ALGEBRAIC AND SCIENTIFIC THINKING VIA SPREADSHEETS: THE UNSTACKING COINS MODEL

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Abstract

By starting with a stack of coins and removing four at a time, we construct a model with negative correlation. This investigation can have the linear regression model with its mathematical nature converted to the scientific nature (physical meaning of variables). Error can be investigated in a spreadsheet simulation to see how it influences results including model predictions. We can validate the model using US Mint data along with pooled group results for some interesting discussion.

Introduction

Mathematical modeling with spreadsheets is an elegant way to unite the disciplines of mathematics and the sciences as outlined in both the *Common Core Standards in Mathematics* and the *Next Generation Science Standards*. Model verification is a common practice but somewhat limited in beginning efforts. Considering errors and trying to evaluate the accuracy of models is a way to bring verification into the classroom for novice modelers. Plus if we develop simple models that novice learners can understand, we can add error to show how the model can deteriorate in its ability to make correct predictions. Dealing with error can be done by implementing it experimentally in measurements (Sinex, Gage, & Beck, 2007; Sinex, Chambers, & Halpern, 2012) and/or building it into spreadsheet simulations for numerical experimentation (Sinex, 2013).

Over the last decade, we have developed a variety of simple modeling activities using manipulatives such as cookies, nested Styrofoam cups, nuts and bolts, bricks and brick walls, nested cubes, and now coins. Table 1 summarizes the activities and the science process that each develops (building down the table). All of these activities combine algebraic and scientific thinking and support the experimental data > mathematical model > simulation pathway plus enhance computational spreadsheet skills. These models are an excellent introduction to metrology, the science of measurement, and help to develop a mindset to address errors.

Table 1: Modeling Activities and Science Process

Activity and Reference	Linear Form and What's Measured	Science Process	Error Simulation
Stacking Cookies (Sinex, Gage, & Beck, 2007; Sinex, 2012)	y = mx height (ruler to 0.1 cm)	What does slope mean & cause of y-intercept in model	Ruler error (positioning of zero); Random error
Stacking Nested Styrofoam Cups (Sinex, 2008)	y = mx + b height (ruler to 0.1 cm)	Relate model parameters to parts of cup; what does y-intercept mean	Ruler error (positioning of zero); Cup part errors; Random error
Nuts and Bolts (Sinex, Chambers, & Halpern, 2011)	y = mx + b mass (balance to 0.01g)	Extrapolation & it's dangers (indirect determination of bolt mass & %error)	Balance error (zero offset); Random error
Stacking Bricks and Brick Walls (Sinex, 2017)	y = mx height (ruler to 0.1 cm)	Test model on wall & revise for mortar (add variable) & examines real variation	Ruler error; Random error including measured brick & mortar variation
Unstacking Pennies or other US coins (This paper)	y = b - mx height (ruler to 0.1 cm) or mass* (balance to 0.01g)	Meaning of both x- & y-intercepts; accuracy via US Mint standards (%error)	Ruler or balance error; Random error; Proportional systematic error
Stacking Nested Cubes as Tower (Sinex, 2015)	y = mx then $y = ax^2 + bx$ height (ruler to 0.1 cm)	Adding more variables; introduce curvature & how to handle	Fixed quadratic regression (c = 0); Ruler error; Random error

*for massing pennies be sure all are minted after 1982

This article introduces a linear model with negative correlation by unstacking (removing) coins from an unknown number of coins in a stack and measuring either decreasing stack height or mass. The mathematical model developed brings physical meaning to the slope, and both the x- and y-intercepts. Furthermore since the US Mint provides official coin thickness and masses (i.e.- a standard to compare), we can introduce the calculation of percent error to judge accuracy. Then via a spreadsheet simulation, which can reinforce model understanding, both random and systematic errors can be introduced into the model and their influence studied by numerical experimentation. The simulation can be pre-built or constructed by students depending on their background and time available. Honey and Hilton (2011) have discussed the advantages of simulations to produce an engaging pedagogy and deeper learning.

