SOLVING QUARTIC EQUATIONS BY *EXCEL*

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We often encounter the quartic equation in our lectures and research, but we avoid solving it because it is complicated. But there are formulas that will solve the general quartic equation, called Ferrari's formula. The formula published in 1545 by Cardano was discovered by his student, Lodovico Ferrari. Historically, this was significant because it extended the mathematician's achievement to solve polynomial equations beyond the quadratic and the cubic.

The roots of a quadratic equation, $ax^2 + bx + c = 0$ are:

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The quantity $\Delta = b^2 - 4ac$ is known as the discriminant:

$-$, both roots complex;

$\Delta =$ 0, one real root;

+,   both roots real and unequal.

There is a general solution to the quartic equation,

(1) $$ax^4 + bx^3 + cx^2 + dx + e = 0,$$

In Eq. (1), let $x = y - \frac{b}{4a}$, with $a = 1$, then

(2) $$y^4 + py^2 + qy + r = 0,$$

called the reduced quartic,

where $p = \frac{c}{a} - \frac{3b^2}{8a^2}$, $q = \frac{d}{a} - \frac{bc}{2a^2} + \frac{b^3}{8a^3}$, and $r = \frac{e}{a} - \frac{db}{4a^2} + \frac{b^2c}{16a^3} - \frac{3b^4}{256a^4}$.

Write $y^4 = -py^2 - qy - r$, and add $y^2z + \frac{z^2}{4}$ to both sides:

(3) $$\left(y^2 + \frac{1}{2}z\right)^2 = (z-p)y^2 - qy + \left(\frac{1}{4}z^2 - r\right).$$

If the RHS can be put in the form $(my+k)^2$, then the roots of (3) are the roots of

(4a) $$y^2 + \frac{1}{2}z = my + k$$

(4b) $$y^2 + \frac{1}{2}z = -my - k.$$

Since the RHS is a quadratic in y, it can be put in the form $(my+k)^2$ iff its discriminant is zero, i.e,

$$q^2 - 4(z-p)\left(\frac{z^2}{4} - r\right) = 0,$$

Or,

(5) $$z^3 - 3pz^2 - 4rz + (4pr - q^2) = 0,$$

called the resolvent cubic equation.

Any real root of (5) can be used to solve the reduced quartic (2).

For example, the equation

$$y^4 + 3y^2 - 2y + 3 = 0$$

has the resolvent cubic

$$z^3 - 3z^2 - 12z + 32 = 0,$$

which has the root $z = 4$. Thus Eq. (5) becomes

$$(y^2+2)^2 = y^2 + 2y + 1 = (y+1)^2,$$

so, either $y^2 + 2 = y + 1$ and then $y = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$,

or else $y^2 + 2 = -y - 1$ and then $y = -\frac{1}{2} \pm \frac{\sqrt{11}}{2}i$.

Ex.1: All four real roots.

$$2x^4 + 9x^3 - 9x^2 - 46x + 24 = 0, \text{ with roots: } 2,\ \frac{1}{2},\ -3,\ -4.$$

Taking $a = 1$:

$$x^4 + \frac{9}{2}x^3 - \frac{9}{2}x^2 - 23x + 12 = 0$$

The reduced quartic is

$$y^4 - \frac{387}{32}y^2 - \frac{95}{64}y + \frac{112125}{4096} = 0$$

Then,

$$\left(y^2 + \frac{1}{2}z\right)^2 = \left(z + \frac{387}{32}\right)y^2 + \frac{95}{64}y + \left(\frac{z^2}{4} - \frac{112125}{4096}\right).$$

The resolvent equation is;

$$z^3 + \frac{387}{32}z^2 - \frac{112125}{1024}z - \frac{43464575}{32768} = 0,$$

and $z = 10.4687499996378$ is a root of the resolvent equation.

Thus, $(y^2 + 5.2343749998189)^2 = 22.5625y^2 + 1.484375y + 0.024414061 = (4.75y + 0.156249995)^2 \Longrightarrow m = 4.75$ & $k = 0.156249995$ in Eqs. (4).

$$\Longrightarrow \quad y^2 + 5.2343749998189 = 4.75y + 0.156249995$$

$$\Longrightarrow \quad y^2 + 5.2343749998189 = -4.75y - 0.156249995$$

Which leads us to two quadratic equations

$y^2 - 4.75y + 5.078125005 = 0$ & $y^2 + 4.75y + 5.390624995 = 0$

We get four solutions of the above two equations which are as follows;

$y_1 = 3.124999997,$ $\quad$ $y_3 = -1.874999995$

$y_2 = 1.625000003,$ $\quad$ $y_4 = -5.750000010$

$x_1 = 3.124999997 - \frac{9}{8} \approx 2,$ $\quad$ $x_2 = 1.625000003 - \frac{9}{8} \approx \frac{1}{2},$

$x_3 = -1.874999995 - \frac{9}{8} \approx -3,$ $\quad$ $x_4 = -2.875000005 - \frac{9}{8} \approx -4.$

The solutions by *Excel* are done using *Solver.* Here, we calculate the RHS of Eq.(3) by randomizing the value of *z* until we get a discriminant zero. (We can do this because the resolvent cubic always has a real solution.)

In Figure 1, we show the solutions by *Excel*.

