EVOLUTION OF SOLITONS VIA *EXCEL*

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(*Abstract*.) After a brief history on the discovery of solitary waves, we follow the evolution of these solitons using the Korteweg-de Vries equation via *Excel*. We will focus on the basic properties of these waves, like their velocity of propagation as a function of height, the collision of two solitons, their dispersion, and their eventual relaxation to some steady profile. There are advantages to using *Excel* because the finite difference equations are not difficult to set up, and the graphing routines are readily implemented. One very real-world application is in the description of tsunamis, which are very destructive solitons.

1. Introduction
    1.1 The discovery of solitons, Scott Russell 1834
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3. Properties of solitons
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    3.3 – opposite directions
    3.4 Decomposition of an initial soliton
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Solitary waves were first discovered by the Scottish mathematician and engineer John Scott Russell in August, 1834:

> I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped – not so the mass of water … which it had set in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, it rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. (JS Russell, British Association for the Advancement of Science, 1844.)

Russell built a wave tank in his laboratory to study his observations (Figure 1). He found that he could reproduce the single-humped wave of translation, and that its square of velocity was proportional to the mean depth of the water (1844). Russell's ideas were received unenthusiastically by the scientific establishment of his day. In particular, the great mathematical physicist George Stokes stated that Russell's theory was completely inaccurate, that only sinusoidal waves could be propagated with constant velocity and without change of form (1847), and another prominent mathematical physicist George

Airy derived a different formula for the speed of propagation (1845). [Stokes is even well-known today for the Navier-Stokes equation for viscid fluid flow, which since its formulation in 1822 and 1942 has been insolvable except for the simplest cases. The problem is now recognized as one of the seven Millennium Problems from the Clay Mathematics Institute.]

In modern terms, waves propagate so that their phase speed $c = \omega/\kappa$ varies with the wavenumber (inverse wavelength λ), where the angular frequency ω and wavenumber κ are related by

(1) $$\omega^2 = g\kappa \ \tanh \kappa H,$$

Figure 1. Russell's solitary wave: development and parameters
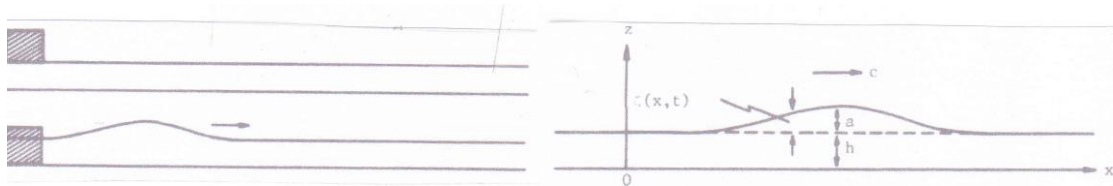
g = gravity acceleration, and H = mean water depth. Thus, waves of different wavelengths, starting from the same place will move away at different speeds and will disperse and spread out, as are often observed with ocean waves generated by a distant storm. The group velocity is given by (Figure 2): $c_g = d\omega/d\kappa$.

Figure 2. Phase and group velocity
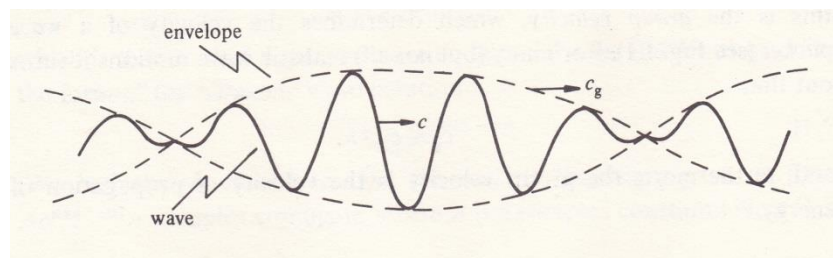
But there are two different limits to the dispersion relation depending on how κ relates to H. (i) For the case of *short waves*, i.e., $\kappa^{-1} \ll H$, (1) becomes

$$\omega^2 = g\kappa.$$

These waves are also called *deep-water waves* because their propagation is unaffected by by the bottom. (ii) For the case of *long waves*, $\kappa^{-1} \gg H$, (1) becomes