Here are the objectives of this activity and accompanying interactive spreadsheet:

- Derive an experimental mathematical model and judge its goodness-of-fit via r-squared from the unstacking of pennies (removing pennies from a set starting stack of unknown number) by measuring stack height to the nearest 0.1cm (or stack mass to 0.01g);
- From the model, develop a conceptual understanding of the physical meaning of the scientific variables for the slope, y-intercept, and x-intercept;
- Consider both random and systematic error in the model by examining an interactive spreadsheet simulation and through numerical experimentation;
- Evaluate the accuracy by computing the percentage error using the US Mint values as a standard or true value;
- Test predictions of the model by examining a variety of errors in this multivariable system, and;
- Collect via Google Sheets and interpret the pooled data for groups.

The activity with instructor notes which includes links to the Google Sheets simulation and pooling data spreadsheets (go to File > Make a copy... to get your own editable copy) can be obtained at https://goo.gl/mwL6u3.

Experimental Investigation

Each group should be given 40 to 50 pennies with the number varying between groups and stack them without counting the number. We prepared pre-counted stacks in plastic penny coin tubes. Students measure the stack height to the nearest 0.01cm with a centimeter ruler (or mass the stack to the nearest 0.01g on a balance). Then students remove four pennies and repeat the measurement and do so until they remove a total of 32 pennies. Data should be recorded in a table and then plotted in Google Sheets. A linear regression should be performed. The regression equation in terms of the variable studies (not x and y) along with r-squared should be recorded. As with any modeling activity, we are dealing with multiple symbolic representations (Table 2) and are trying to get students to transition between them in a smooth fashion. Students need to develop an understanding of what the model parameters represent (slope, y-intercept, and x-intercept). This model has the capability to have the parameters verified including the percent accuracy, in this situation as closeness to a true value (thickness or mass) from the US Mint.

After students derive their model, they can turn to the Unstacking Coins simulation constructed in Google Sheets to experiment with the parameters to reinforce model understanding. Here we are keeping the errors set to zero to deal with an ideal model. This model can be constructed by either measuring the stack height to the nearest 0.1 cm (Figure 1) or the stack mass to the nearest 0.01g (Figure 2). If students are going to measure mass, instructors will need to be sure that all the pennies are minted after 1982; the US Mint in 1982 changed the metal composition and hence the mass of a penny.

Table 2: Symbolic Representations Oh My!

Model	Equation	Slope	y-intercept	x-intercept
Mathematical	y = -mx + b = b - mx	-m	b	x = b/-m
Scientific - height measured	$H = H_o - tn$ H = height n = number of coins	-t t = -slope thickness of coin	H_o initial height of stack	$n_o = H_o/-t$ initial number of coins in stack
Scientific - mass measured	$M = M_o - M_c n$ M = mass n = number of coins	$-M_c$ M_c = -slope mass of coin	M_o initial mass of stack	$n_o = M_o/-M_c$ initial number of coins in stack
Spreadsheetese (computational formula)	in cell B7 =-$G$3*A7+$D$3*$G$3 (= -mx + b) where $D$3*$G$3 = n_o*t or n_o*M_c (computed)	Place thickness or mass in G3, so Slope = -G3	computed from slope function↟ in cell I15	computed from intercept function↟ in cell I20

↟to allow for recalculation with various errors added

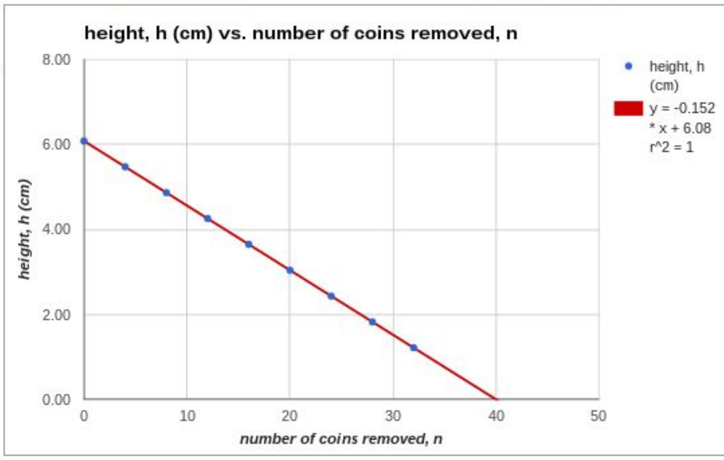
Unstacking Coins

Number of coins in initial stack = 40

-slope
coin thickness = 0.152 cm

Keep errors at zero
for ideal model

number of coins removed, n	height, h (cm)
0	6.08
4	5.47
8	4.86
12	4.26
16	3.65
20	3.04
24	2.43
28	1.82
32	1.22



