

## GEODESICS ON SURFACES BY VARIATIONAL CALCULUS

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Geodesics are curves of shortest distance on a given surface. Apart from their intrinsic interest, they are of practical importance in the transport of goods and passengers at minimal expense of time and energy. They are also of paramount importance as escape routes during flights. Finding geodesics can be accomplished using the methods of differential geometry. We will use

instead the calculus of variation, which we have used before in solving the brachistochrone problem.

The fundamental equation in the calculus of variations is the Euler-Lagrange equation:

$$(1) \quad \frac{\partial f}{\partial y} - \frac{d}{dx} \left( \frac{\partial f}{\partial y'} \right) = 0, \quad x_1 \leq x \leq x_2.$$

In differential calculus, we are looking for those values of  $x$  which give some function  $f(x)$  its maximum or minimum values. In the calculus of variation, we are seeking the function  $f$  itself that makes some integral of  $f$ , satisfying certain conditions, a maximum or minimum. In the geodesic problem, we wish to find that curve  $f = f(x, y, y')$  that joins two points on a given surface such that the distance between them is minimized. Thus, the problem is to find that integrand  $f$  which minimizes the integral of the arclength:

$$(2) \quad L = \int ds = \int_{x_1}^{x_2} \sqrt{1 + (y')^2} dx, \quad f = \sqrt{1 + (y')^2}.$$

Here,  $f = f(x, y, y')$ ,

$$\text{and, } \frac{d}{dx} \left( \frac{\partial f}{\partial y'} \right) = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y'} \right) + \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y'} \right) + \frac{\partial}{\partial y'} \left( \frac{\partial f}{\partial y'} \right).$$

Hence, (1) is equivalent to:

$$(3) \quad f_y - f_{y'x} f_{y'y'} - f_{y'y} y'' = 0.$$

There are first integrals of (1) based on two simplifying assumptions: (i) If  $f$  is independent of  $y$ , i.e.,  $\frac{\partial f}{\partial y} = 0$ . Then

$$(4) \quad \frac{d}{dx} \left( \frac{\partial f}{\partial y'} \right) = 0 \Rightarrow \frac{\partial f}{\partial y'} = C$$

a first-order degree 1 differential equation.

(ii) If  $f$  is independent of  $x$ :

$$(5) \quad \frac{d}{dx} \left( y' \frac{\partial f}{\partial y'} - f \right) = y' \frac{d}{dx} \left( \frac{\partial f}{\partial y'} \right) - \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} y' = -y' \left[ \frac{\partial f}{\partial y} - \frac{d}{dx} \left( \frac{\partial f}{\partial y'} \right) \right] - \frac{\partial f}{\partial x}$$

The last term = 0,  $\Rightarrow$

$$(5') \quad y' \frac{\partial f}{\partial y'} - f = C.$$

The geodesic problem may be recast in curvilinear coordinates. The surface  $g(x, y, z) = 0$  in parametric form:

$$(6) \quad x = x(u, v), \quad y = y(u, v), \quad z = z(u, v).$$

The arclength  $ds$  is:

$$(7) \quad (ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 = Pdu^2 + 2Qdudv + Rdv^2$$

$$\text{with} \quad P = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2, \quad R = \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2,$$

$$Q = \frac{\partial x}{\partial u} \frac{\partial x}{\partial v} + \frac{\partial y}{\partial u} \frac{\partial y}{\partial v} + \frac{\partial z}{\partial u} \frac{\partial z}{\partial v}.$$

And the integral (2) to minimize is:

$$(8) \quad L = \int_{u_1}^{u_2} [P + 2Qv' + Rv'^2]^{1/2} du.$$

The first integrals (4) and (5') become (9) and (10):

$$\text{with} \quad x \rightarrow u, \quad y \rightarrow v, \quad v' = \frac{dv}{du};$$

$$\frac{\frac{\partial P}{\partial v} + 2v' \frac{\partial Q}{\partial v} + v'^2 \frac{\partial R}{\partial v}}{2\sqrt{P+2Qv'+Rv'^2}} - \frac{d}{du} \left( \frac{Q+Rv'}{\sqrt{P+2Qv'+Rv'^2}} \right) = 0.$$