$$\omega^2 = g\kappa^2 H,$$

i.e.,
$$c^2 = gH.$$

These are also called *shallow-water waves* because their propagation is affected by the bottom. But a consequence of this is that such waves are nondispersive, or that their phase speed does not depend on wavenumber. This means that in this limit, waves of different frequencies or wavelengths move at the same speed and thus travel as one with no dispersion (Figure 3). This was the result that Russell found but without mathematical basis.
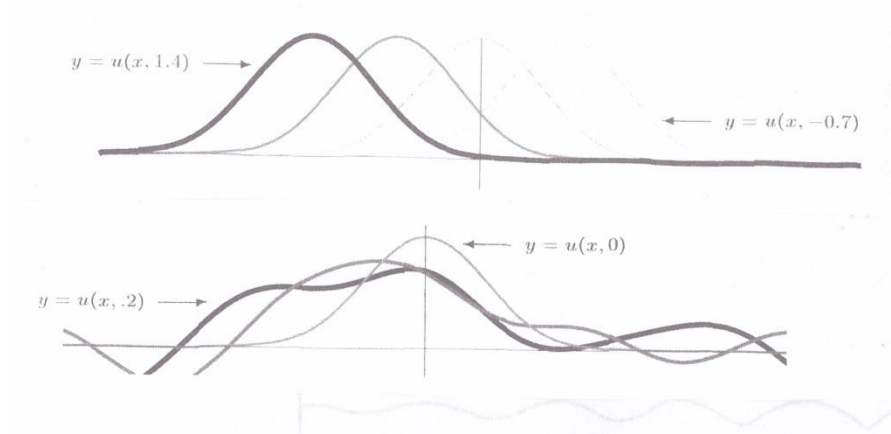
Figure 3. Dispersion effects for a group of waves

The Navier-Stokes equations

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = f - \nabla p + \nu \Delta u, \ \ \nabla \cdot u = 0$$

describe the motion of a fluid of viscosity ν, given the force f, i.e., to find the velocity u (in three dimensions) and the pressure p for all time t. [This is the Millennium Problem #4.] In 1895, the Dutch mathematicians Diederik Korteweg and Gustav de Vries made simplifying assumptions and derived a partial differential equation that describes waves in one dimension in a narrow canal with constant shallow depth. The KdV equation

$$u_t = \frac{3}{2}uu_x + \frac{1}{4}u_{xxx}$$

is a nonlinear partial differential equation that follows the evolution of the wave in time. It was known before that the first term on the RHS leads to distortion of the wave, and the second term to dispersion, both effects resulting to dissipation. Yet somehow the combination of the two terms leads to a closed form of the solution that yields a wave of a single hump persistent through time and propagating at constant velocity, the soliton.

This is what we plan to show in this simulation. We will use *Excel* to track the KdV equation as an initial wave propagates in time along one dimension. We calculate the speed $u(x, t)$ as a function of distance x and time t. We use the finite elements:

$$u_t \to \frac{u_{t1}-u_{t0}}{\tau},$$

$$u_x \to \frac{u_{x1}-u_{x0}}{\Delta},$$

$$u_{xx} \to \frac{u_{x2}-2u_{x1}+u_{x0}}{\Delta^2}$$

$$u_{xxx} \to \frac{u_{x3}-3u_{x2}+3u_{x1}-u_{x0}}{\Delta^3}.$$

For an initial waveform we pick the familiar hyperbolic secant-squared profile. The height is twice the speed, i.e., the speed is proportional to the height. Also, the width of the waveform is inversely to the height. Thus,

$$u(x, t) = h \, sech^2(x - ct)$$

is a localized hump that travels to the right at speed c, proportional to h.

Numerical experiments:

(1) For our first experiment, we show a single soliton with height 2 units and speed 1 unit at times 0 and 10 (Figure 4).
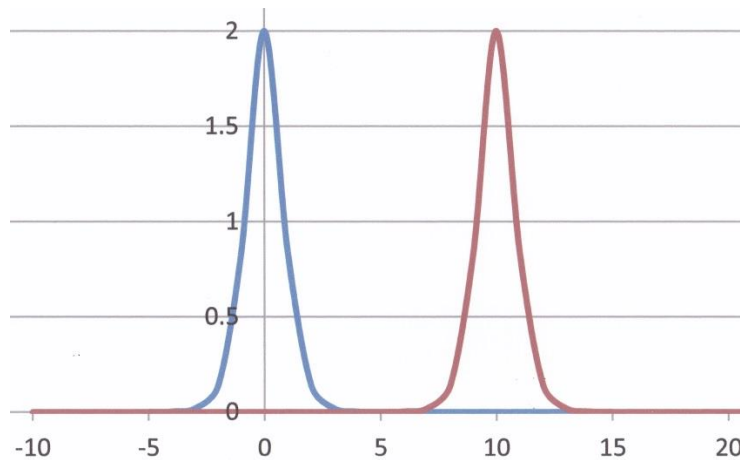

Figure 4. A soliton propagating to the right at unit speed.

