ELLIPTIC INTEGRALS AND SOME APPLICATIONS

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Elliptic integrals arose from the attempts to find the perimeter of an ellipse:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \quad x=a\cos\theta, \;\; y=b\sin\theta$$

$$L=\int ds=2\int_{-a}^{a}\sqrt{1+y'^2}dx=\int_0^{2\pi}\sqrt{1+b^2cos^2\theta}(-asin\theta)d\theta,$$

which last integral *cannot* be evaluated by elementary functions, viz., trigonometric, exponential, or logarithmic functions.

The incomplete elliptic integral of the first kind is defined as

$$u=F(k,\phi)=\int_0^{\phi}\frac{d\theta}{\sqrt{1-k^2sin^2\theta}}, \;\; 0<k<1,$$

where ϕ is the amplitude of $F(k,\phi)$ or u, written $\phi=\mathrm{am}\,u$, and k is the modulus, $k=\mathrm{mod}\,u$. The integral is also called *Legendre's form for the elliptic integral of the first kind*. If $\phi=\pi/2$, the integral is called the complete integral of the first kind, denoted by $K(k)$, or simply *K*.

The incomplete elliptic integral of the second kind is defined by

$$E(k,\phi)=\int_0^{\phi}\sqrt{1-k^2sin^2\theta}\,d\theta, \;\; 0<k<1,$$

also called *Legendre's form for the elliptic integral of the second kind.* If $\phi=\pi/2$, the integral is called the complete elliptic integral of the second kind, denoted by $E(k)$, or simply *E*. This is the form that arises in the determination of the length of arc of an ellipse.

The incomplete elliptic integral of the third kind is defined by

$$H(k,n,\phi)=\int_0^{\phi}\frac{d\theta}{(1+nsin^2\theta)\sqrt{1-k^2sin^2\theta}}, \quad 0<k<1, n\neq 0,$$

also called *Legendre's form for the elliptic integral of the third kind.*

If the transformation $v=\sin\theta$ is made in the Legendre forms, we obtain the following integrals, with $x=\sin\phi$

$$F_1(k,x)=\int_0^{x}\frac{dv}{\sqrt{(1-v^2)(1-k^2v^2)}},$$

$$E_1(k,x)=\int_0^{x}\sqrt{\frac{1-k^2v^2}{1-v^2}}\,dv,$$

$$H_1(k,n,x)=\int_0^{x}\frac{dv}{(1+nv^2)\sqrt{(1-v^2)(1-k^2v^2)}},$$

called *Jacobi's forms for the elliptic integrals* of the first, second, and the third kinds respectively. These are complete integrals if $x = 1$.

Using *Landen's transformation*

$$\tan\phi = \frac{\sin 2\phi_1}{k+\cos 2\phi_1} \qquad \text{or} \qquad k\sin\phi = \sin(2\phi_1 - \phi),$$

we can rewrite

$$F(k,\phi) = \frac{2}{1+k}F(k_1,\phi_1) = \frac{2}{1+k}\int_0^{\phi_1}\frac{d\phi_1}{\sqrt{1-k_1^2 sin^2\phi_1}}, \text{ where } k_1 = \frac{2\sqrt{k}}{1+k}, \; k < k_1 < 1.$$

By successive applications of the transformation, we obtain a sequence of moduli

$$k_n, n = 1,2,3,\ldots, \quad k_i = \frac{2\sqrt{k_{i-1}}}{1+k_{i-1}}, \; \lim_{n\to\infty} k_n = 1,$$

from which we can write

$$F(k,\phi) = \sqrt{\frac{k_1k_2k_3\ldots}{k}}\int_0^{\phi}\frac{d\theta}{\sqrt{1-sin^2\theta}} = \sqrt{\frac{k_1k_2k_3\ldots}{k}}\ln tan\left(\frac{\pi}{4}+\frac{\Phi}{2}\right), \; \Phi = \lim_{n\to\infty}\phi_n.$$

In practice, accurate results are obtained after only a few applications of the transformation.

Many integrals are reducible to elliptic type. If $R(x,y)$ is a rational algebraic function of x and y, i.e., the quotient of two polynomials in x and y, then $\int R(x,y)dx$ can be evaluated in terms of the usual elementary functions (algebraic, trigonometric, inverse trigonometric, exponential, and logarithmic) if $y = \sqrt{ax+b}$ or $y = \sqrt{ax^2+bx+c}$, with a,b,c constants. If $y = P(x)$, with $P(x)$ a cubic or quartic polynomial, the integral R can be evaluated in terms of elliptic integrals of first, second, or third kinds, or for special cases in terms of elementary functions. If P is a polynomial of degree > 4, then R may be evaluated with the aid of hyper-elliptic functions.