If  $P, Q, R$  are explicit functions of  $u$  alone, the first term is zero, and further if  $Q = 0$ , then

$$\frac{Rv'}{\sqrt{P+Rv'^2}} = C$$

And the solution is:

$$(9) \quad v = C \int \frac{\sqrt{P}}{\sqrt{R^2 - C^2 R}} du.$$

If  $Q = 0$  and  $P, R$  are explicit functions of  $v$  alone:

$$v' \frac{\partial f}{\partial v'} - f = C,$$

$$\text{or,} \quad \frac{Rv'^2}{\sqrt{P+Rv'^2}} - \sqrt{P+Rv'^2} = C,$$

with the solution:

$$(10) \quad u = C \int \frac{\sqrt{R}}{\sqrt{P^2 - C^2 P}} dv.$$

Examples of geodesics. Surface 1: A plane. Our problem is:  $L = \int_{x_1}^{x_2} f(x, y, y') dx$ ,

with  $f = \sqrt{1 + y'^2}$ ,

independent of  $y$ , thus, we have solution (4):

$$\frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1+y'^2}} = C$$

$$\Rightarrow y'^2 = C^2(1 + y'^2) \Rightarrow y'^2 = \frac{C^2}{1-C^2} = C'^2 \Rightarrow \frac{dy}{dx} = C'$$

$$(11) \quad \therefore y = C'x + C''.$$

Thus,  $y$  is a linear function of  $x$ , with the constants  $C'$  and  $C''$  determined by joining  $x_1$  to  $x_2$ . The geodesic on a plane is a *straight line* between the two points (Figure 1).

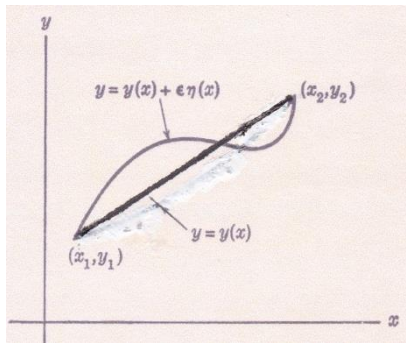


Figure 1. Geodesic on a plane

### Surface 2: Sphere

$$x = a \sin v \cos u, \quad y = a \sin v \sin u, \quad z = a \cos v,$$

with  $u =$  longitude,  $v =$  colatitude,  $a =$  radius.

Then,  $P = a^2 \sin^2 v$ ,  $Q = 0$ ,  $R = a^2$ , and  $f = \sqrt{P + 2Qv' + Rv'^2}$ .

The solution is given by (10):

$$\begin{aligned} u &= C \int \frac{\sqrt{R}}{\sqrt{P^2 - C^2 P}} dv = C \int \frac{adv}{\sqrt{a^4 \sin^4 v - C^2 a^2 \sin^2 v}} = \int \frac{csc^2 v dv}{\sqrt{[(a/c)^2 - 1] - cot^2 v}} \\ &= -\sin^{-1} \frac{cot v}{\sqrt{(a/c)^2 - 1}} + C'. \end{aligned}$$

The last equation may be rewritten:

$$(\sin C')a \sin v \cos u - (\cos C')a \sin v \sin u - \frac{a \cos v}{\sqrt{(a/c)^2 - 1}} = 0$$

$$(12) \quad \therefore \quad x \sin C' - y \cos C' - \frac{z}{\sqrt{(a/c)^2 - 1}} = 0.$$

The geodesic is the intersection of the sphere with a plane through its center connecting the two points on its surface – a *great circle*.