(2) In Figures 5a,b,c we show the solution of the KdV equation for 2 solitons of heights 3 and 1, both moving to the right. The taller (faster) wave overtakes the shorter (slower) wave, and after merging momentarily, separates again as if no collision happened.
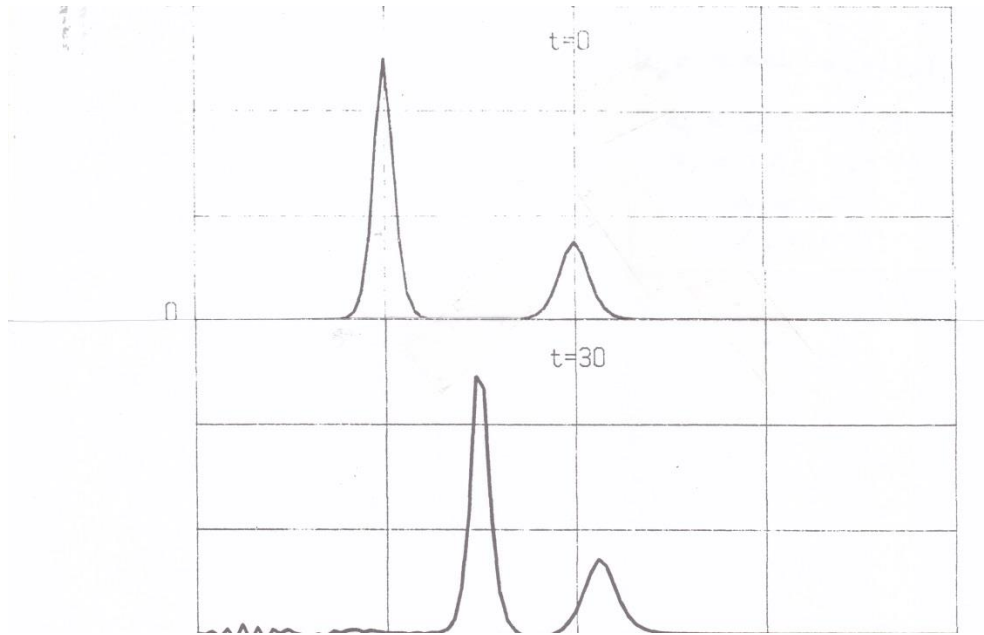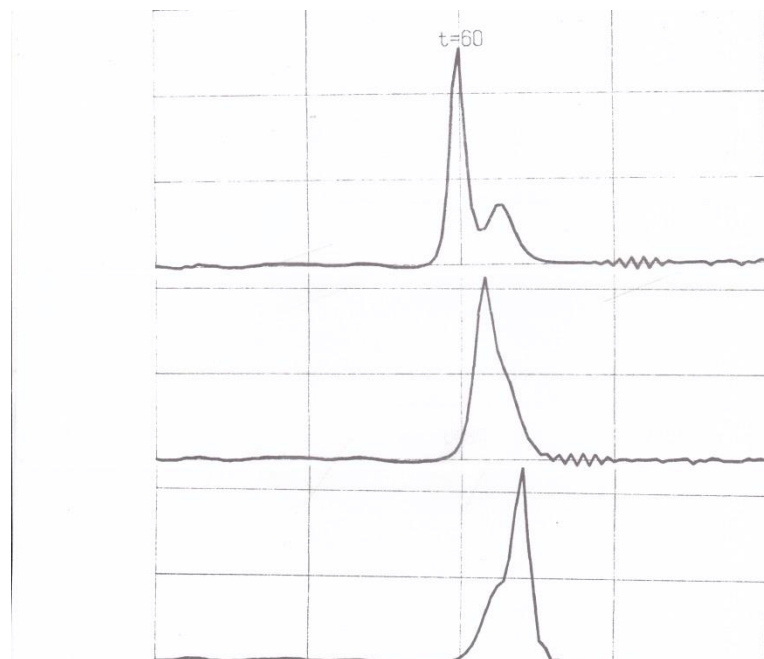

Figure 5a.

