THE EFFECT OF WIND ON THE TRAJECTORY OF GOLF BALLS

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The problem that we shall examine in this paper is that of the effect of wind on the trajectory of objects such as golf balls and baseballs. We will approximate their flight trajectories, their maximum heights, and distance traveled under different wind conditions where the wind shall be implemented as a vector field in three-dimensional space.

Using the mass and cross-sectional areas of the objects, the trajectories of the objects under the effect of wind are a set of differential equations whose solution will be approximated by numerical methods. These solutions will then be rendered in a 3D graphics package allowing students to study the effects of wind on the flight paths of objects from any position and orientation in three-dimensional space.

By changing the parameters-such as the speed and direction of the wind, the models and animations will allow us to see the effects of the parameters both numerically and visually.

This talk is geared towards people teaching differential equations, numerical analysis, and mathematical modeling.

Approximating the Object

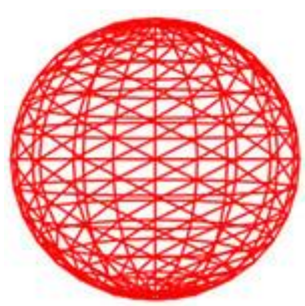

Figure 1-A wireframe sphere created in the HTML5 canvas

Needed to Create a Mathematical Model of Flight:

- Gravity
- Mass
- Cross-Sectional Area
- Shape of Surface
- Initial Velocity Vector
- Coefficient of Drag
- Rotation of Object
- Vector Field of Wind
- Rotation of Earth

We shall first consider a model that only deals with gravity, then build in the effects of atmospheric drag, then add in the effects of wind.

Gravity Only Model

Where v is the velocity vector, a is the acceleration vector, F_x, F_y and F_z are the forces in the x, y, and z-directions respectively:

$$\frac{dv}{dt} = a$$

$$\frac{ds}{dt} = v$$

$$F_x=0$$

$$F_y=0$$

$$F_z=-mg$$

where g is the force of gravity.

As $F_i=ma_i$

$$a_x=0$$

$$a_y=0$$

$$a_z=-g$$

$$v(t)-v_0=\int_{t_0}^{t} a(t)dt$$

Where v_x, v_y, and v_z are the velocities in the given directions, assuming constant acceleration:

$$v_x=v_{x0}$$

$$v_y=v_{y0}$$

$$v_z=v_{z0}-gt$$

As $s-s_0=\int_{t0}^{t} v(t)dt$

The position (x, y, z) of the object at time t given the initial position vector $<x_0, y_0, z_0>$ is given by:

$$x=x_0+v_{x0}*t \tag{1}$$

$$y=y_0+v_{y0}*t \tag{2}$$

$$z=z_0+v_{z0}*t-\frac{1}{2}g*t^2 \tag{3}$$

Considering Drag Effects Due to the Atmosphere

The force on an object due to the drag of the atmosphere is given as follows:

$$F_D=\frac{1}{2}\rho^2 v^2 A c_D$$

where:

ρ= the density of the fluid
v=the velocity of the object
A=the cross-sectional area
C_D=the coefficient of drag

c_D is a function of the density of the fluid, the object's velocity and the size/shape of the object. It is not constant.

$c_D = c_D (R_e)$ where R_e is the Reynold's number which is a function of: the fluid's density, the velocity of the object, the body length parallel to the direction of fluid flow, and the viscosity (thickness of the fluid)

$$R_e = \frac{\rho v L}{\mu}$$

At sea-level the air density is ρ=1.225 kg/m^3.

Total velocity

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

Forces on an object due to drag:

$$F_{Dx} = -F_D \frac{v_x}{v}$$

$$F_{Dy} = -F_D \frac{v_y}{v}$$

$$F_{Dz} = -F_D \frac{v_z}{v}$$

Force equations in x, y, and z:

$$F_x = -F_D \frac{v_x}{v}$$

$$F_y = -F_D \frac{v_y}{v}$$

$$F_z = -mg - F_D \frac{v_z}{v}$$

$$F_D = \frac{1}{2} \rho^2 v^2 A c_D$$

This implies:

$$a_x = \frac{-F_D v_x}{mv}$$

$$a_y = \frac{-F_D v_y}{mv}$$

$$a_z = -g - \frac{-F_D v_z}{mv}$$

Velocity equations:

$$\frac{dv_x}{dt} = a_x = \frac{-F_D v_x}{mv} \tag{4}$$

$$\frac{dv_y}{dt} = a_y = \frac{-F_D v_y}{mv} \tag{5}$$

$$\frac{dv_z}{dt} = a_z = -g - \frac{-F_D v_z}{mv} \tag{6}$$

Location equations:

$$\frac{dx}{dt} = v_x \tag{7}$$

$$\frac{dy}{dt} = v_y \tag{8}$$

$$\frac{dz}{dt} = v_z \tag{9}$$

Effect of the Wind

The wind effects the amount of drag on the object. Consider the apparent object speed relative to the fluid:

Define the velocity of the wind as follows:

$$v_w = \begin{bmatrix} v_{wx} \\ v_{wy} \\ v_{wz} \end{bmatrix}(\mathrm{t}) \tag{10}$$

This gives us the equations for the velocity and acceleration of the object under the effect of wind.

$$v_{ax} = v_x + v_{wx} \tag{11}$$

$$v_{ay} = v_y + v_{wy} \tag{12}$$

$$v_{az} = v_z + v_{wz} \tag{13}$$

Hence:

$$a_x = \frac{-F_D v_{ax}}{m v_a} = \frac{dv_x}{dt} \tag{14}$$

$$a_y = \frac{-F_D v_{ay}}{m v_a} = \frac{dv_y}{dt} \tag{15}$$

$$a_z = -g - \frac{-F_D v_{az}}{m v_a} = \frac{dv_z}{dt} \tag{16}$$

$$F_D = \frac{1}{2}\rho^2 v_a^2 A c_D \tag{17}$$

Where v_a is defined as follows:

$$v_a = \sqrt{v_{xa}^2 + v_{ya}^2 + v_{za}^2} \tag{18}$$

Approximating the solution of the above set of differential equations (7)-(18) one may approximate the position vector (x, y, z) at any time. Standard techniques are Euler’s method and the Runge-Kutta methods.

Side Note: One can also consider the Magnus force of a spinning object where lift is in direction orthogonal to direction of flight and spin axis.

Creating a 3D Animation of the Trajectory

I used a simple Euler’s approximation of the above flight equations and wrote this as a Python program. This program was then run inside the 3D graphics package Poser. Poser allows us to run the simulation and view the trajectory of the object from any point of view in 3-space.

The program also allows us to change the initial speed and angle of launch and the vector field of wind to study the effects on the object.

The program can also be run as a stand-alone to compute the coordinates of the object at any point during flight.

Below are some screen shots of the flight of a pumpkin as shot out of a cannon. (I chose a pumpkin because a golf ball was too hard to see on a screen.

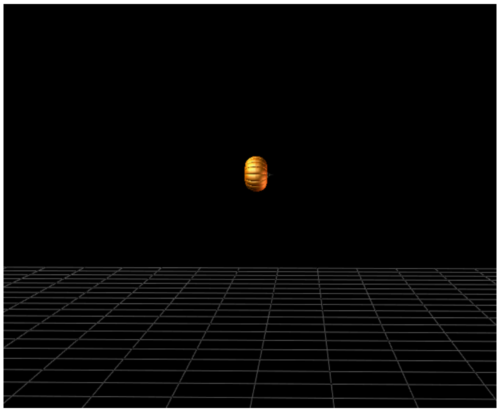

Figure 2: Side View of Flight

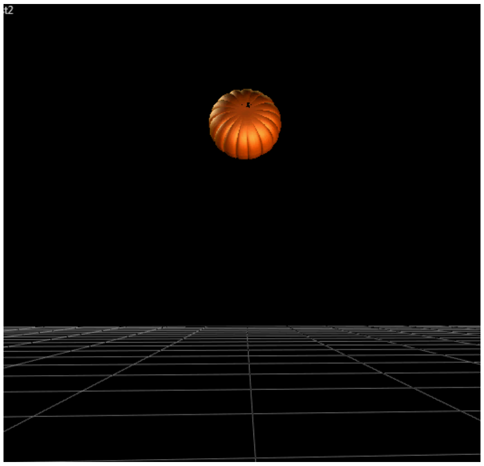

Figure 3-Front View of Flight

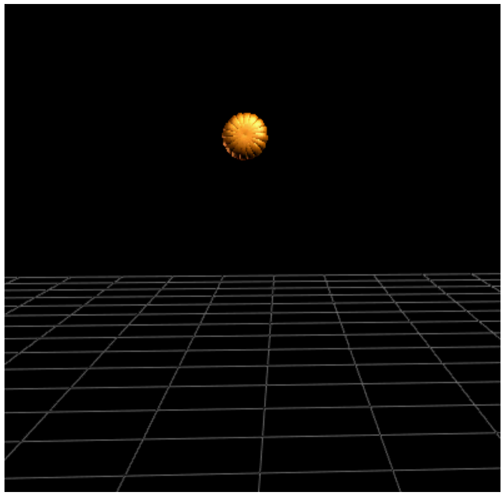

Figure 4-Reverse View of Flight

Allowing students to change the parameters such as angle of launch and wind speed:

- Shows how much of an effect wind can really has on an object.
- Shows that to get a maximum distance one needs to fire an object off at an angle of about 33°, not 45°, due to the effect of the atmosphere.

References:

[1] James Stewart, *Calculus* 7th Edition, Brooks-Cole, Belmont, CA, 2012.