Dealing with error:
random error
0
ruler error
0 cm

What math means:
initial stack height
y-intercept
6.080 cm
number of coins initially in stack
x-intercept
40

Figure 1 - Unstacking Coins set for US Pennies by measuring height (by height tab)

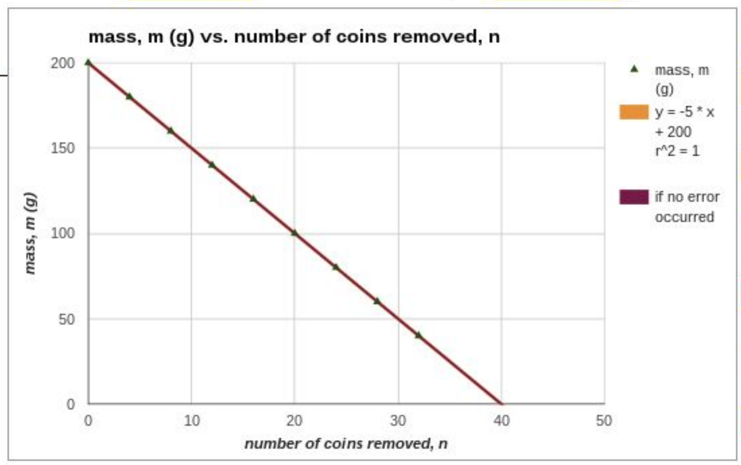
Unstacking Coins - by mass

Number of coins in initial stack = 40

-slope
coin mass, m = 5 g

Keep errors at zero
for ideal model

number of coins removed, n	mass, m (g)
0	200.0
4	180.0
8	160.0
12	140.0
16	120.0
20	100.0
24	80.0
28	60.0
32	40.0



Dealing with error:
random error
0
balance offset
0 g

What math means:
initial stack height
y-intercept
200 g
number of coins initially in stack
x-intercept
40

Figure 2 - Unstacking Coins set for US Pennies by measuring mass (by mass tab)

For the mass measurements, we can measure the mass of the coins removed and develop a directly proportional model (Figure 3). How are the lines in these two models (Figures 2 and 3) related? The lines are just intersecting lines, where the slope of both lines yields the average coin thickness.