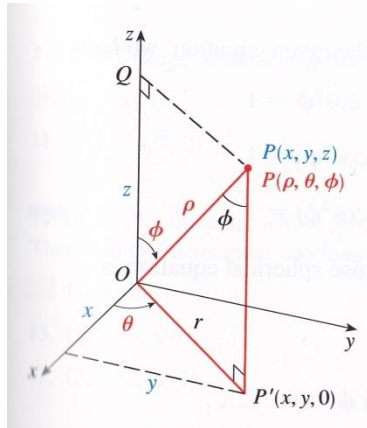


Figure 2. Spherical coordinates ( $\theta \rightarrow u, \phi \rightarrow v$ )

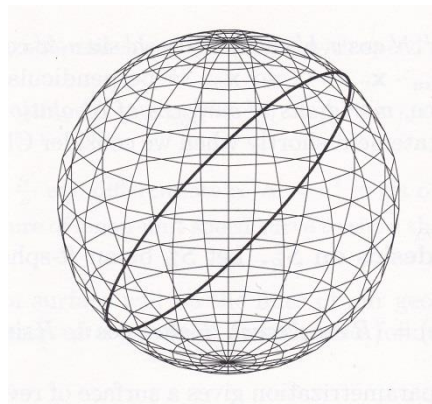


Figure 3. Geodesic on a sphere: a great circle

Surface 3: Right circular cylinder

$$x = a \cos \theta, \quad y = a \sin \theta, \quad z = u, \quad a = \text{radius of the cylinder};$$

$$ds^2 = dx^2 + dy^2 + dz^2 = (-a \sin \theta)^2 + (a \cos \theta)^2 + dz^2 = (a d\theta)^2 + dz^2.$$

With  $z \rightarrow u$ ,  $\theta \rightarrow v$ ,  $\theta' \rightarrow v'$ :  $L = \int ds = \int \sqrt{a^2 v'^2 + 1} du \Rightarrow f = \sqrt{a^2 v'^2 + 1}$ .

Here,  $f$  is independent of  $u$  and  $v$ , and we have solution (9):  $v' = C' \Rightarrow$

$$(13) \quad \theta = C'z + C''.$$

The geodesic on a right circular cylinder is a cylindrical spiral – a *helix*.

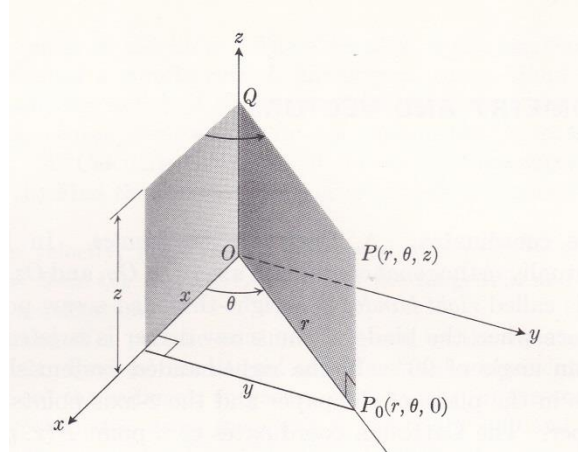


Figure 4. Cylindrical coordinates ( $z \rightarrow u$ ,  $\theta \rightarrow v$ )

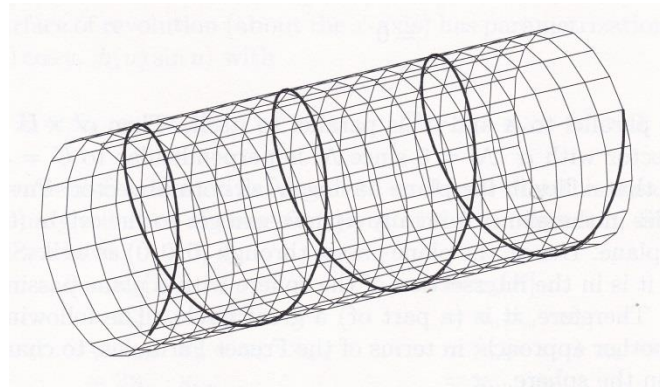


Figure 5. Cylindrical spiral

Surface 4: Right circular cone. Here,

$$x = u \sin \alpha \cos v, \quad y = u \sin \alpha \sin v, \quad z = u \cos v$$

$$ds^2 = du^2 + (u \sin \alpha)^2 dv^2, \quad r \rightarrow u, \quad \phi \rightarrow v, \quad \alpha = \text{apex angle},$$

$$L = \int ds = \int \sqrt{1 + (u \sin \alpha)^2 v'^2} du, \quad f = \sqrt{1 + (u \sin \alpha)^2 v'^2}, \text{ independent of } v.$$