Example 1. $\int_0^2 \frac{dx}{\sqrt{(4-x^2)(9-x^2)}}$. Let $x = 2\sin\theta$. The integral becomes

$$\int_0^{\pi/2}\frac{d\theta}{\sqrt{9-4sin^2\theta}} = \frac{1}{3}\int_0^{\pi/2}\frac{d\theta}{\sqrt{1-\frac{4}{9}sin^2\theta}} = \frac{1}{3}F\left(\frac{2}{3},\frac{\pi}{2}\right).$$

Example 2. $\int_0^1 \frac{dx}{\sqrt{(1+x^2)(1+2x^2)}}$. Let $x = \tan\theta$. The integral becomes

$$\int_0^{\pi/4} \frac{sec^2\theta d\theta}{\sqrt{1+tan^2\theta}\sqrt{1+2tan^2\theta}} = \int_0^{\pi/4} \frac{d\theta}{\sqrt{cos^2\theta+2sin^2\theta}} = \int_0^{\pi/4} \frac{d\theta}{\sqrt{2-cos^2\theta}} = \frac{1}{\sqrt{2}} \int_0^{\pi/4} \frac{d\theta}{\sqrt{1-\frac{1}{2}cos^2\theta}}.$$

Let $\phi = \pi/2 - \theta$. The integral becomes

$$\frac{1}{\sqrt{2}} \int_{\pi/4}^{\pi/2} \frac{d\phi}{\sqrt{1-\frac{1}{2}sin^2\phi}} = \frac{1}{\sqrt{2}} \left\{ F\left(\frac{1}{\sqrt{2}}, \frac{\pi}{2}\right) - F\left(\frac{1}{\sqrt{2}}, \frac{\pi}{4}\right) \right\}.$$

Example 3. $\int_4^6 \frac{dx}{\sqrt{(x-1)(x-2)(x-3)}}$. Let $u = \sqrt{x-3}$ or $x = 3 + u^2$. The integral becomes

$$2\int_1^{\sqrt{3}} \frac{du}{\sqrt{(u^2+2)(u^2+1)}}.$$ Let $u = \tan\theta$. The integral becomes

$$2\int_{\pi/4}^{\pi/3} \frac{d\theta}{\sqrt{2cos^2\theta+sin^2\theta}} = 2\int_{\pi/4}^{\pi/3} \frac{d\theta}{\sqrt{2-sin^2\theta}} = \sqrt{2}\int_{\pi/4}^{\pi/3} \frac{d\theta}{\sqrt{1-\frac{1}{2}sin^2\theta}}$$

$$= \sqrt{2}\left\{ F\left(\frac{1}{\sqrt{2}}, \frac{\pi}{3}\right) - F\left(\frac{1}{\sqrt{2}}, \frac{\pi}{4}\right) \right\}.$$

In general, $\int \frac{dx}{\sqrt{P}}$, where P is a 3rd or 4th degree polynomial, can be evaluated by elliptic integrals.

The elliptic functions are defined via the elliptic integrals. The upper limit x in the Jacobi form of the elliptic integral of the first kind is related to the upper limit ϕ in the Legendre form by $x = \sin\phi$. Since $\phi = \text{am}\, u$, it follows that $x = \sin(\text{am}\, u)$. We define the elliptic functions

$$\begin{aligned} x &= \sin(\text{am}\, u) &&= sn\, u \\ \sqrt{1-x^2} &= \cos(am\, u) &&= cn\, u \\ \sqrt{1-k^2x^2} &= \sqrt{1-k^2sn^2u} &&= dn\, u \\ \frac{x}{\sqrt{1-x^2}} &= \frac{sn\, u}{cn\, u} &&= tn\, u. \end{aligned}$$

It is also possible to define *inverse elliptic functions*. For example, from $x = sn\, u$, we define $u = sn^{-1}x$, or $u = sn^{-1}(x,k) = sn^{-1}x, \text{mod}\, k$, to show the dependence of u on k.