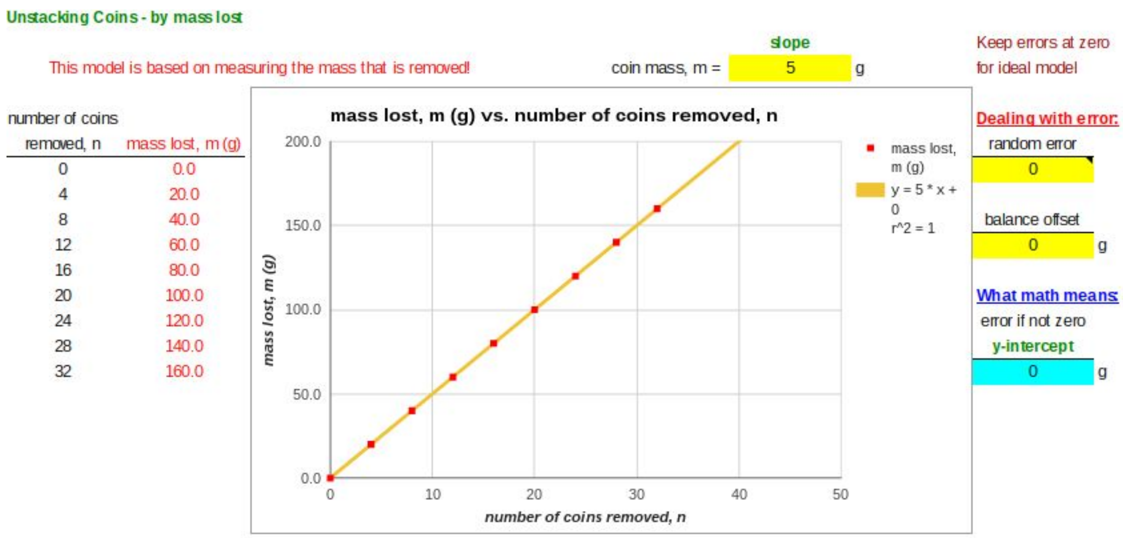


Figure 3 - Measuring the mass of coins removed from the stack (by mass lost)

Since we have the thickness (or mass) of coins from the US Mint, we can calculate the percent error for the average thickness determined from the slope of the model.

% error = (measured thickness - US Mint thickness)*100/US Mint thickness

The sign of the error tells you if the error is above the mint value (positive %error) and if the values is below the mint value (negative %error). We examine percent error further in the pooled data section.

Exploring Experimental Error via a Spreadsheet Simulation

All the tabs in the spreadsheet allow random error to be explored. The addition of random error increases the scatter of the data points and will randomly change the slope, y-intercept, and x-intercept. Students will need to numerically experiment to discover this. In Google Sheets if you type CTRL + R, the spreadsheet recomputes the random numbers. Notes, like comment boxes in Excel, are used to set some limits or the range of values on the size of the random error and other errors as well. Repeating this a number of times will allow them to see the random variation as seen in Figure 4. The spreadsheet shows the initial model parameters for comparison.

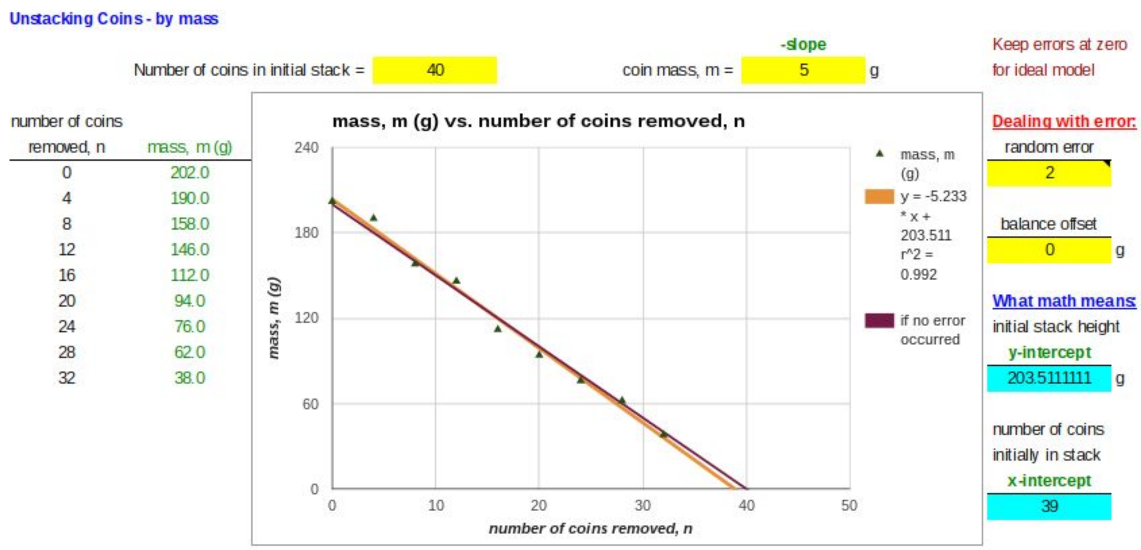


Figure 4 - Adding random variation, scatter, to the data (by mass tab)

Constant systematic error as ruler error (zero mark not at the end of the ruler) for the stack height or balance offset (zero mass is off) for the stack mass can be explored. Instructors can find rulers with sizable ruler errors to see how students deal with it and can adjust most pan balances to knock the balance off the zero mark (miscalibrated), to cause either a positive or negative constant error. See Sinex (2005) for a discussion of the various types of errors. The constant systematic errors only influence the x- and y-intercepts. The new line with error is parallel to the original data.