The solution is given by (9):  $\frac{\partial f}{\partial v'} = C$

$$\Rightarrow (u \sin \alpha) v' = \frac{C}{\sqrt{(u \sin \alpha)^2 - C^2}}$$

$$(14) \Rightarrow \phi = \int \frac{dr}{r \sin \alpha \sqrt{(r \sin \alpha / c)^2 - 1}} = \frac{1}{\sin \alpha} \sec^{-1} \left[ \left( \frac{\sin \alpha}{c} \right) r + C' \right].$$

Thus, the geodesics are *spirals* on the surface of the cone.

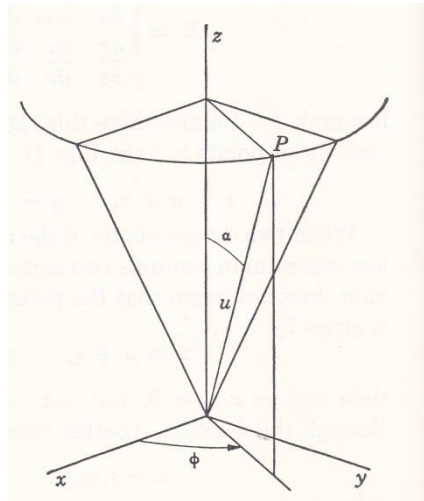


Figure 6. Right circular conical coordinates

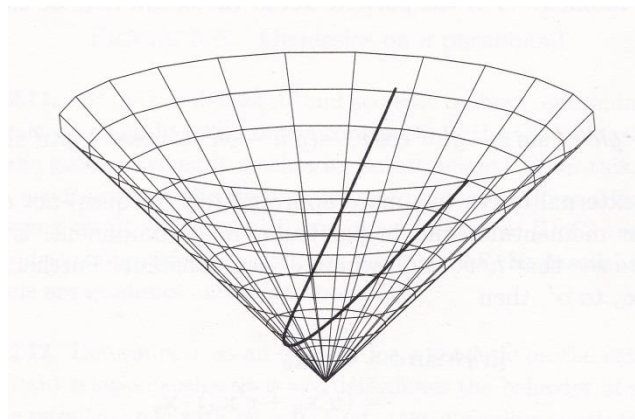


Figure 7. Cone geodesic

Surface 5: Hyperbolic paraboloid. This is the saddle-point surface:

$$x = \sinh u, \quad y = \cosh u, \quad z = v,$$

with parabolas for cross-sections in the  $xz$  – plane and  $yz$  – plane, and hyperbolas in the  $xy$  – plane. We have:

$$ds^2 = (\cosh^2 u + \sinh^2 u)du^2 + dv^2 = \cosh 2u du^2 + dv^2$$

$$L = \int ds = \int \sqrt{\cosh 2u + v'^2} du, \quad f = \sqrt{\cosh 2u + v'^2}, \text{ independent of } v.$$

The solution is given by (9):  $\frac{\partial f}{\partial v'} = C$

$$(15) \Rightarrow v = C' \int \sqrt{\cosh 2u} du.$$

Alas, there is no closed form to this integral. It turns out that this integral can be put in the form of an incomplete elliptic integral of the second kind, with solution:

$$v = -iE(iu|2) + C''.$$

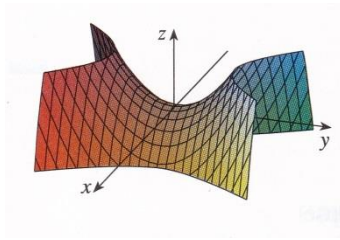


Figure 8. Hyperbolic paraboloid:  $z = y^2 - x^2$

For this less familiar surface, it is known that the sum of the interior angles of a triangle on the surface is less than  $180^\circ$ . It would have been interesting to see what kind of geodesics it has, but they could only be displayed parametrically.