These functions have many important properties analogous to those of trigonometric functions. One special property is the fact that these functions are doubly periodic, one period is real, the other is complex. If

$$K = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2 sin^2\theta}},$$

then
$$sn(u+4K) = sn\,u$$
$$cn(u+4K) = cn\,u$$
$$dn(u+2K) = dn\,u$$
$$tn(u+2K) = tn\,u.$$

These functions also have other periods, which are complex. If

$$K' = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k'^2 sin^2\theta}}, \quad k' = \sqrt{1-k^2},$$

Then $sn\,u$ has periods $4K$ and $2iK'$; $cn\,u$ has periods $4K$ and $2K+2iK'$; and $dn\,u$ has periods $2K$ and $4iK'$. For this reason, the elliptic functions are called *doubly-periodic*.

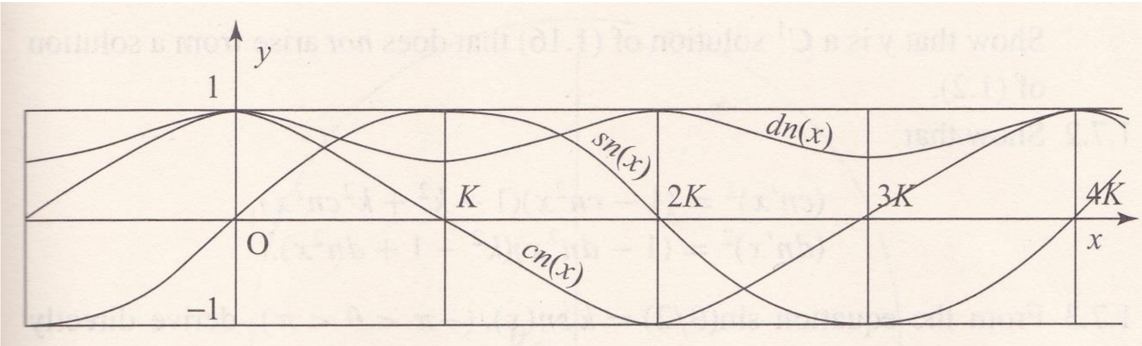

Figure 1. Graphs of *sn* (*x*), *cn* (*x*), *dn* (*x*) ($k^2 = 0.7$)

Application 1. Perimeter of an ellipse.

The ellipse $x = a\cos\theta,\ \ y = b\sin\theta,\ \ a > b > 0,$ has length

$$L = 4\int_0^{\pi/2} \sqrt{dx^2+dy^2} = 4\int_0^{\pi/2} \sqrt{a^2cos^2\theta + b^2sin^2\theta}d\theta$$

$$= 4\int_0^{\pi/2} \sqrt{a^2-(a^2-b^2)sin^2\theta}d\theta = 4a\int_0^{\pi/2} \sqrt{1-e^2sin^2\theta}d\theta,$$

where $e^2 = (a^2-b^2)/a^2 = c^2/a^2$ is the square of the eccentricity of the ellipse.

The result can be written as $L = 4aE\left(e, \frac{\pi}{2}\right) = 4aE(e)$. For the special case of a circle, $a = b = r$, i.e., $e = 0$, and $E(0) = \pi/2$, and we recover the circumference of a circle: $L = 2\pi r$. The term *elliptic integral* was coined by Count Fagnano (1682 – 1766) in 1750. He discovered that the arclength of the lemniscate can be expressed in terms of an elliptic integral of the first kind.

Application 2. Arclength of a lemniscate.

The lemniscate is the figure 8 curve: $(x^2 + y^2)^2 = a^2(x^2 - y^2)^2$, or in polar form

$$r^2 = a^2 \cos 2\theta.$$

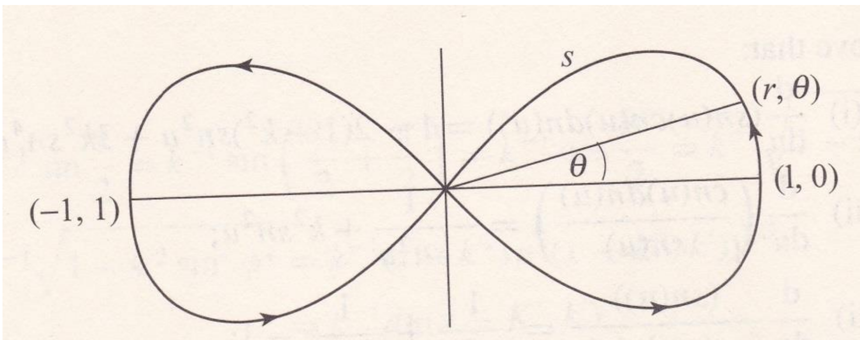

Figure 2. The lemniscate $r^2 = \cos 2\theta$.