Proportional systematic error is error that can occur with balances and is illustrated in Figure 5. This type of error influences the slope and the y-intercept. The y-intercept is influenced due to being a measured quantity. The x-intercept is not influenced as it is not measured (assumed to be zero mass and found by extrapolation). The errant line rotates around the fixed x-intercept.

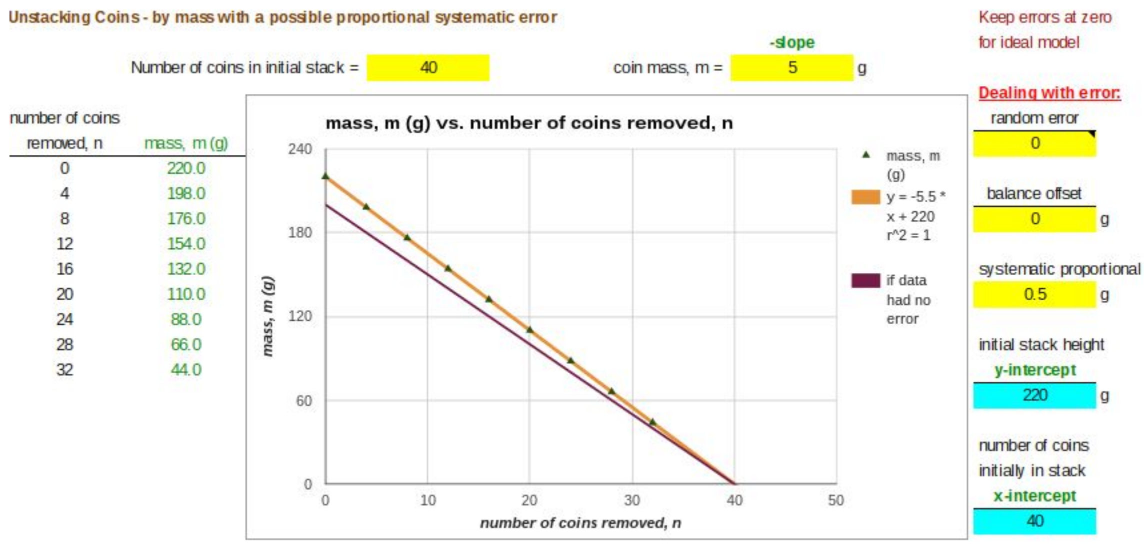


Figure 5 - Proportional systematic error for mass model (by mass - systematic proportional tab)

The variety of errors for this negative correlation linear model is summarized in Table 3. Ideally, you want students to discover the behavior of the various errors via the simulation. For student to see the effects of random error they will need to numerically experiment. The constant systematic errors do not influence the slope and hence, the percent error. From a scientific point of view, the percent error determination gives the investigator some credence and is a common practice in science, if a standard for comparison, the US Mint values, is available.

Evaluating Model Predictions

How do the various errors influence model predictions? Figure 6 illustrates the unstacking coins model by height where the various errors can be induced and compares the results of with and without error. This is setup to allow students to explore a variety of errors and get feedback on behavior.

