must choose the directed graph and the grading group. There needs to be enough gradings to ensure that the secret message is uniquely determined. One possible choice for the grading is offered in Definition 10.

Definition 10 *Suppose G is a group and D is a weakly connected directed graph with k vertices, which contains no loops and no repeated arrows. An* ***elementary grading*** *is obtained by numbering the vertices of D from 1 to k, choosing $g_1, \dots, g_k \in G$, and declaring that for all i, j, $1 \le i, j \le k$, if an arrow from i to j exists, the grade of ij is $g_j - g_i$.*

Remark 11 *If (i, j, k) is a transitive triple, then the transitivity rule holds since*

$$(g_j - g_i) + (g_k - g_j) = g_k - g_i.$$

Remark 12 *For all i, j, $1 \le i \le j \le k$, we have*

$$g_j - g_i = (g_j - g_{j-1}) + (g_{j-1} - g_{j-2}) + \cdots + (g_{i+1} - g_i)$$

Therefore the elements $g_2 - g_1, g_3 - g_2, \dots, g_n - g_{n-1}$ completely determine the elementary grading. If G is finite and D contains no isolated vertices, then there are $|G|^{k-1}$ elementary gradings.

5.2 Combinatorial Results

Notation 13 *Suppose G is a finite group and D is a finite and weakly connected directed graph, which contains no loops and no repeated arrows. The number of gradings on D over G is denoted by $C_G(D)$.*

Remark 14 *By remark 12, if D contains no isolated vertices and every grading of D is elementary, then $C_G(D) = |G|^{k-1}$, where k is the number of vertices. Conditions for every grading of D to be elementary are provided in [3, Example 2] and [5, Proposition 4.8].*

The author has several results for computing $C_G(D)$, but in a slightly different context (see [1, Notation 1.2]). Under certain conditions, Theorem 15 offers upper and lower bounds on $C_G(D)$.

Theorem 15 *Suppose G is a finite group and D is a finite and weakly connected directed graph, which contains no loops, no repeated arrows, and no isolated vertices. If G has k vertices and n arrows, then $|G|^{k-1} \le C_G(D) \le |G|^n$.*

Proof. We have $|G|^{k-1} \le C_G(D)$ since there are at least $|G|^{k-1}$ elementary gradings. There are $|G|^n$ ways to label the arrows with elements of G, so $C_G(D) \le |G|^n$. ■

If S is a G-grading set for D, then $C_G(D) = |G|^{|S|}$. However, $C_G(D)$ is not always a power of $|G|$ (see [1, Theorem 2.2]). Sufficient conditions on the directed graph for a G-grading set to exist are provided in [5]. Sufficient conditions on the group G are provided in [1, Theorem 2.4]. The latter result has argument is elementary argument, which is provided in the proof of Theorem 16.

Theorem 16 *Suppose D is a directed graph and G is a finite cyclic group of order p, where p is a prime number. Then D contains a G-grading set.*

Proof. We may assume $G = \mathbb{Z}_p$, the group of integers modulo p. Since $\mathbb{Z}_p$ is a field under addition and multiplication modulo p there is a reduced row echelon form of the transitivity matrix for D over $\mathbb{Z}_p$. The free variables will correspond to the elements of a $\mathbb{Z}_p$-grading set. ■

Remark 17 *The number of free variables is $n-r$, where n is the number of arrows of D and r is the rank of the transitivity matrix calculated over $\mathbb{Z}_p$. Thus $C_G(D) = p^{n-r}$ in the situation of Theorem 16.*