Figure 5b.

Figure 5c. Two solitons: overtaking collision.

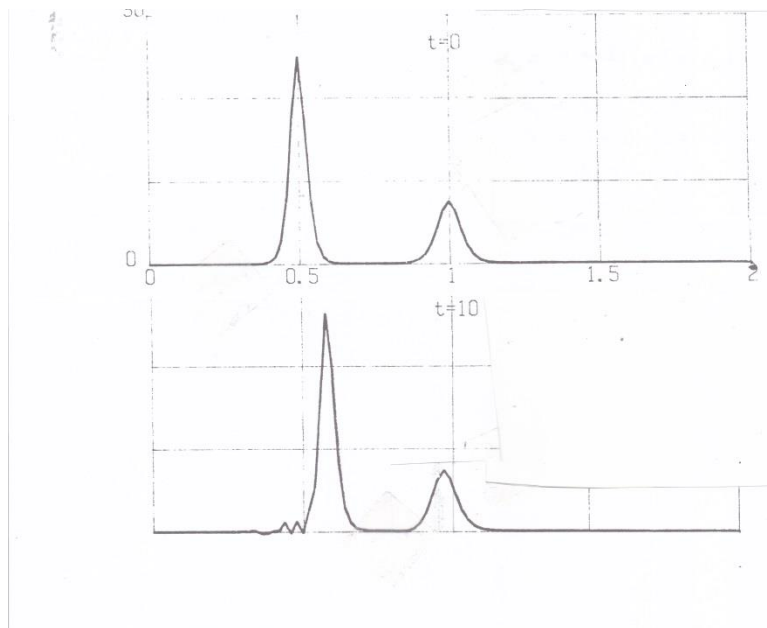(3) In Figure 6a,b,c the two waves approach each other and collide. After collision, the waves separate with no apparent effect on their shapes or speeds.
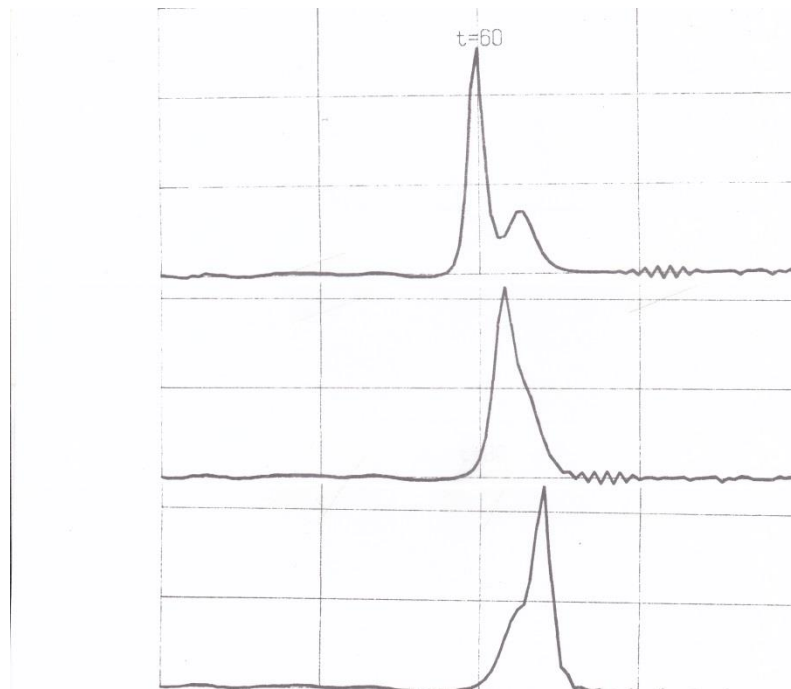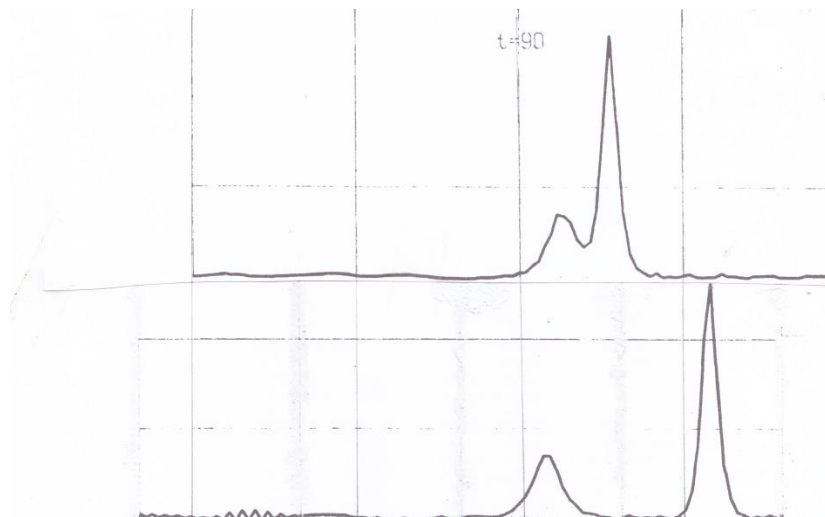