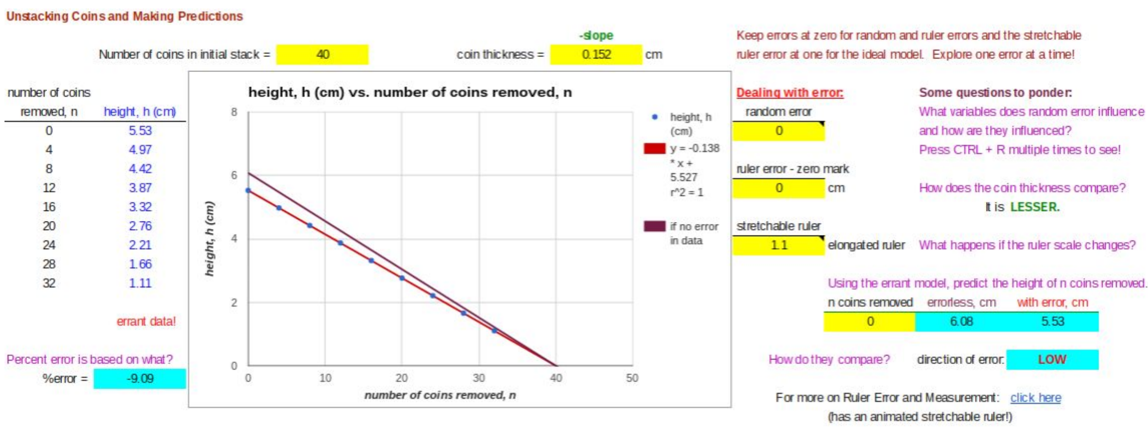


Figure 6 - Unstacking coin model by height and making predictions (by height - predictions tab)

Table 3: Influence of Errors on the Unstacking Coins Model

<table>
<tr><th colspan="2">Type of Error</th><th>Slope (average thickness = -slope)</th><th>Percent error on average thickness</th><th>y-intercept (initial stack height)</th><th>x-intercept (number of coins in stack)</th><th>r^2 (goodness of fit)</th></tr>
<tr><td colspan="2">Random (To visualize random effects, numerical experimentation is required)</td><td>Changes randomly – plus or minus*</td><td>Changes randomly – more positive or more negative*</td><td>Changes randomly – plus or minus*</td><td>Changes randomly – plus or minus*</td><td>Decrease (adds scatter to data points)</td></tr>
<tr><td rowspan="2">Systematic-Constant</td><td>+ve</td><td rowspan="2">No effect</td><td rowspan="2">No change</td><td>Increase</td><td>Increase</td><td rowspan="2">No effect</td></tr>
<tr><td>-ve</td><td>Decrease</td><td>Decrease</td></tr>
<tr><td rowspan="2">Systematic-Proportional (depends on magnitude of height or mass)</td><td>+ve</td><td>Increase - greater negative slope</td><td>More positive error</td><td>Increase**</td><td rowspan="2">No effect (height or mass is zero, not measured, so no error)</td><td rowspan="2">No effect</td></tr>
<tr><td>-ve</td><td>Decrease - lesser negative slope</td><td>More negative error</td><td>Decrease**</td></tr>
</table>

*need to vary random error (press CTRL + R keys in Sheets) to see this occur
**due to negative correlation (does not occur with positive correlation models)

Pooling Your Class Data

This should not be your first data pooling experience as it is very different and you do not get an improved average class model, such as pooling the cookie stack height data (Sinex, Chambers, & Halpern, 2016) or stacking bricks model. This model produces a series of near-parallel data sets on the pooled plot (if no or very little experimental error) that are dependent on the initial number of coins in the stack (x-intercept varies). Figure 7 illustrates a set of data for seven groups with five different initial stack heights. It should lead to a nice class discussion as they submit data into a shared spreadsheet that automatically plots the data. A link is provided in the activity for a pooling data spreadsheet (Go to File > Make a copy... and use your version with students).

