THE MATHEMATICS OF VIDEO POKER

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Video poker [1] is a casino game based on five-card draw poker. It is played on a device similar in size to a slot machine. However, unlike slot machines, it is possible to compute the probability of winning the amount of money listed on the front of the machine in the payout table.

Game Basics:

The player inserts either money or a bar-coded house card with credit. After placing a bet, usually $1 to $5 per hand, the player presses the Deal button. The player is "dealt" 5 cards from a randomized deck of cards and is allowed to choose the cards they wish to discard. Pressing Deal again replaces these cards with new cards from the remaining deck. At which point the machine pays out the amount on the payout table if the players hand matches any of the winning hands.

Random Number Generators (RNG):

Briefly, a RNG "shuffles" a standard deck of cards until a person hits the Deal button. At that point the player is dealt 5 cards. The RNG continues to shuffle the remaining 47 cards until the Deal button is pressed again-at which point the discarded cards are replaced. The RNG insures there is no such thing as a "hot" or "cold" machine. Note: the topics of how to create RNGs and the various ways of shuffling decks of cards are interesting to study in their own right.

Variants:

There are many types of video poker games: Joker's Wild, Jacks or Better, Tens or Better, Deuces Wild, Bonus Poker, Double Bonus Poker… Within each of these games there are subvariants-the winning hands are the same but the payouts are different depending on the machine. In this paper we shall analysis the Jacks or Better game as it is one of the oldest and most popular of all video poker games.

Strategy:

Unlike slot machines, a player is able to compute the probability of obtaining the winning hands as shown on the pay table. Hence, an optimal strategy can be devised to maximize the expected return on every dollar bet. Computing the expected values and standard deviations for a given type of video poker allows the user to maximize their chance of winning, minimize their losses, and estimate the initial bankroll needed to achieve various outcomes. The interesting thing about video poker is that some variants, when played optimally, have an expected payout of almost $1 for every dollar bet. Hence well-played, video poker has a much better return than regular slot machines.

This talk is geared towards people teaching probability and statistics.

A Quick Look at Regular 5-Card Poker

Figure 1 5-card draw poker

We shall briefly look at regular 5-card draw poker as this is what is generally studied in a first level course on probability.

The Probability of Standard Poker Hands

Hand	Method	Probability
One-Pair	[13*C(4,2)*C(12,3)*C(4,1)^3]/C(52,5)	.4225690276
Two-Pair	[C(13,2)*C(4,2)^2*C(11,1)*4]/C(52,5)	.0475390156
Three of a Kind	[C(4,3)*13*C(12,2)*C(4,1)^2]/C(52,2)	.0211284514
Full House	[13*C(4,3)*12*C(4,2)]/C(52,5)	.0014405762
Four of a Kind	[13*C(4,4)*48]/C(52,5)	.00024009600384
Straight	10*4^5/C(52,5)	.0039246468
Flush	C(13,5)*C(4,1)/C(52,5)	.0019654015
Straight Flush	4*10/C(52,5)	.00001385169452
Royal Flush	4/C(52,5)	.000001539077169

Figure 2-Standard Poker Probabilities

One thing I think is interesting to do at this point in a course is to do a simulation, dealing say n=1,000,000 poker hands and seeing how well the above theory matches up with "reality"

A Simulation of Size

n=1,000,000

run:
Percent of hands with Pairs is 0.42255000
Percent of hands with Two Pairs is 0.04773800
Percent of hands with Three of a Kind is 0.02118300
Percent of hands with Full House is 0.00145600
Percent of hands with Four of a Kind is 0.00023300
Percent of hands with Straight is 0.00389400
Percent of hands with Flush is 0.00199300
Percent of hands with Straight Flush is 0.00001300
Percent of hands with Royal Flush is 0.00000200

As we see there appears to be a very good match. We can also use such simulations with a Chi-Square goodness-of-fit test to see if our simulation is a good model of reality.