From $ds^2 = dx^2 + dy^2 = dr^2 + r^2 d\theta^2$,

$$L = 4\int_{r=0}^{\pi/4} ds = 4a\int_0^1 \frac{1}{\sqrt{1-r^4}}dr, \quad r = \tan\theta \Rightarrow$$

$$\int_0^1 \frac{1}{\sqrt{1-r^4}}dr = \int_0^{\pi/4} \frac{sec^2\theta d\theta}{sec\theta\sqrt{1-tan^2\theta}} = \int_0^{\pi/4} \frac{d\theta}{\sqrt{\cos 2\theta}}, \quad (\cos 2\theta = cos^2 u) \Rightarrow$$

$$= \int_0^{\pi/2} \frac{du}{\sqrt{2-sin^2 u}} = \frac{1}{\sqrt{2}}\int_0^{\pi/2} \frac{du}{\sqrt{1-\frac{1}{2}sin^2 u}} = \frac{1}{\sqrt{2}} \cdot K\left(\frac{1}{\sqrt{2}}\right).$$

Thus, $L = 4a \cdot \frac{1}{\sqrt{2}}K\left(\frac{1}{\sqrt{2}}\right) = a \cdot 2\sqrt{2}(1.85407) = a(5.244102)$.

(*Historical note:* The rectification of the lemniscate was first done by Fagnano in 1718. The lemniscatus, *L.*‘decorated by ribbons’, was first studied in astronomy in 1680 by Cassini, known as the *ovals of Cassini* (Figure 3), but his book was published in 1749, many years after his death. The curves were popularized by the Bernoulli brothers in

1694.) Cassini considered more general forms of the lemniscate for whose points the products of the distances to two foci is a constant:

$$d_1 d_2 = b^2,$$

$$b^4 = r^4 + \frac{a^2}{4} - r^2 a^2 \cos 2\theta.$$

When $b = \frac{a}{\sqrt{2}}$, centered at the origin, we get the ribbon-shaped curve.

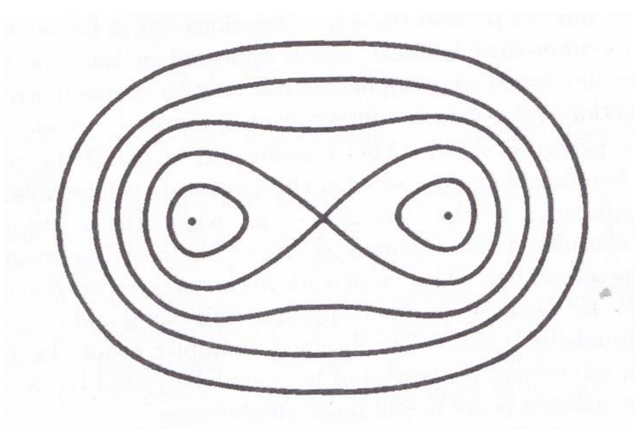

FIGURE 3

Application 3. Finite-amplitude pendulum.

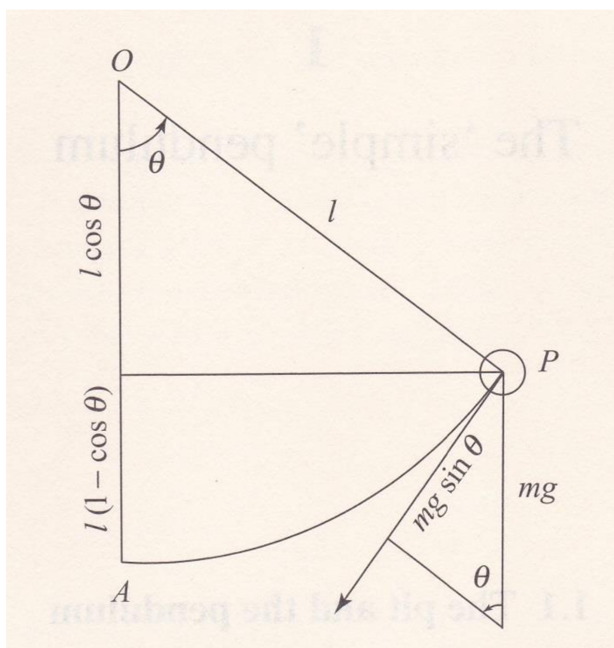

Figure 4. The simple pendulum.

