

MATHEMATICAL MODELING USING R PROGRAMMING ENVIRONMENT – SOME EXAMPLES

Leslie Chandrakantha

John Jay College of Criminal Justice of CUNY
Mathematics and Computer Science Department
524 West 59th Street, New York, NY 10019
lchandra@jjay.cuny.edu

Abstract

In this paper, we explain how to use simulation to study mathematical modeling. The simulations are performed using the R programming environment. We use two examples to demonstrate the modeling activities. For each case, the description of the model, simulation steps, and the solution are discussed. R codes for each case are also provided.

Introduction

A mathematical model is a description of a real situation using mathematical concepts. The process of creating a mathematical model for a given problem is called mathematical modeling. Many mathematical models relate to real life problems and that are interdisciplinary in nature. Blomhoj and Jensen [5] have given the following six steps to formulate and solve problems using mathematical modeling. These include: formulation, selection of relevant objects and relations, translation of these objects and relation to mathematics, use of mathematical methods to obtain the results, interpretation of the results to make conclusions, and evaluation of the validity of the models. Students often fail to connect and apply what they have learnt in introductory mathematics courses to other subjects, sometimes leading to the belief that mathematics is not relevant to them. Due to this lack of problem solving skills, teaching mathematical modeling is a challenging task to many instructors [14]. Sometimes due to the unpredictable and unexpected outcomes of mathematical modeling tasks, instructors may be discouraged to introduce modeling activities in the classroom [4].

Technology and technology based environment will allow instructors and students to alleviate these shortcomings and lack of problem solving skills in mathematical modeling [1]. Technology should not replace the entire modeling process, but only provide temporary means to overcome the difficulties. The approaches of teaching mathematical modeling have been influenced by the development and introduction of technologies such as graphing calculators and computer software [10]. Many researchers have reported the

successful use of technology in introducing mathematical ideas. For instance, the use of a spreadsheet to explore mathematical concepts has been discussed by Chua and Wu [7] for a secondary classroom, and by Beare [3] at college level. Ang and Awyong [2] reported that the use of computer algebra systems such as Maple in some tertiary courses has been well received. Not surprisingly, the use of technology continues to prevail in the mathematics classroom at all levels. The acquisition of the knowledge of the technologies will be an essential part before applying them in mathematical modeling. We believe the use of technology continues to prevail in the mathematics classroom at all levels.

In this paper, we describe how to use R with some programming or coding capabilities in solving mathematical models. We use a simulation approach to find the solutions. R is increasingly being used as a tool for mathematical and statistical education. Many mathematics instructors use R to teach and perform calculations. Although it is bit challenging to write statements in the command line, R can be used in simulation effectively. A valuable introduction to R for introductory courses is given in [12]. We use two examples to demonstrate mathematical modeling and solving. In coming section, we give introduction to simulation modeling, a brief overview of R, and modeling and solving two examples. We end with some concluding remarks.

Simulation Modeling

A simulation model is a mathematical model, which combines both mathematical and logical concepts that try to imitate the operations of a real life process or system through use of computer software. The computer simulation approach has been used in mathematical modeling and solving complex problems [13]. In recent times, many instructors use simulation in their classroom to explain the difficult concepts [6]. In certain situations, relevant data are not readily available or cannot be obtained. In these situations, simulation modeling provides ways to study such problems.

In the simulation process, a computer program or a software application is used to generate steps that are formulated to model and solve the given problem. Based on the assumptions and the rules of the problem, outcomes or the values of the variables of the model are calculated iteratively until desired results are obtained. There are many technological tools and computer programming languages available to perform such simulations. In this paper, we use R for this process. R is freely available to instructors and students.

Brief Overview of R

R is a free software environment used for working with data. R can be used to create sophisticated graphs, carry out statistical analyses, and run simulations. It is also a programming language with a set of built-in-functions. With some knowledge of coding,

students can write their own codes for mathematical and statistical computations. For computationally intensive tasks, one can incorporate functions written in other languages such as C, C++, and FORTRAN. R compiles and runs on Windows, MacOS, and a wide variety of UNIX platforms. The examples of R used in this paper come from the most recent version of R, R 3.4.0. R is available from <http://www.r-project.org>. To install R 3.4.0 on your operating system, download R from the site above using the closest mirror site to your location and choose the appropriate link for your operating system.