Jacks or Better Video Poker

Royal Flush	250	500	750	1000	4000
Straight Flush	50	100	150	200	250
Four of a Kind	25	50	75	100	125
Full House	9	18	27	36	45
Flush	6	12	18	24	30
Straight	4	8	12	16	20
Three of a Kind	3	6	9	12	15
Two Pair	2	4	6	8	10
Jacks or Better	1	2	3	4	5

Figure 3-A Simulated Jacks or Better Screen

Video Poker Strategy:

The “simple” strategy is as follows [2]:

Upon the initial deal, hold the cards in the following list-the higher hands in the list take priority over lower ranked hands. The ordering in the list is derived from computing the expected value of each hand and ordering the hands in terms of which has the highest expected payout per dollar bet (as shown in parentheses). The probabilities to derive these expected values will be shown below.

Note: via expected values, it would appear that #13 and #14 in the list should be reversed. However, the standard ordering that I have seen on various web sites [2] are as listed and the probabilities used to derive 13 and 14 were approximations (more on this shortly) so small errors were possible.

1. Four of a kind (25), straight flush (50), royal flush (800)
2. 4 to a royal flush (18.84)
3. Three of a kind (4.31), straight (4), flush (6), full house (9)
4. 4 to a straight flush (2.94)
5. Two pair (2.60)
6. High pair (1.54)
7. 3 to a royal flush (1.42)
8. 4 to a flush (1.15)
9. Low pair (.824)
10. 4 to an outside straight (.745)
11. 2 suited high cards (.599)
12. 3 to a straight flush (.536)
13. 2 unsuited high cards (if more than 2 then pick the lowest 2) (.485)
14. Suited 10/J, 10/Q, or 10/K (.494)
15. One high card (.472)
16. Discard everything (.361)

For example, in a hand with three of a kind hold the three like cards and discard the remaining two.

Terms:

High card: A jack, queen, king, or ace.

Outside straight: An open-ended straight that can be completed at either end, such as the cards 3,4,5,6.

Inside straight: A straight with a 10, J, Q, K, A also count as inside straights.

We shall call the above list the simple list

The above list is called the “simple” strategy. There are other strategies whose list can number up to 60 hands. As these are generally memorized many people stick with the simple strategy as it yields only about $.001 less in expected value as the more complex (harder to remember) strategy.

Finding (Estimating) the Probabilities of the Final Hands

Royal Flush	250	500	750	1000	4000
Straight Flush	50	100	150	200	250
Four of a Kind	25	50	75	100	125
Full House	9	18	27	36	45
Flush	6	12	18	24	30
Straight	4	8	12	16	20
Three of a Kind	3	6	9	12	15
Two Pair	2	4	6	8	10
Jacks or Better	1	2	3	4	5

Figure 4-The winning hands and payouts for a $1-$5 bet

Two Parts:

I: After the initial deal find the probability of each of the hands in the above simple list.

II: Find (estimate) the conditional probabilities of each of the winning hands given the initial hands in the above simple list.

Part I is easy-just write a program and run through all the possibilities. Part II is the problem as there are so many subcases.

Before we look at these problems, a logical question to ask is "Why not just write a program and run through all the cases? This will give us an exact answer."

The reason is as follows:

There are C(52,5)=2,598,960 possible hands on the first deal. For each of these hands there are

$$C(5,0)*C(47,5)+C(5,1)*C(47,4)+C(5,2)*C(47,3)+C(5,3)*C(47,2)+C(5,4)*C(47,1)+$$

$$C(5,5)*C(47,5)=2,598,960$$

possible hands in the second deal. Hence to run through all possibilities there are

$$2,598,960\cdot 2,598,960=6,754,593,081,600 \text{ possibilities}$$

At 1,000,000 hands a second, a computer would need 79 days of nonstop running to run through all the possibilities.

By using equivalence classes, initial hands whose chances of winning a certain amount of money after the second deal are equal, the number of possibilities can be reduced a good deal-but is still a computationally prohibitive problem.