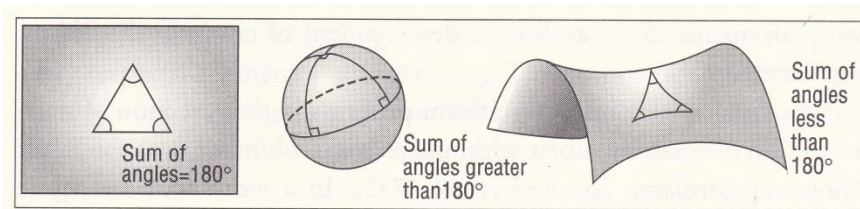


Figure 9. Triangles on a plane, a sphere, and a saddle-point surface



As an application, consider the calculation of great circle routes on a sphere, in particular, for any two cities on the Earth. The airlines have long discovered this, with flight paths between cities as sections of planes through the Earth's center.

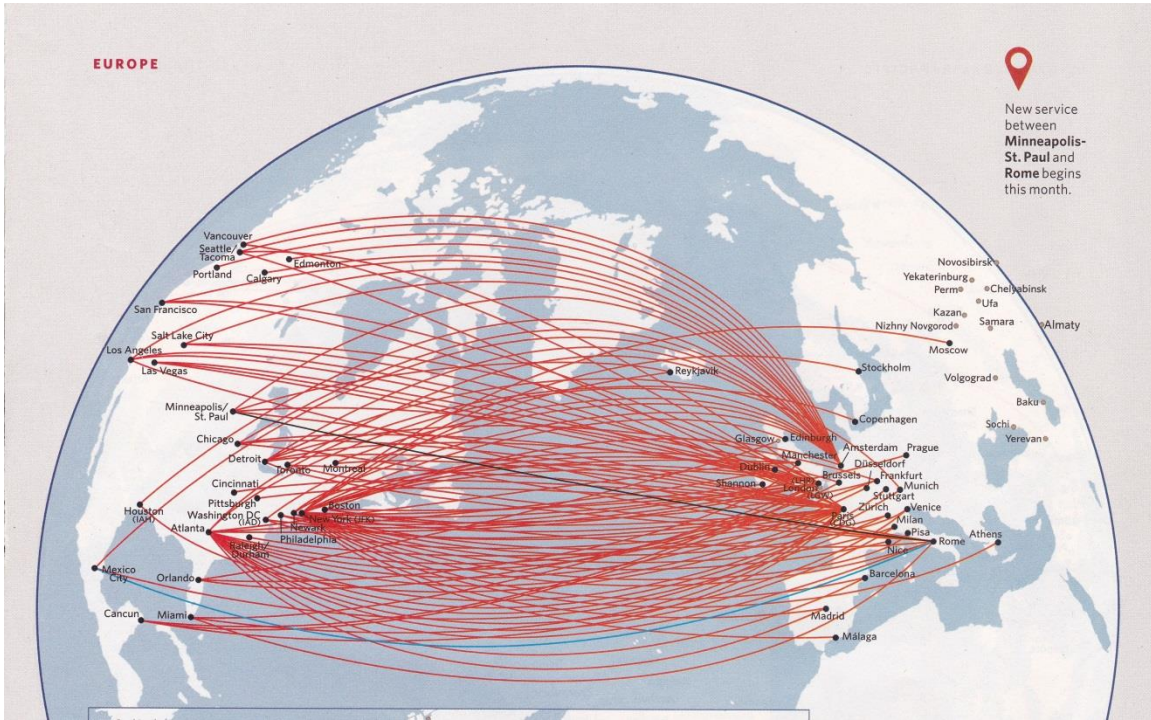


Figure 10. Approximate great circle routes for airlines

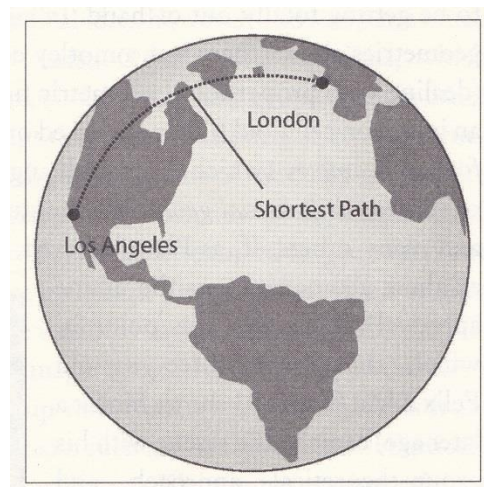


Figure 11. Great circle path between two cities on the Earth

Let  $P$  and  $Q$  represent any two cities on the Earth's surface. The dot product of the vectors  $\overrightarrow{OP}$  and  $\overrightarrow{OQ}$  from the origin  $O$  gives the central angle  $\alpha$  between  $\overrightarrow{OP}$  and  $\overrightarrow{OQ}$ , in radians, and the formula  $s = R\alpha$  gives the arclength  $s$  along a great circle between points  $P$  and  $Q$ .  $R = 3960$  *mi*, the Earth's radius.

$\overrightarrow{OP} = (x_1, y_1, z_1)$ ,  $\overrightarrow{OQ} = (x_2, y_2, z_2)$ . Convert to spherical coordinates:

$$x = R \sin \phi \cos \theta, \quad y = R \sin \phi \sin \theta, \quad z = R,$$

$\theta =$  longitude

$\phi =$  colatitude

$\rightarrow \theta$ , if East of Greenwich

$\rightarrow 90^\circ - \phi$ , if North Hemisphere

$\rightarrow 360^\circ - \theta$ , if West of Greenwich

$\rightarrow 90^\circ + \phi$ , if South Hemisphere

$$\overrightarrow{OP} \cdot \overrightarrow{OQ} = R^2 \cos \alpha, \quad \text{and} \quad \therefore s = R\alpha, \text{ miles.}$$