R is a relatively simple syntax-driven and case-sensitive language. Even though the syntax for writing instructions may be somewhat difficult initially, most students with little or no prior programming experience have become comfortable using R. In this particular course about 50% of the students are Computer Information Systems majors and they have taken at least one computer programming course prior to taking this course. The other 50% are from quantitative disciplines such as sciences and economics, and have been exposed to some sort of computer logic. R is installed into the school computers, located in the computer labs which also serve as the location for this course.

R is an object-oriented program that works with data structures such as vectors (one dimensional array) and data frames (two dimensional arrays). A vector contains a list of values. When R is started, we will see a window that is called the R console. This is where we type our commands and see the text results. Graphics appear in a separate window. The `>` is called the prompt, where R commands are written. To quit R we type `> q()`.

R can be used as a calculator. At the prompt, we enter the mathematical expression and by hitting “enter”, it will calculate the result and display it. The standard arithmetic operators ‘+’, ‘-’, ‘*’, and ‘/’ are used in expressions and ‘^’ is used for exponentiation. The following example demonstrates this:

```
> 2*3 -10  
[1] -4
```

The results of a calculation can be assigned to a variable (object in R) using `<-` or `=`. In this paper, we will use `<-`. Even though we can work with single numbers (scalars), R is primarily designed to work with vectors and functions. In R, a vector is a sequence of data values of the same type. The function, `c`, is used to create vectors from scalars. The following statement creates a vector:

```
> x <- c(2, 4, 6, 8, 10)  
> x  
[1] 2 4 6 8 10
```

Once we have a vector of numbers, we can apply built-in functions to get useful statistical summaries and visual displays. R also provides functions for generating random samples from various probability distributions.

Random Numbers using *sample* function

The *sample* function in R generates a sample of specified size from a set of values using with or without replacement. Let's suppose values are stored in a vector named *x*. To take a random sample of size *n* without replacement from the set *x*, we use following R command:

```
> sample(x, n)
```

To obtain a sample of size *n* with replacement, we use following command

```
> sample(x, n, replace = TRUE)
```

We can use the *sample* function to obtain a random number from a set of numbers, say 1 through 10, in following way:

```
> sample(1:10,1)
[1] 2
```

Control Structures

R has the standard control structures such as *if*, *while*, and *for*, which can be used to control the flow of an R code. We will demonstrate the use of control structures in R using the following code segment. Let's assume that we have stored 1000 numbers in the vector named *x*. The following code will compute the average of the nonnegative numbers in vector *x*. The symbol *#* is used to write comments.

```
> sum <- 0          # variable sum initialized to 0
> count <- 0       # variable count initialized to 0
> for(i in 1:1000){
+   if(x[i] >= 0) {
+     count <- count + 1      # count the positive values
+     sum <- sum + x[i] }     # add the positive values
+ }
> average <- sum/count      # compute the average
```

In the above code segment, the *for* loop iterates 1000 times, selecting only nonnegative numbers using the *if* statement. It computes the average as well. The variable named *count* counts the number of nonnegative numbers stored in *x*. We will use the control structures when we discuss simulations in the following sections.

Examples

In this section, we illustrate two examples that use simulation modeling. All the simulations are performed using R. For each case, we provide the R code that performs the simulations.

Example One: Chaos Game

We consider a game called the “chaos game”. The game is played as follows. First pick three points and draw a triangle. It can be any triangle. Label the vertices as A, B, and C. Start with any point, say P, inside the triangle. Next take a three sided die with side 1, side 2, and side 3, and roll. There will be three possible outcomes (side 1, 2, or 3) with equal probability. Then continue the game based on the each of the possible outcomes given below:

- If the die rolled side 1, find the midpoint between P and A.
- If the die rolled side 2, find the midpoint between P and B.
- If the die rolled side 3, find the midpoint between P and C.

Use this midpoint and repeat this process to get next point and continue this process to get the subsequent points. *Figure 1* shows the results of simulation of this game with 10 rolls of the die. We want to find what will happen if we repeat this process a large number of times.