To find the probabilities of getting each of the hands in the simple list after the first deal (after the Deal button is pressed for the first time) I wrote a program and ran through all C(52,5)=2,598,960 cases and got the following results:

Percent of hands with Pair High is 0.12984578
Percent of hands with Two Pairs is 0.04753902
Percent of hands with Three of a Kind is 0.02112845
Percent of hands with Full House is 0.00144058
Percent of hands with Four of a Kind is 0.00024010
Percent of hands with Straight is 0.00389233
Percent of hands with Flush is 0.00190692
Percent of hands with Straight Flush is 0.00001385
Percent of hands with Royal Flush is 0.00000154
Percent of hands with 4 to Royal is 0.00035860
Percent of hands with 3 to Royal is 0.01083664
Percent of hands with 4 to a straight flush is 0.00198079
Percent of hands with 4 to a flush is 0.03290239
Percent of hands with Pairs Low is 0.28218672
Percent of hands with 4 to an outside straight is 0.02528396
Percent of hands with 2 High Suited Cards is 0.06603103
Percent of hands with Three to a Straight flush is 0.02528396
Percent of hands with Two or More Unsuited High Cards is 0.15516206
Percent of hands with Ten and a Face Card is 0.01218949
Percent of hands with Exactly One High Card is 0.14931665
Percent of hands with no cards to keep is 0.03245914

Now we need to find all the conditional probabilities:

$$P(\text{final hand}|\text{cards kept after the first draw})$$

As there are 21 types of hands in the simple list (after the first draw) and nine possible winning hands, there are 189 conditional probabilities to compute. Fortunately, many of these conditional probabilities are 0 so do not need to be computed. An example of this would be

$$P(\text{final hand is two pair}|\text{hand after first deal is three of a kind})=0$$

The following tables give all the conditional probabilities. The headers in the columns are the cards kept after the initial deal. The hand in the first column of each table is the final hand after the second deal. The numbers next to the final hands is the payout for each $1 bet (if a person bets the maximum of $5 each hand). The value under each initial hand is the expected value derived from holding those cards after the initial deal.

		High Pair	Two Pair	Three Kind	Full House	Four Kind
		0.12984578	0.04753902	0.02112845	0.00144058	0.0002401
Royal Flush	800	0.000000000000	0.000000000000	0.000000000000	0.000000000000	0.000000000000
Straight Flush	50	0.000000000000	0.000000000000	0.000000000000	0.000000000000	0.000000000000
Four Kind	25	0.002775208141	0.000000000000	0.042553191489	0.000000000000	1.000000000000
Full House	9	0.010175763182	0.085106382979	0.061954579093	1.000000000000	0.000000000000
Flush	6	0.000000000000	0.000000000000	0.000000000000	0.000000000000	0.000000000000
Straight	4	0.000000000000	0.000000000000	0.000000000000	0.000000000000	0.000000000000
Three Kind	3	0.114338575393	0.000000000000	0.896392229417	0.000000000000	0.000000000000
Two Pairs	2	0.159851988899	0.914893617021	0.000000000000	0.000000000000	0.000000000000
Jacks Better	1	0.712858464385	0.000000000000	0.000000000000	0.000000000000	0.000000000000
		1.536540241	2.595744681	4.310597687	9	25

Figure 5 i

		Staight	Flush	St Flush	Royal Flush	4 to Royal
		0.00389233	0.00190692	0.00001385	0.00000154	0.0003586
Royal Flush	800	0.000000000000	0.000000000000	0.000000000000	1.000000000000	0.021276574470
Straight Flush	50	0.000000000000	0.000000000000	1.000000000000	0.000000000000	0.004255319149
Four Kind	25	0.000000000000	0.000000000000	0.000000000000	0.000000000000	0.000000000000
Full House	9	0.000000000000	0.000000000000	0.000000000000	0.000000000000	0.000000000000
Flush	6	0.000000000000	1.000000000000	0.000000000000	0.000000000000	0.165957446809
Straight	4	1.000000000000	0.000000000000	0.000000000000	0.000000000000	0.102127659574
Three Kind	3	0.000000000000	0.000000000000	0.000000000000	0.000000000000	0.000000000000
Two Pairs	2	0.000000000000	0.000000000000	0.000000000000	0.000000000000	0.000000000000
Jacks Better	1	0.000000000000	0.000000000000	0.000000000000	0.000000000000	0.204255319149
		4	6	50	800	18.84253617