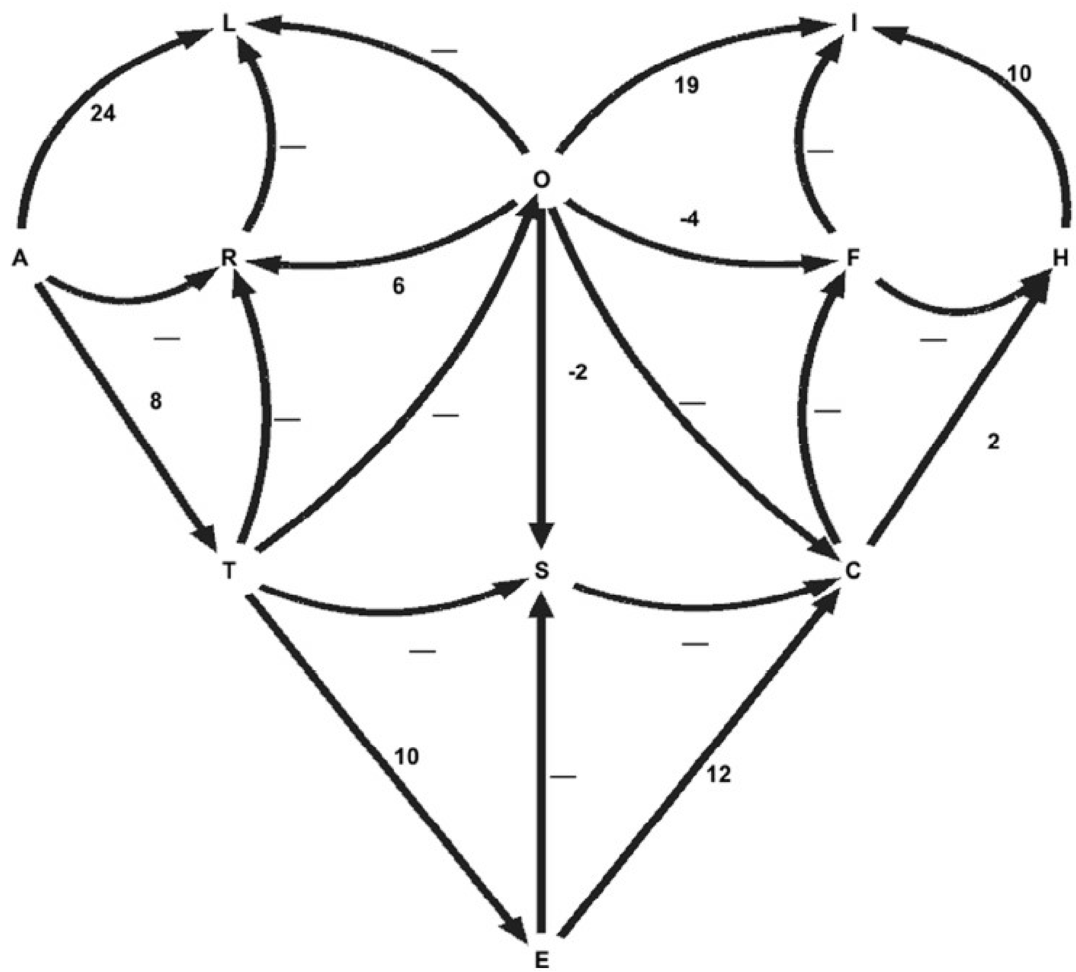
Problem 18 *There are directed graphs that do not contain G-grading sets (see [1, Theorem 2.2]). However, the author is unaware of any directed graphs that do not contain a G-grading set when $|G|$ is odd.*

6 Two Arrowgrams

The arrowgrams contained in this section were created using the software described in section 4. The first one has a heart-shaped directed graph and uses the integers for its grading group. The other arrowgram describes yellow things and uses addition mod 30.