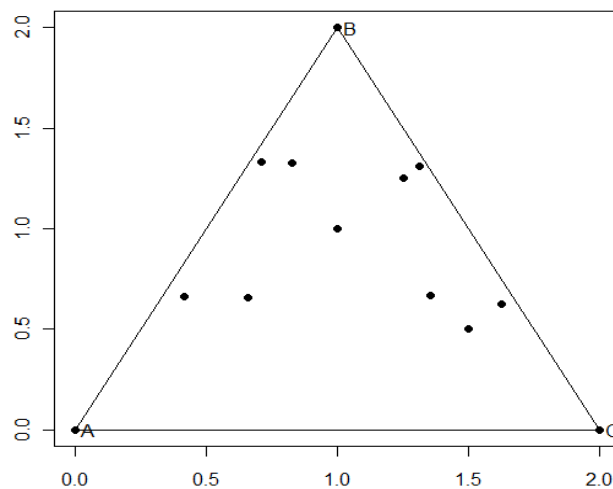


Figure 1: Results of 10 rolls of the game

The following R code simulates this process 10,000 times.

```
> plot(0:2,0:2,type = "n")
> x <- c(0,1,2)
> y <- c(0,2,0)
> points(x,y,pch = 19)
> labels <- c("A","B","C")
> text(x+.05,y,labels)
> segments(0,0,2,0)
> segments(0,0,1,2)
> segments(1,2,2,0)
> points(1,1,pch = 19)
> X <- c()
> Y <- c()
> X[1] <- 1
> Y[1] <- 1
> for(i in 2:10000){
+   r <- sample(1:3,1)
+   if(r == 1){
+     X[i] <- (X[i-1]+0)/2
+     Y[i] <- (Y[i-1]+0)/2}
+   else if(r == 2){
+     X[i] <- (X[i-1]+1)/2
+     Y[i] <- (Y[i-1]+2)/2}
+   else {
+     X[i] <- (X[i-1]+2)/2
+     Y[i] <- (Y[i-1]+0)/2}
+   points(X[i],Y[i],pch = 19)}
```

In beginning of the code, *plot* function with `type = "n"` provides a 2 by 2 empty plot. The *points* function with `pch = 19` plots three solid points (0, 0), (1, 2), and (2,0). The *text* function labels those points A, B, and C. The *segments* function connects the three vertices of the triangle. Then the game starts with a point inside the triangle. In this case, we start at the point (1, 1) using *points*(1,1, `pch = 19`). Then we create X and Y vectors to hold the points created inside the triangle in each iteration of the simulation. The for loop generates and plots 9,999 points (first of 10,000 points is created before for loop begins) inside the triangle. The *sample* function simulates the rolling the die by generating a random integer number from 1 to 3. *Figure 2* shows the results of simulating the game 10,000 times.

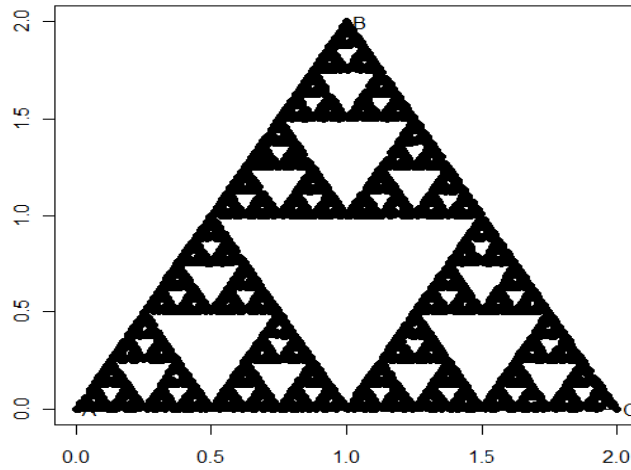


Figure 2: Results of simulating 10,000 times

The goal of the chaos game is to roll the die many times and predict what the resulting pattern of points will be. Most students who are unfamiliar with the game guess that the resulting image will be a random smear of points. Others predict that the points will eventually fill the entire triangle. The resulting image is anything but a random smear that the points form what mathematicians call the Sierpinski triangle [8] shown in above figure. Even though this example does not involve a real life problem, it illustrates the use of simulation for problem solving. It is impossible to predict the outcome or the emerging pattern of this game without simulation process.

Example Two: Secretary Problem

The Secretary problem is also known as the best choice problem that has been examined extensively in mathematics [11]. Imagine an administrator wanting to hire a secretary from n applicants with the following conditions:

- n applicants are ranked from best to worst without ties.
- The applicants are interviewed one by one in random order.
- The decision about each applicant is to be made immediately after the interview.
- Once rejected, an applicant cannot be recalled.
- During the interview, the administrator can rank the applicant among all applicants interviewed so far, but is unaware of the quality of yet unseen applicants.

We propose the following strategy that will maximize the probability of choosing the best applicant from the set of n applicants. Selecting the first applicant is not a wise strategy because the first applicant has no one to be compared with. Since the decision must be made immediately after interviewing an applicant, if we wait until last applicant is

interviewed, we may miss out earlier better applicants. A reasonable strategy is to reject a certain number of applicants, say $k-1$ of the n applicants, and then choose the first applicant that is better than all the previous applicants. If no such applicant exists, then we accept the last applicant. *Figure 3* illustrates this strategy.

