CREATING A PATH OF FASTEST DESCENT THROUGH A GRID

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I Introduction

A rectangle in R^2 is broken into a grid of n rows and m columns. On a given cell, c(i, j), an object can move with a speed of s(i, j). The goal is to traverse from the top-left to the bottom-right of the grid in a minimum amount of time. Newton's method will be used to solve a system of nonlinear equations to find the optimal path.

The problem of finding a path of fastest descent on a grid has applications where a person/particle/army has to move between two points on a map in the shortest amount of time. Different terrains and/or obstacles can cause objects to be able to move at different speeds at different points. The problem may be viewed as a generalization of the Brachistochrone problem [1-3].

In this note, we shall consider the case of a grid consisting of two rows and two columns. The general problem is handled in a similar way.

II Possible Paths

Consider a 2x2 grid defined by the points (0, 0), (x_1, y_1), and (x_2, y_2). We shall consider positive y in the downwards direction.

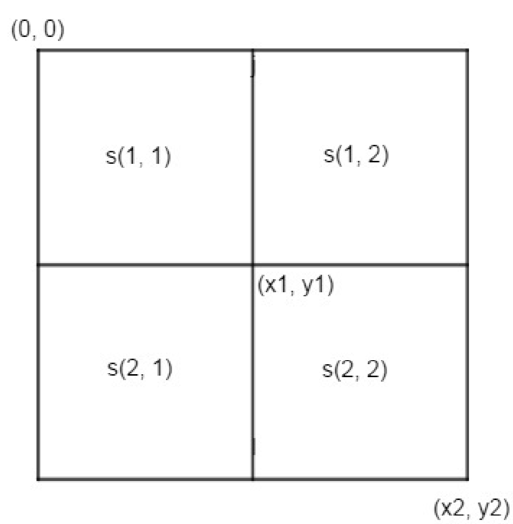


Figure 1

The objective is to move from (0, 0) to (x2, y2) in the minimal amount of time. We define s(i, j) to be the speed an object can move on cell c(i, j) i, j=1, 2.
The possible paths are given by:

Case I:

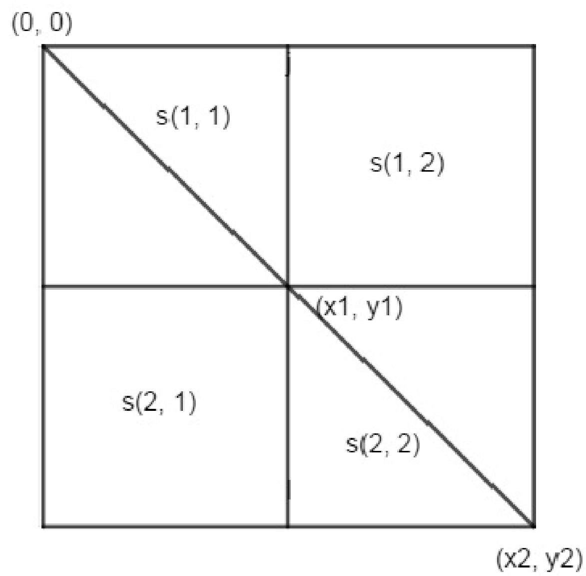


Figure 2a

Case II:

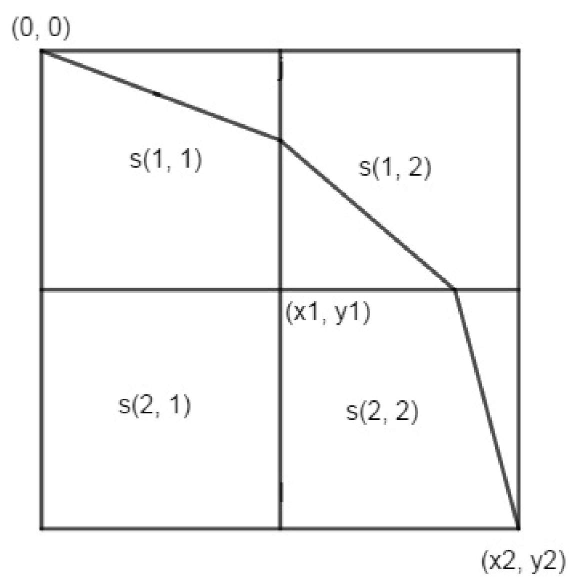


Figure 2b

Case III:

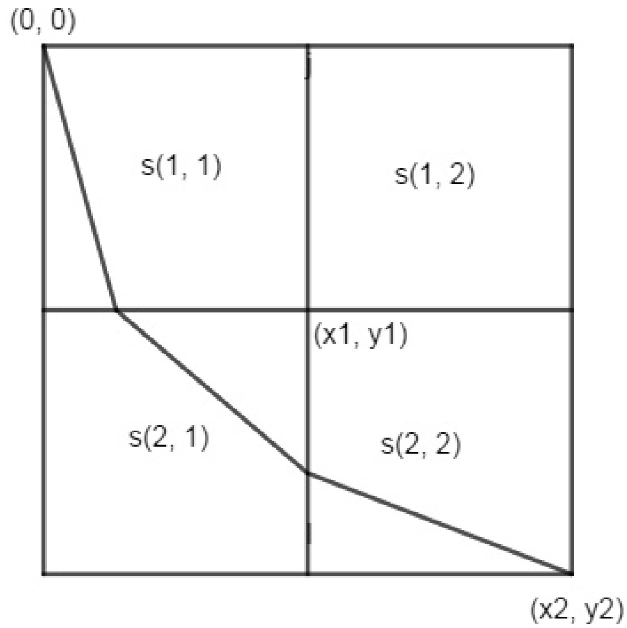


Figure 2c

As case I is trivial and mathematically cases II and III are symmetric, we only need to consider the mathematics of case II. (Of course, if $s(1, 2) > s(2, 1)$ we would use the second path and if $s(2, 1) < s(1, 2)$ we would use path 3.

If $s(1, 2) \leq \min\{s(1, 1), s(2, 2)\}$ the fastest path is just a straight path from (0, 0) to (x2, y2). So to avoid the trivial case assume $s(1, 2) > \min\{s(1, 1), s(2, 2)\}$.

III Finding the Fastest Path

The path C between (0, 0) and (x2, y2) is defined by two parameters, d1 and d2, as shown in Figure 3 below:

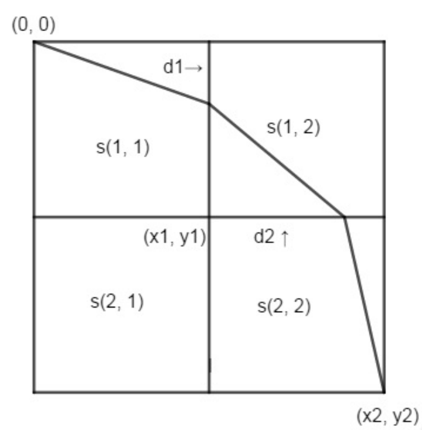


Figure 3

The time to get from (0, 0) to (x2, y2) is given by

$$T(d1, d2) = \frac{\sqrt{x1^2 + d1^2}}{s(1,1)} + \frac{\sqrt{(y1 - d1)^2 + d2^2}}{s(1,2)} + \frac{\sqrt{(x2 - x1 - d2)^2 + (y2 - y1)^2}}{s(2,2)}$$

(1)

Goal: Minimize T(d1, d2) over d1ϵ[0,y1] and d2ϵ[0, x2-x1].

Our approach utilizes two steps:

1) To get an initial estimate of d1 and d2 we write a program breaking the intervals [0, y1] and [0, x2-x1] into, say, 1000 pieces each, and out of the 1,000,000 possible paths find the one that yields the minimal time. Call the optimal d1 and d2 $d1^{(0)}$ and $d2^{(0)}$
2) Use ($d1^{(0)}$, $d2^{(0)}$) as an initial estimate of the solution of the nonlinear equations

$$\frac{\partial T}{d1} = 0 \; and \; \frac{\partial T}{d2} = 0 \tag{2}$$

in Newton's method [4]:

$$d^{(i)} = d^{(i-1)} - [J(d^{(i-1)})]^{-1} F(d^{(i-1)}) \tag{3}$$

where $d^{(i)} = col\left(d1^{(i)}, d2^{(i)}\right)$, J is the Jacobian of T(d1, d2) and $F = col\left(\frac{\partial T}{\partial d1}, \frac{\partial T}{\partial d2}\right)$.

Using (2)-(3), with our initial estimate $d^{(0)} = col(d1^{(0)}, d2^{(0)})$ Newton's method yields the values of d1 and d2 which will define our path of fastest descent.

IV Example

Consider the 2x2 grid consisting of 100x100 cells and s(1, 1)=10, s(1, 2)=50, s(2, 1)=30, and s(2, 2)=20.

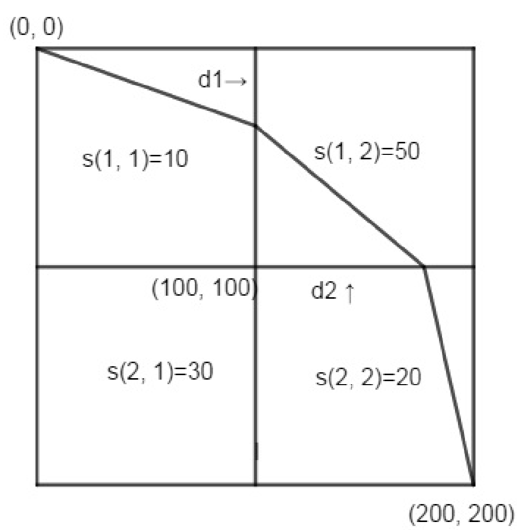


Figure 4

Writing a program breaking [0, 100] into 1000 pieces for both d1 and d2, out of the 1,000,000 possible paths, the values of d1 and d2 which give the minimum time are:

$$d1^{(0)} = 15.3 \; and \; d2^{(0)} = 73 \tag{4}$$

The time of this path is given by T=17.53175658 time units.