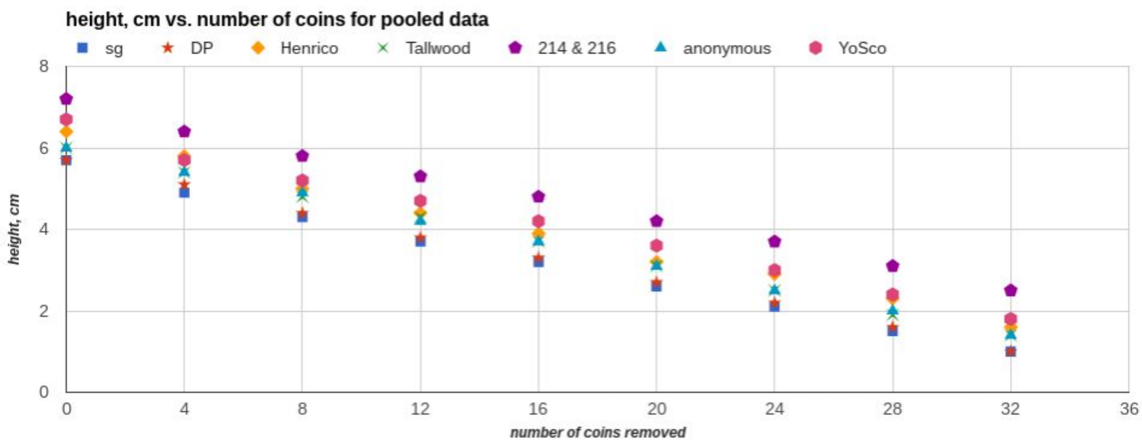
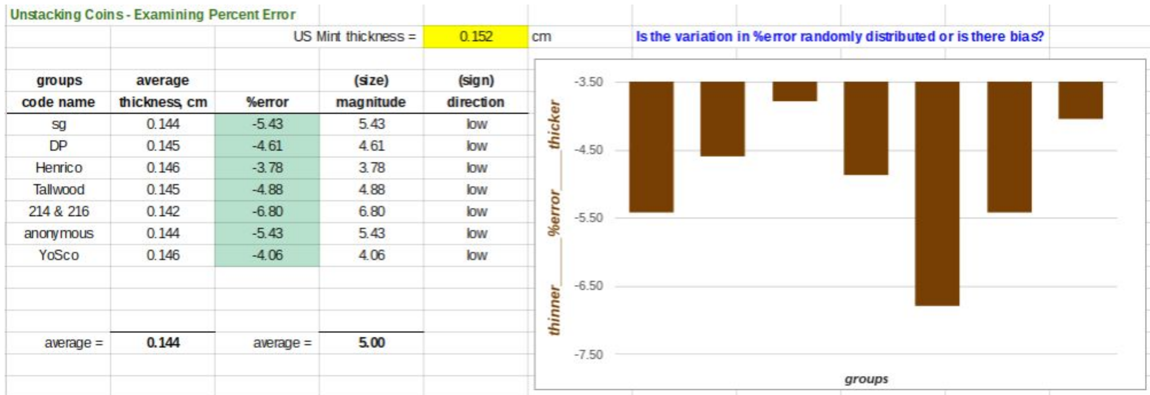


Figure 7 - Unstacking pennies pooled data (7 groups with 5 different initial stack heights)

The percent error for each group is calculated and displayed graphically in Figure 8. The teachers noticed that all groups had a negative error, hence bias results (all groups were thinner than the US Mint value). We had a lively discussion about wear (0.01 cm or 0.1 mm of wear on average) on the penny due to circulation and that it might depend on the age (date) on the penny. Teachers decided this could be tested by student's measuring stacks of the same year or measuring newly minted pennies. Teachers also considered contamination by Canadian pennies which are 0.145 cm thick since 1980 according to the Royal Canadian Mint. Ruler error (zero mark off from end of ruler) was eliminated, since it does not influence the slope. So, yes, we had a lively discussion and proposed error causes that are testable.

The "variety of stacks" tab, which is hidden on the simulation spreadsheet, allows instructors to demonstrate the pooled data and how the initial stack height influences the series of near parallel lines. To unhide a tab: go to View on the menu bar > select Hidden Sheets > variety of stacks.



Unstacking Coins - Examining Percent Error

US Mint thickness = 0.152 cm

groups code name	average thickness, cm	%error	(size) magnitude	(sign) direction
sg	0.144	-5.43	5.43	low
DP	0.145	-4.61	4.61	low
Henrico	0.146	-3.78	3.78	low
Tallwood	0.145	-4.88	4.88	low
214 & 216	0.142	-6.80	6.80	low
anonymous	0.144	-5.43	5.43	low
YoSco	0.146	-4.06	4.06	low
average =	0.144	average =	5.00	

Figure 8 - Unstacking pennies pooled data showing percent error

Feedback from Teachers

Eight teachers provided feedback and gave the overall use of Unstacking Coins Simulation an average score of 4.4 out of 5 (87.5% agreed it was easy to use). When asked if the cookie stacking model (Sinex 2012) helped, 52.5% agreed it did. Half of them were surprised by the pooling results for the unstacking of pennies model.