Inputs		$a = 1$	Coef reduced quartic	Estimated y	Soln Cubic	Roots	Quartic
a	2	1	1	4.32751E-09	10.46875	x1	2
b	9	4.5	0	y1	0	x2	0.5
c	-9	-4.5	-12.093750000000	y	4.32751E-09	x3	-3
d	-46	-23	-1.484375000000	Δ	1.87273E-17	x4	-4
e	24	12	27.374267578125				

Figure 1. Solutions by *Excel,* Ex.1

Ex. 2: Two Real Roots

$$x^4 + x^3 - x^2 + x - 2 = 0, \text{ with roots: } 1, \ -2, i, \ \& -i.$$

The reduced quartic is

$$y^4 - \frac{11}{8}y^2 + \frac{13}{8}y - \frac{595}{256} = 0,$$

and $$\left(y^2 + \frac{1}{2}z\right)^2 = \left(z + \frac{11}{8}\right)y^2 - \frac{13}{8}y + \left(\frac{z^2}{4} + \frac{595}{256}\right).$$

The resolvent equation is;

(5) $$z^3 + \frac{11}{8}z^2 + \frac{595}{64}z + \frac{5193}{512} = 0$$

$z = -\frac{9}{8}$ is a root of the resolvent Eq. (5).

Thus, $\left(y^2 - \frac{9}{16}\right)^2 = \frac{1}{4}y^2 - \frac{13}{8}y + \frac{676}{256} = \left(\frac{1}{2}y - \frac{13}{8}\right)^2 \implies m = \frac{1}{2}$ $and\ k = -\frac{13}{8}$

$$\implies \quad y^2 - \frac{9}{16} = \frac{1}{2}y - \frac{13}{8} \quad \& \quad y^2 - \frac{9}{16} = -\frac{1}{2}y + \frac{13}{8},$$

which leads us to two quadratic equations

$$y^2 - \frac{1}{2}y + \frac{17}{16} = 0 \quad \& \quad y^2 + \frac{1}{2}y - \frac{35}{16} = 0.$$

We get four solutions of the above two equations which are as follows;

$$y_1 = \frac{1}{4} + i, \qquad y_3 = \frac{5}{4}$$

$$y_2 = \frac{1}{4} - i, \qquad y_4 = -\frac{7}{4}.$$

Thus, the four roots of the given quartic are

$$x_1 = \frac{1}{4} + i - \frac{1}{4} = i, \qquad x_3 = \frac{5}{4} - \frac{1}{4} = 1,$$

$$x_2 = \frac{1}{4} - i - \frac{1}{4} = -i, \qquad x_4 = -\frac{7}{4} - \frac{1}{4} = -2$$

In Figure 2, we show the solutions by *Excel*.

Inputs		$a = 1$	Coef reduced quartic	Estimated y	Soln Cubic	Roots	Quartic
a	1	1	1	-0.000288743	-1.124971126	x1	i
b	1	1	0	y1	0	x2	-i
c	-1	-1	-1.375	y	-0.000288743	x3	1
d	1	1	1.625	Δ	8.33723E-08	x4	-2
e	-2	-2	-2.32421875				

Figure 2. Solutions by *Excel,* Ex. 2

Ex. 3: No Real Roots

$x^4 - 2x^3 + 3x^2 - 2x + 2 = 0,$ with roots: $1 + i,\ 1 - i,\ i,\ \& -i.$

The reduced quartic equation is;

$$y^4 + \frac{3}{2}y^2 + \frac{25}{16} = 0.$$

or, $$\left(y^2 + \frac{1}{2}z\right)^2 = y^2\left(z - \frac{3}{2}\right) + \left(\frac{z^2}{4} - \frac{25}{16}\right).$$

The resolvent cubic equation is;

(1) $$z^3 - \frac{3}{2}z^2 - \frac{25}{4}z + \frac{75}{8} = 0$$

$z = \frac{5}{2}$ is a root of the resolvent equation.

Thus, $$\left(y^2 + \frac{5}{4}\right)^2 = y^2$$

$$\Longrightarrow \quad y^2 + \frac{5}{4} = y \quad \& \quad y^2 + \frac{5}{4} = -y,$$

which leads us to two quadratic equations

$$y^2 - y + \frac{5}{4} = 0 \quad \& \quad y^2 + y + \frac{5}{4} = 0.$$

We get four solutions of the above two equations which are as follows;

$$y_1 = \frac{1}{2} + i, \qquad y_3 = -\frac{1}{2} + i$$

$$y_2 = \frac{1}{2} - i, \qquad y_4 = -\frac{1}{2} - i.$$

Thus, the four roots of the given quartic are

$$x_1 = \frac{1}{2} + i + \frac{1}{2} = 1 + i, \qquad x_2 = \frac{1}{2} - i + \frac{1}{2} = 1 - i$$

$$x_3 = -\frac{1}{2} + i + \frac{1}{2} = i, \qquad x_4 = -\frac{1}{2} - i + \frac{1}{2} = -i.$$

In Figure 3, we show the solutions by *Excel*.

Inputs		$a = 1$	Coef reduced quartic	Estimated y	Soln Cubic	Roots	Quartic
a	1	1	1	-1.1303E-05	-1.124971126	x1	1+i
b	-2	-2	0	y1	0	x2	1-i
c	3	3	1.500	y	-1.1303E-05	x3	i
d	-2	-2	0.000	Δ	1.27758E-10	x4	-i
e	2	2	1.56250000				

Figure 3. Solutions by *Excel,* Ex. 3

Conclusions:

- There is a formula to solve a quartic polynomial, called Ferrari's method. (Descartes has an alternate method, but we did not show it here);
- The method is similar to solving a cubic equation where, first we reduce the equation to one where the cubic term is missing, and then we define parameters so that the remaining quartic equation becomes equivalent to two quadratic equations;
- There are three cases for the roots of a quartic equation: (i) When all four roots are real, (ii) when two roots are real, and (iii) when there are no real roots;
- We verified our solutions by *Excel* using *Solver.*

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