6.1 Heart-Shaped Arrowgram

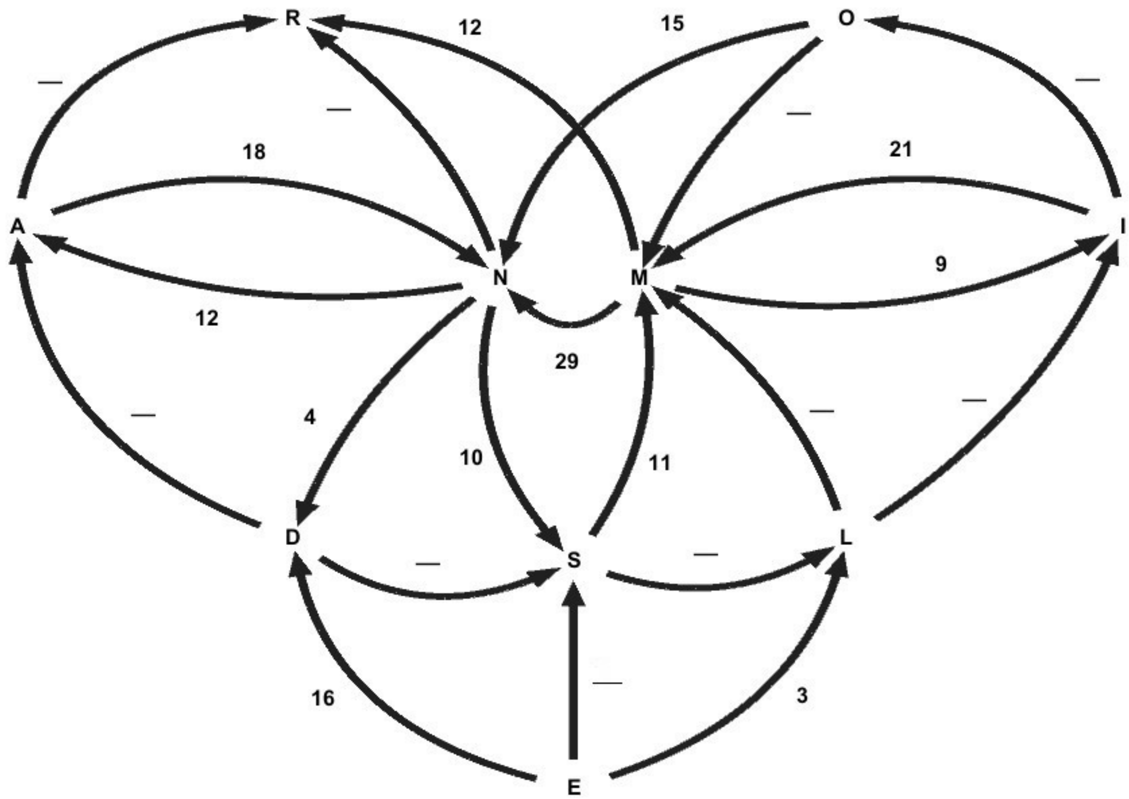
This secret message is something to be shared. There are two repeated numbers, X and Y, with X less than Y.



$\overline{}$ $Y - X - 1$	$\overline{}$ $X - 3$	$\overline{}$ $X - Y + 4$	$\overline{}$ $X - 2$	$\overline{}$ $Y - X$

6.2 Yellow-Things Arrowgram

The secret message lists yellow things. All operations use addition mod 30. There are two repeated grades, X and Y, with $X < Y$.



______	______	______	______	______
$X - 4 \bmod 30$	$Y - 9 \bmod 30$	$Y + 20 \bmod 30$	$48 - y \bmod 30$	$xy - 26 \bmod 30$
______	______	______	______	______
$X - 5 \bmod 30$	$Y + 2 \bmod 30$	$X - 1 \bmod 30$	$2x - 6 \bmod 30$	$Y + 7 \bmod 30$

6.3 Solutions

6.3.1 Heart-Shaped Arrowgram

Something to be shared is “chocolates.” In the solution $X = 10$ and $Y = 13$.

CH	OC	OL	AT	ES
2	7	1	8	3

6.3.2 Yellow-Things Arrowgram

Yellow things are “dandelions” and “lion’s manes.” In this case, $X = 12$ and $Y = 13$.

DA	ND	EL	IO	NS
8	4	3	5	10

LI	ON	SM	AN	ES
7	‘5	11	18	20

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