Figure 5 ii

			3 to Royal	4 to St Flush	4 to Flush	Low Pair	4 to outside st
			0.01083664	0.00198079	0.03290239	0.28218672	0.02528396
Royal Flush	800		0.000925069380	0.000000000000	0.000000000000	0.000000000000	0.000000000000
Straight Flush	50		0.000370027752	0.031914893617	0.000000000000	0.000000000000	0.000000000000
Four Kind	25		0.000000000000	0.000000000000	0.000000000000	0.002775208141	0.000000000000
Full House	9		0.000000000000	0.000000000000	0.000000000000	0.010175763182	0.000000000000
Flush	6		0.040205180389	0.159574468085	0.191489361702	0.000000000000	0.000000000000
Straight	4		0.019703977798	0.095744680851	0.000000000000	0.000000000000	0.170212765957
Three Kind	3		0.008325624422	0.000000000000	0.000000000000	0.114338575393	0.000000000000
Two Pairs	2		0.024976873266	0.000000000000	0.000000000000	0.159851988899	0.000000000000
Jacks Better	1		0.261979648474	0.000000000000	0.000000000000	0.000000000000	0.063829787234
			1.415514154	2.936170213	1.14893617	0.823681776	0.744680851

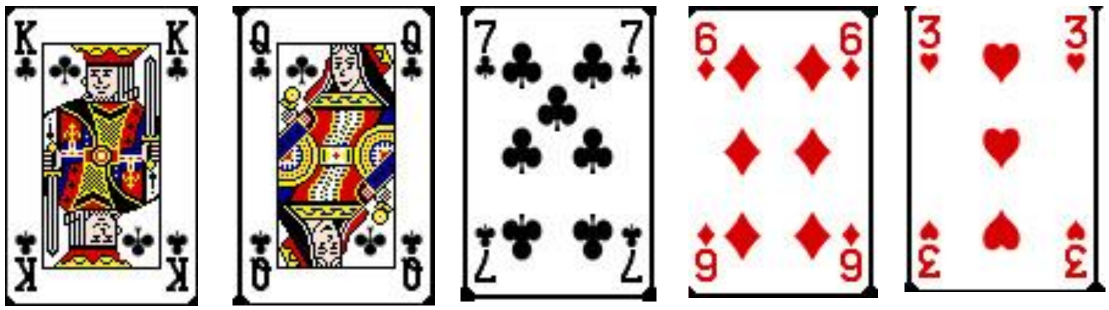
Figure 5 iii

			2 suited high	3 to St Flush	two or more unsuited high	10 & face matching	Exactly 1 high	Discard All
			0.06603103	0.02528396	0.15516206	0.01218949	0.14931665	0.03245914
Royal Flush	800		0.000061671292	0.000000000000	0.000000000000	0.000061671292	0.000005606481	0.000002607666
Straight Flush	50		0.000041343669	0.001850138760	0.000000000000	0.000185013876	0.000009811342	0.000012386412
Four Kind	25		0.000123342584	0.000000000000	0.000123342584	0.000123342584	0.000291537017	0.000224259244
Full House	9		0.001110083256	0.000000000000	0.001110083256	0.001110083256	0.001614666555	0.001384670446
Flush	6		0.010072977696	0.039777983349	0.000000000000	0.007153869874	0.001833319317	0.002020940859
Straight	4		0.006146572104	0.027012025902	0.005632963306	0.014554424915	0.002803240546	0.003781767072
Three Kind	3		0.017329633056	0.008325624422	0.017329633056	0.017329633056	0.022997785440	0.020536670624
Two Pairs	2		0.043848288622	0.024976873266	0.043848288622	0.043848288622	0.049751913212	0.046808901788
Jacks Better	1		0.309713228492	0.022201665125	0.309713228492	0.182238667900	0.254848204524	0.157555157017
			0.59890139	0.536355227	0.485004872	0.494727105	0.472354442	0.360809654