Figure 6a.

Figure 6b.

Figure 6c. Two solitons: head-on collision.

(4) For the *N*-soliton case, we start with three solitons of different heights (speeds), the tallest at the left. It overtakes the two waves to its right, collides with them, and separates. As the waves reach the right end, we apply a periodic boundary condition -- $u(0, t + 1) = u(L, t)$ – so that the waves scroll back to the left side and we get the effect of multiple waves colliding with each other. It is clear that the waves retain their shape and speed in spite of the collisions (Figure 7).
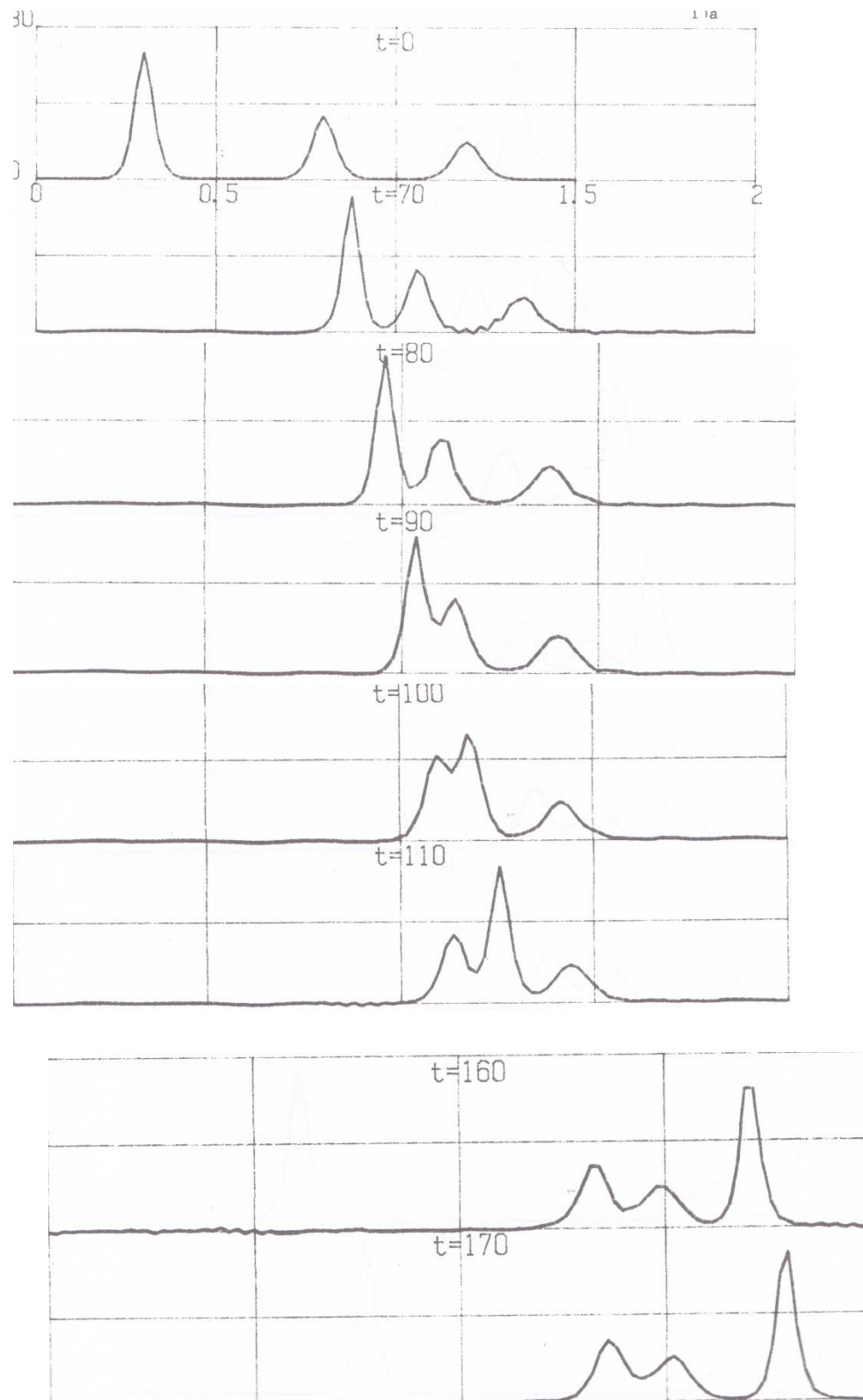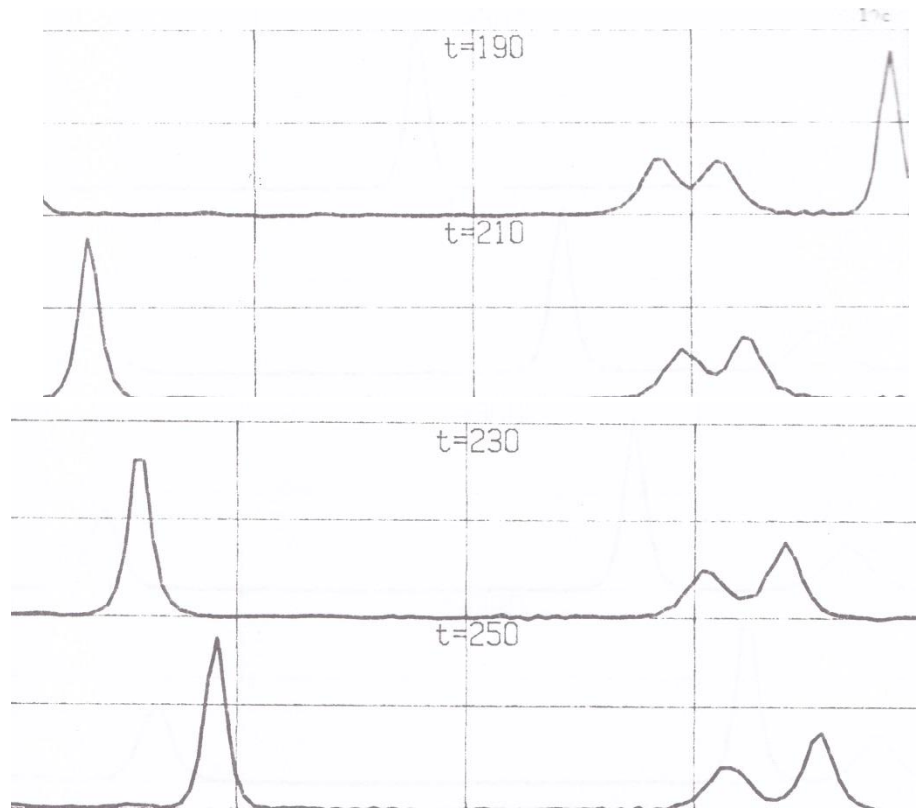
Figure 7a. *N* solitons.

Figure 7b: A periodic boundary condition scrolls the wave on the right to the left.