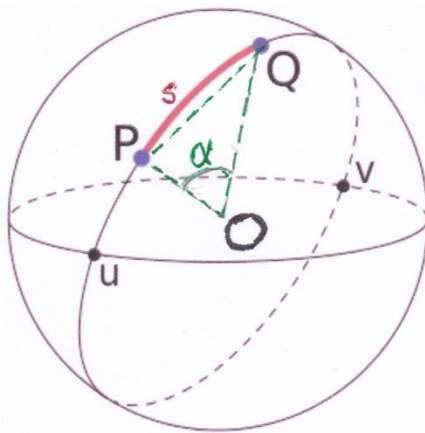


Figure 12. Great circle distance  $s$  between  $P$  and  $Q$

Example 1. Miami ( $25^\circ 47'N, 80^\circ 13'W$ ), Manila ( $14^\circ 35'N, 120^\circ 58'N$ )  $\rightarrow s = 9,308$  mi

Example 2. Chicago ( $41^\circ 53'N, 87^\circ 38'W$ ), Tokyo ( $35^\circ 41'N, 139^\circ 42'E$ )  $\rightarrow s = 6,306$  mi

Example 3. NP ( $90^\circ N, 0^\circ E$ ), Quito ( $0^\circ 15'S, 78^\circ 35'W$ )  $\rightarrow 6,238$  mi.

This last example serves as a check: between the North Pole and the Equator is a quarter of the Earth's circumference,  $s = 2\pi R/4 = 6,220$  mi, the difference between the values to account for the slight deviation of Quito into the South Hemisphere.

Geodesics are minimal arcs between two points on a surface. There are several ways to calculate geodesics. Here we found them directly by the calculus of variations. The main process is solving the Euler-Lagrange partial differential equation for the function that minimizes the arclength between two points on the surface. The integral is the parametric equation of the geodesic. For familiar surfaces, like the plane, sphere, cylinder, and cone, the results were also familiar because the integrals of the Euler-Lagrange equation could be put in standard forms and worked out nicely. For a less familiar surface, such as the hyperbolic paraboloid, the integration could get very complicated and foils our attempt at getting a simple classical standard form. For application, the calculation of great circle distances between any two cities on the Earth was very straightforward.