The equation of motion is:

$$ml\ddot{\theta} = -mg\sin\theta. \quad \text{Let } p = \dot{\theta} \rightarrow p\frac{dp}{d\theta} \dot{=} -\frac{g}{l}\sin\theta$$

$$\Rightarrow \frac{p^2}{2} = \frac{g}{l}\cos\theta + C.$$

I.C.: At $t = 0$: $\theta = \theta_0$, $\dot{\theta} = 0 \quad \Rightarrow \quad \frac{d\theta}{dt} = -\sqrt{\frac{2g}{l}}\sqrt{\cos\theta - \cos\theta_0}$.

The period, T, is given by

$$\frac{T}{4} = \sqrt{\frac{l}{2g}}\int_{\theta_0}^{0}\frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}},$$

or,

$$T = 4\sqrt{\frac{l}{2g}}\int_{0}^{\theta_0}\frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} = 2\sqrt{\frac{l}{g}}\int_{0}^{\theta_0}\frac{d\theta}{\sqrt{sin^2\left({}^{\theta_0}\!/_2\right) - sin^2\left({}^{\theta}\!/_2\right)}}$$

$$= 4\sqrt{\frac{l}{g}}\int_{0}^{\pi/2}\frac{du}{\sqrt{1-k^2 sin^2 u}}, \quad \sin\left(\frac{\theta}{2}\right) = \sin\frac{\theta_0}{2}\cdot\sin u, \quad k = \sin\left(\frac{\theta_0}{2}\right)$$

$$\therefore \quad T = 4\sqrt{\frac{l}{g}}\cdot F(k), \text{ an elliptic integral.}$$

For the special case of *small* oscillations, $k = 0$, we get the classical result:

$$T = 2\pi\sqrt{\frac{l}{g}}.$$

Application 4. Perihelion of Mercury.

According to Newton's law, where the force on a particle per unit mass is μ/r^2, a planet moves around the Sun in an ellipse and, if there are no other planets disturbing it, the ellipse remains the same forever. According to Einstein's law, the curvature of space appears as a correction term proportional to $1/r^4$, so that the gravitational force per unit mass is of the form

$$\mu\left(\frac{1}{r^2} + \frac{3h^2}{c^2 r^4}\right),$$

where h is the angular momentum per unit mass of the planet about the Sun, and c is the speed of light (cf. Lawden, 1989, or Armitage and Eberlein, 2006, for more details). The prediction of Einstein is that the path is very nearly an ellipse, but it does not quite close up. In the next revolution, the path will have advanced slightly ahead in which the planet is moving, and so the orbit is an ellipse which very slowly precesses. The advance of Mercury is *the perihelion of Mercury*, and is indeed one of the tests of the theory of relativity. Elliptic functions help to explain all that.

Mercury's orbit has period T = 88 days, its semimajor axis is a = 57,909,050 km, and eccentricity e = 0.205630. In 1859, Leverrier of France calculated 574" of arc/century by perturbations due to other planets; 531" were measured. To acccount for the 42.98"/century, Leverrier postulated the existence of an inner planet Vulcan inside Mercury's orbit, but which was never found. Einstein's prediction came in 1915.

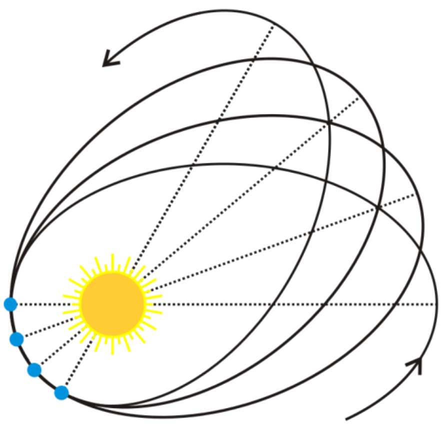

Figure 5. Perihelion of Mercury

The equations of motion, in polar coordinates (r, θ), are

$$\frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - \mu\left(\frac{1}{r} + \frac{h^2}{c^2r^3}\right) = E, \;\; r^2\dot{\theta} = h,$$

where E is the energy. Let $u = 1/r$, the orbit equation becomes

$$\left(\frac{du}{d\theta}\right)^2 = \frac{2\mu}{h^2}u - u^2 + \frac{2\mu}{c^2}u^3 + \frac{2E}{h^2}.$$