(5) In the final simulation, we start with an initial profile, first, with a cosine wave. As it propagates in time, it spontaneously breaks up into separate wavelets, undergoes multiple collisions, and manages to recover its original profile after very long times. The behavior may be described as decomposition of an initial profile, then long-time recurrence of the same initial profile. When we tried a different initial profile – a sine wave – the same effects are observed, a spontaneous decomposition to individual wavelets, then their eventual relaxation to the initial profile. Near the end though, there is a more pronounced upward slope of the wavelets to the right for the sine profile than was seen in the cosine profile (Figures 8 and 9).
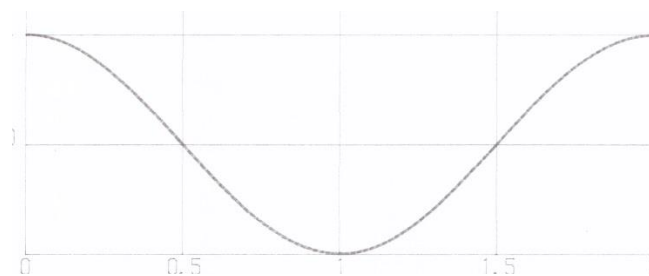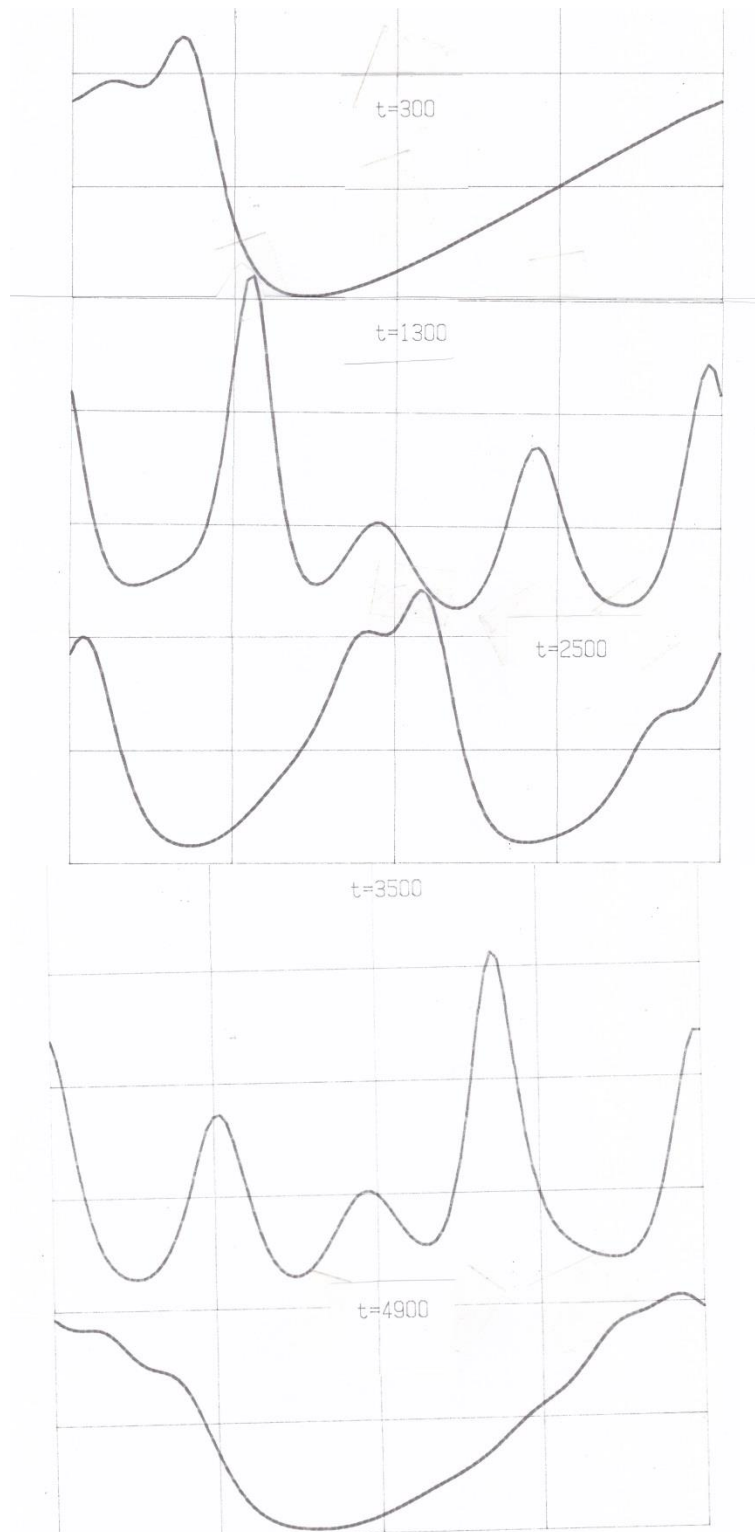
Figure 8. Decomposition and recurrence of an initial profile.
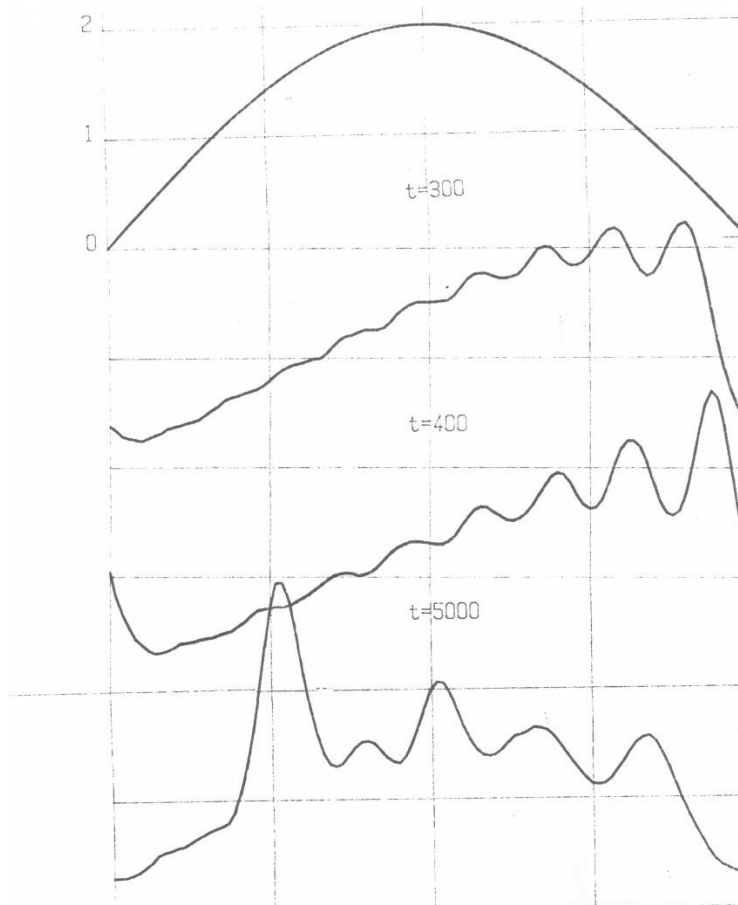
$$u(x, 0) = \cos\pi x, \quad [0, 2].$$