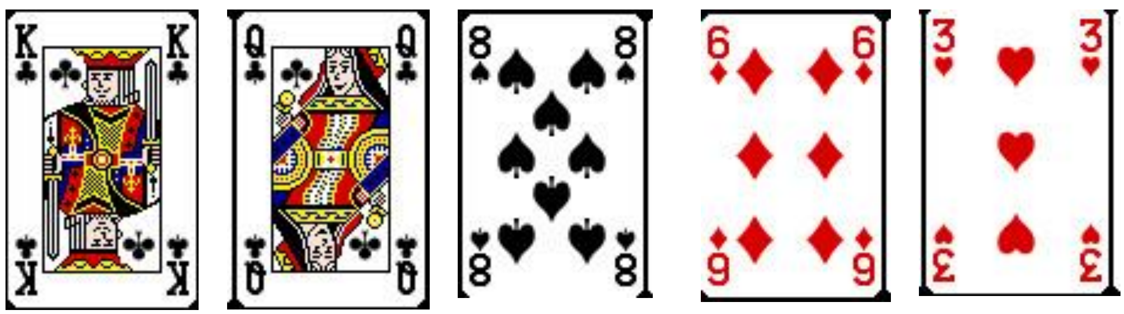
Figure 5 iv

While many of these conditional probabilities are exact, several are (good) approximations for the following reason:

If a person's initial hand is given by



They will hold the King and Queen, whereas if their initial hand is



They will hold the same two cards, however their chance of getting a flush after the second deal will have increased. To cover all such cases-which can be viewed as equivalence classes-would take a great deal of time, often leading to little increase in the accuracy of our results.

Hence, in deriving the above conditional probabilities I occasionally grouped equivalence classes together where there would be little difference in their probabilities in order to make the problem more trackable.

The Final Result

Using the probabilities of getting each of the initial hands in the simple list and then deriving the conditional probabilities of each hand we can derive the chance of getting each winning hand in the payout table after the second deal. In the following table, the prob column is the probability of ending up with the given hand on the left and the prob*payoff column is how much this outcome contributes to our final expected value:

	Payoff		prob	prob*payoff
Royal Flush	800		2.49402E-05	0.019952134
Straight Flush	50		0.000136234	0.00681168
Four Kind	25		0.002362256	0.059056388
Full House	9		0.011533319	0.103799874
Flush	6		0.011116082	0.06669649
Straight	4		0.011317373	0.045269491
Three Kind	3		0.074496303	0.22348891
Two Pairs	2		0.129441133	0.258882265
Pair of Jacks or Better	1		0.211544112	0.211544112
			Mean:	0.995501345
			Standard Deviation:	.044388

Figure 6*

Again, the above probabilities should be viewed as approximations with a margin of error at most +/-.005. As the largest errors (those hands with the greatest number of equivalence classes) are associated small payouts-and the errors offset-our mean should be accurate to within +/.001.

As we see, playing the simple strategy will return approximately 99.5 cents for every dollar bet if a person never makes a mistake.

Using the above probabilities one can then analyze problems such as the mean and standard deviation of returns if a certain number of hands are played

In 10,000 runs where 600 hands are played per run:
Overall expected value is $0.996673166667 Overall standard deviation is $0.183364531321

Here we see the expected return is about what we expected however, the standard deviation is so large that is all but impossible to analyze the returns if a small number of hands are played.

We can also estimate the probability that if a person plays a fixed number of times the probability that they will lose money. As expected, since the expected value is less than 1, the more times a person plays the higher the chance that they will lose money.

In 10,000 runs
How many plays? 100 Percent of times lost money is 0.5905 How many plays? 1000 Percent of times lost money is 0.6456 How many plays? 10000 Percent of times lost money is 0.6939

Problems such as how much money should a person have to start with in order the avoid going bankrupt after so many plays can also be studied.

References:

[1] Video Poker http://en.wikipedia.org/wiki/Video_poker.

[2] Jacks or Better Simple Strategy https://wizardofodds.com/games/video-poker/strategy/jacks-or-better/9-6/simple/