Comments were very positive and a few are given below:

- A good way to derive and illustrate negative correlation
- How to question the analysis further and in context what do the numbers mean
- Good example of open ended questions
- How to use Google Sheets

How to Build the Simulation

From the "by mass - systematic proportional" tab (Figure 5), if you click on cell B7, you will find the following formula that computes the mass as part of an if statement:

=if ($D$3 = 0, 0, Y)

```
Y =-($G$3+$I$12)*A7+$D$3*($G$3+$I$12)+$I$9+$I$6*randbetween(-5,5)
```

where $_$_ is an absolute cell reference as they will not change on dragging a formula down a column. The if statement above is needed to avoid negative values along with conditional formatting to turn the cell blank if less than or equal to zero and the y-axis minimum on the graph is set to zero. Because this is a negative correlation, both the slope and y-intercept, a measured quantity, are influenced by the proportional systematic error. Table 4 gives the anatomy of the formula above in a piece-by-piece fashion.

For more on errors and the mechanics of building them into a spreadsheet to accomplish error analysis, see Sinex (2005, 2013). See Sinex (2016) for an interactive spreadsheet in Google Sheets that addresses how to induce the errors. The range of values for an error must be determined for each error and to keep the graphs looking realistic in nature (avoid complete loss of trend in the data).

Coin Capers Assessment

To see if students truly understand the model, the coin caper tab has students derive a model and address some questions. From the height of a stack of coins as a function of the number of coins removed, students are asked to identify the coin, how many are in the initial stack, and how much money is involved in the task (sum three stacks). The coin caper assessment data simulation can be used by instructors to generate different data sets for individual students or online collaborative groups (https://goo.gl/jAfXqi, go to File > Make a copy… to get an interactive copy). These can easily be shared via Google Drive and results even pooled for a class discussion (see Sinex et al. 2016).

Table 4: Anatomy of the Computational Formula including Errors

Part of formula	Math	What it does and *why in italics*
-($G$3+$I$12)	-m	-(coin mass + proportional systematic error) *This changes the proportion of the coin mass! This is the proportional systematic error!*
-($G$3+$I$12)*A7	-mx	-(coin mass + proportional systematic error)* number of coins *= mass of stack (proportionally adjusted)*
+$D$3*($G$3+$I$12)	b	Number of coins in initial stack*(coin mass + proportional systematic error) *The y-intercept mass must be changed proportionally.*
+$I$9	error	Balance offset *[This is constant systematic error!]*
+$I$6*randbetween(-5,5)	error	Adjustable variable * function that generates a random number between -5 to 5 *This computes the random error!*

Some Final Thoughts

The unstacking coins model provides a negative correlation model that allows students to collect data and derive a mathematical model. The model results can then be put in turns of the scientific variables measured (number of coins, height). Students find the scientific meaning of the slope, y-intercept, and the x-intercept. Using US coins, we can validate the model by assessing the percent error. The unstacking coins simulation in Google Sheets allows the exploration of a variety of errors and their influence on predictions. This gives the modeling activity a very real-world scientific appeal and helps strengthen data analysis and interpretation skills.

The pooling of data via a shared Google Sheets spreadsheet produces a series of near parallel lines that differ by their x-intercepts, the initial number of coins in the stacks. All our group's slopes were below the US Mint penny thickness (negative percent error), suggesting that wear during circulation may be a factor. For instructors interested in data pooling, Sinex et al. (2016) review the online collaborative capabilities of Google Sheets.

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Classroom Resources

"Making Money" Ep. 03: Circulating Coins,
https://www.youtube.com/watch?v=mqPvKxJXC_Y

US Mint - Coin Speciifications
https://www.usmint.gov/about_the_mint/index583f.html?action=coin_specifications

Royal Canadian Mint - Canadian Penny
http://www.mint.ca/store/mint/about-the-mint/1-cent-5300004#.V-6TpTMrLnB