For the planets in the Solar System, the 3rd term on the RHS is very small, define the dimensionless variable v by

$$u = \mu v/h^2.$$

The orbit equation becomes

$$\left(\frac{dv}{d\theta}\right)^2 = 2v - v^2 + \alpha v^3 - \beta = f(v),$$

where $\quad \alpha = 2(\mu/ch)^2, \quad \beta = -2Eh^2/\mu^2.$

Constraints on the stability of the orbit require

$$0 < \beta \leq 1, \quad \text{thus, } \beta = 1 + \varepsilon,\ \varepsilon > 0.$$

The term α is very small, the largest value being that for Mercury, $\alpha = 5.09 \times 10^{-8}$. Further constraints on $f(v)$ show that the zeros are real and positive and satisfy $0 < v_1 < 1 < v_2 < 2 < v_3$. Accordingly,

$$f(v) = \alpha(v - v_1)(v - v_2)(v - v_3).$$

Since $f(v) \geq 0$, v must lie in the interval $v_1 \leq v \leq v_2$. The case $v \geq v_3$ would lead to $v \to \infty$ as $\theta \to \infty$ (so the planet would fall into the Sun) and so must be excluded.

For small α, the roots of the cubic $f(v)$ may be expanded in ascending powers of α:

$$v_1 = 1 - e - \frac{\alpha}{2e}(1-e)^3 + O(\alpha^2),$$

$$v_2 = 1 + e + \frac{\alpha}{2e}(1+e)^3 + O(\alpha^2),$$

$$v_3 = \frac{1}{\alpha} - 2 + O(\alpha),$$

with $\quad e^2 = 1 - \beta = 1 + 2Eh^2/\mu^2.$

The orbit is given by

$$\alpha^{1/2}\theta = \int \frac{dv}{\sqrt{(v-v_1)(v-v_2)(v-v_3)}}.$$

Let $v = v_1 + 1/t^2 \ \rightarrow$

$$\alpha^{1/2}\theta = -\frac{2}{\sqrt{(v_2-v_1)(v_3-v_1)}} \int \frac{dt}{(t^2-a^2)(t^2-b^2)},$$

where $\quad a^2 = 1/(v_2 - v_1), \ \ b^2 = 1/(v_3 - v_1).$

The last integral is an elliptic integral

$$\alpha^{1/2}\theta = \frac{1}{\sqrt{(v_3-v_1)}}\, sn^{-1}\left(t\sqrt{(v_2 - v_1)}\right),$$

with modulus *k* given by $k^2 = (v_2 - v_1)/(v_3 - v_1).$

Finally, the relativistic orbit is given by

$$v = v_1 + (v_2 - v_1)sn^2\left(\frac{1}{2}\sqrt{\alpha(v_3 - v_1)}\theta\right).$$

In polar coordinates, the orbit is given by

$$\frac{1}{r} = \frac{\mu}{h^2}(A + Bsn^2\eta\theta),$$

where $A = 1 - e - \frac{\alpha}{2e}(1-e)^3 + O(\alpha^2),$

$$B = 2e + \alpha\left(3e + \frac{1}{e}\right) + O(\alpha^2),$$

$$\eta = \frac{1}{2} - \frac{1}{4}\alpha(3-e) + O(\alpha^2), \text{ and}$$

$$k^2 = 2e\alpha + O(\alpha^2).$$

For the special case $\alpha = 0$: $A = 1 - e$, $B = 2e$, $\eta = \frac{1}{2}$, $k = 0$, and we recover the classical orbit

$$\frac{1}{r} = 1 - e\cos\theta,$$

with semilatus rectum $l = h^2/\mu$ and eccentricity e.

Perihelion occurs between $\theta = K/\eta$ and $\theta = 3K/\eta$, or $\Delta\theta = 2K/\eta$ vs 2π for the classical result. Thus, the advance of perihelion per revolution is, with $K = \int_0^{\pi/2}(1 - k^2 sin^2\theta)^{-1/2}d\theta$:

$$\epsilon = \frac{2K}{\eta} - 2\pi = \frac{\pi\left(1+\frac{1}{4}k^2+\cdots\right)}{\frac{1}{2}-\frac{1}{4}\alpha(3-e)+\cdots} = 3\pi a.$$

For Mercury: $\alpha = 5.09 \times 10^{-8}$, $T = 88$ days, and $\epsilon = 43$"/century.

Conclusions:

- Elliptic integrals arose from the attempts to find the perimeter of an ellipse, akin to the circumference of a circle.
- There are several standard forms of elliptic integrals, but they involve radicals of polynomials of degree 3 or 4.
- Familiarity with elliptic integrals allows us to solve interesting problems in mathematics and physics that we have heretofore avoided.