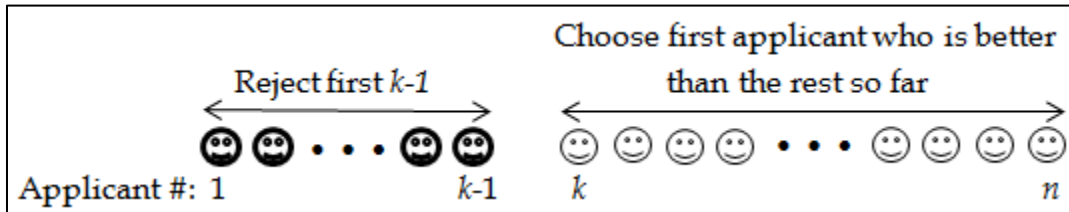


Figure 3: Illustration of rejecting first $k-1$ applicants and selecting next best one

According to this strategy, the probability of selecting the best applicant is always at least $1/e$ (about 37%) (see [9] for details). It further suggests that always rejecting the first n/e applicants that are interviewed (e is the base of the natural logarithm and has the value 2.71828) and then stopping at the first applicant who is better than every applicant interviewed so far (or continuing to the last applicant if this never occurs). This strategy selects the best applicant about 37% of times.

Now we show how to use R to simulate this process with the above strategy and to compute the success rate (or empirical probability) of selecting best applicant. As an example, we take ten applicants ($n = 10$) and assume that we reject the first 3 ($k-1 = 3$) applicants before we consider the rest of them. In each iteration of the simulation process, we randomly rank 10 applicants from 1 to 10. The following R code simulates the process 10 times and finds the selected applicant for each iteration.

```
> count <- 0
> for (i in 1:10){
+   x <- sample(1:10,10)
+   cat("Applicants' Rank: ", x, "\n")
+   y <- c()
+   y[1] <- y[2] <- y[3] <- 0
+   for (j in 4:9){
+     if (max(x[1:j-1]) < x[j] && max(y[1:j-1])==0)
+       y[j] <- x[j]
+     else
+       y[j] <- 0
+     if (max(y[4:9])==0){
+       y[10] <- x[10]}
+     else {
+       y[10] <- 0}
```



```

+ for (j in 4:10){
+   if (y[j] != 0)
+     select <- j}
+ cat("Selected Applicant: ", select, "\n")
+ if (max(x[1:10]) == max(y[4:10]))
+   count <- count + 1}

```

The output of the above code for first three iterations given below:

```

Applicants' Rank: 10 8 5 4 3 9 2 7 6 1
Selected Applicant: 10
Applicants' Rank: 6 3 4 2 1 8 7 9 10 5
Selected Applicant: 6
Applicants' Rank: 9 4 5 7 8 1 10 3 6 2
Selected Applicant: 7

```

In the above R code, the variable `count` is initialized to zero and it counts the number of times the best applicant is selected in the simulation process. Then the `for` loop is used to repeat the simulation 10 times. The `sample` function generates 10 random digits from 1 to 10 randomly without replacement. These 10 numbers are the ranks of the ten applicants. These ranks are displayed using the `cat` function. Then a `y` vector is created and assigns 0 or the rank of the selected applicant. Since first three applicants are not selected, `y[1]`, `y[2]`, and `y[3]` get zeros. Among the rest of the applicants, an applicant is selected if the applicant's rank is better than the previous applicants. Then the value of `y` for that applicant is the corresponding value of `x`, and others get 0. If there is no such applicant, last applicant in the list gets selected. The last `if` statement counts the number of times the best applicant is selected.

We have simulated this process 10,000 times and counted the number of times the best applicant is selected. Dividing this count by 10,000 gives the proportion of times the best applicant is selected in this strategy. Our simulation process gave this proportion as 0.3964. This means our simulation model approximates the best solution to 39.6%. This is close to the 37% that is given in the literature of this problem.

Conclusion

We have demonstrated how to use R to model and solve two examples in mathematical modeling. This simulation approach is useful in the classroom to model and visualize the solutions for situations when it is difficult to construct the real physical experiment. The key concern is that the students need to have a basic understanding of computer coding and logical thinking to be successful in this approach. In many quantitative disciplines,

students take at least one course in basic computer programming. Therefore students who want to learn mathematical modeling using technology would not have much difficulty to succeed using this approach.

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