Figure 9. Dispersion and relaxation of an initial profile:

$$u(x, 0) = 2\sin\frac{\pi}{2}x, \quad [0, 2].$$

Applications:

(1) Since the success of the Korteweg-de Vries equation, there have been a number of evolution equations that describe some nonlinear dynamics of a physical system. Among these are:

(i) Schrödinger equation

$$iu_t + 2uu^2 + u_{xx} = 0,$$

(ii) sine-Gordon equation

$$u_{xt} - \sin u = 0,$$

(iii) Boussinesq equation

$$u_{tt} - (12uu_x + u_{xxx}) = 0.$$

The general characteristics are that solitary waves are solitons only if they are steady progressing pulse-like solutions that preserve their shapes and speeds after interaction.

(2)   In geophysical fluid dynamics, the Great Red Spot and other eddies in the atmosphere of Jupiter have been studied in terms of solitary waves. They persist after many years of collision.

(3) Tsunamis in the ocean are the best illustrations of solitary waves. Just a foot high in the open ocean traveling at hundreds of miles/hour, they rise to hundreds of feet near shore with little loss in speed, leading to devastating effects.

(4) In physics, the collisions of elementary particles are often likened to solitons. Long internal gravity waves have been ascribed to solitary waves. They have been used in studies of wave interactions in dispersive media. Recent studies in elementary particles interacting with magnetic fields have used the theory of solitary waves.

Conclusions:

- Solitons are single-humped waves that propagate with constant velocity for very long times.
- Their dynamics is governed by the KdV equation, a nonlinear pde.
- There are many nonlinear pde's, but the KdV is the simplest. It admits many closed-form solutions. The simplest of these is the $sech-$ squared profile.
- There are a number of numerical solutions to the KdV. The advantage of using *Excel* is its ready availability, and ease in programming. Many interesting results can be derived using *Excel.* [Reference #6 is well recommended for a more versatile animation of solutions using *Mathematica.*]
- Solitons travel with speeds proportional to the wave amplitudes.
- Solitons may collide with little loss in shape or speed.
- An arbitrary pulse, given a long time, may spontaneously decompose into several solitons, but eventually relax to its initial profile.
- Tsunamis are among the best examples of solitons.