Using (4) as our initial values in Newton's method (2)-(3) we get

$$d1 = 15.32871767758 \; and \; d2 = 72.952450225974 \tag{5}$$

after just two iterations. The time of this path is given by T=17.53175564 time units.

As the straight-line path takes

$$\frac{\sqrt{20000}}{10} + \frac{\sqrt{20000}}{20} = 21.2132 \; times \; units \tag{6}$$

The path given by (5) is our optimal path. This is illustrated by the Maple graph of z=T(d1, d2) shown below.

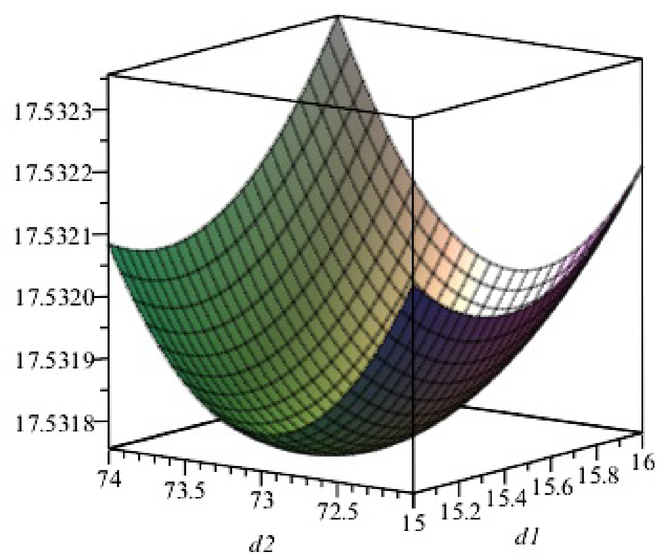


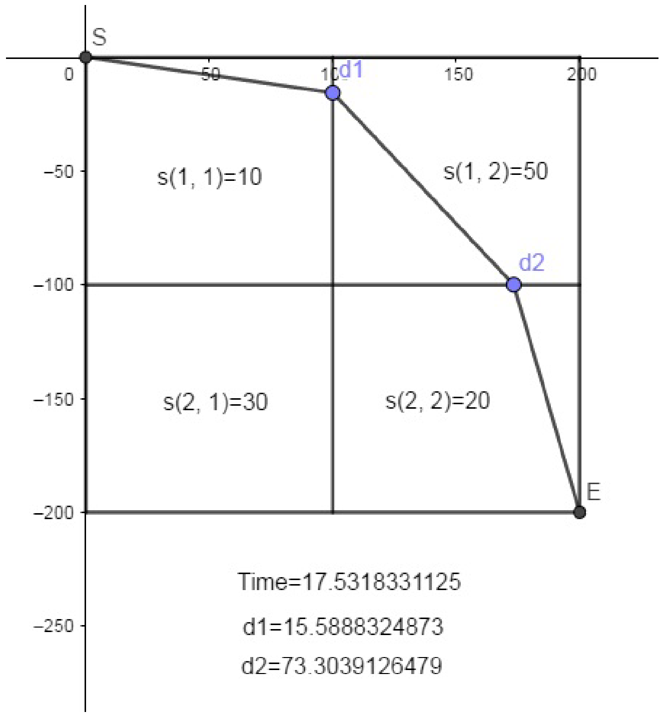
Figure 5

V Remarks on Computations

We can see from (4) and (5) that we gained almost no improvement timewise by using the extra computations from Newton's method. So, for low order cases (number of rows and columns equaling 2) Newton's method may not be necessary.

However, in higher order cases, due to the additional parameters needed to define a minimal time curve, we will not be able to approximate initial values with as much accuracy due to computational complexity. In these cases, Newton's method (or some form of quasi-Newton or steepest-descent method) may be needed to compute an optimal path.

VI Optimization by Geogebra



In lower order cases it is possible to use Geogebra to get a quick estimate of the optimal path by moving the points representing d1 and d2 as shown above. Moving d1 and d2 automatically compute the total time it takes to traverse the path. In this case it only took a minute or so to get a path from (0, 0) to (200, 200) whose time is given by T=17.531883, which only differs from the optimal path by .0001 time units.

VII Conclusions and Future Work

Like the Brachistochrone problem [1-3], the shortest path is in general not a straight line. Future work will generalize the result to a grid with n rows and m columns where n and m are arbitrary. The main problem that will be encountered is that of computational complexity.

References

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