

Explorations & Notes: 1.3 Linear Functions

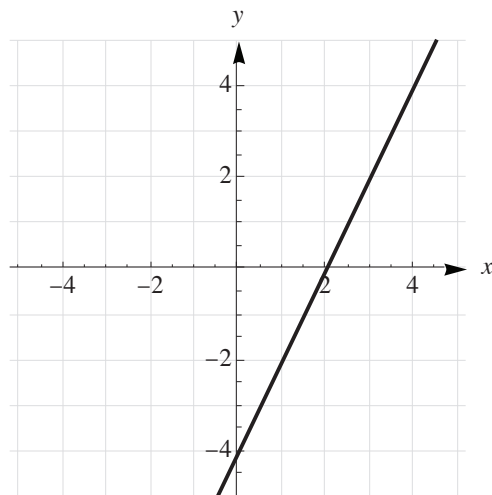
OBJECTIVES

Upon completing this section, you should be able to

- ☐ identify linear and constant functions;
- ☐ explain the difference quotient as a slope;
- ☐ convert between words, points, graphs, and algebraic representations of linear functions;
- ☐ explain when two linear functions are perpendicular;
- ☐ interpret slope as a rate of change;
- ☐ find the average rate of change of a function.

Algebra Check »

1. Find the slope of the line passing through the points $(0, -3)$ and $(2, 5)$.
2. Write the point-slope form for the equation of a line with slope m passing through the point (x_1, y_1) .
3. Find the slope-intercept form for the equation of the line graphed in the figure.



4. Find the y-intercept of the line passing through the points $(-1, 3)$ and $(2, 2)$.
5. Find the average speed, in miles per hour, for a bicyclist who rides 7 miles in 35 minutes.

Linear Functions

DEFINITION Linear and Constant Functions

Any function f that can be written in the form $f(x) = mx + b$ is a **linear function**. The graph of a linear function is a line that is not vertical. A **constant function** has the form $f(x) = b$. The graph of a constant function is a horizontal line.

The defining characteristic of a linear function is that the difference in any two outputs is proportional to the difference in the corresponding inputs. The constant of proportionality is known as the **slope** of the line and is denoted m . Given a linear function with outputs $f(x_1) = y_1$ and $f(x_2) = y_2$, the slope can be written as

$$m = \frac{\overbrace{f(x_2) - f(x_1)}^{\substack{\text{difference} \\ \text{in outputs}}}}{\underbrace{x_2 - x_1}_{\substack{\text{difference} \\ \text{in inputs}}}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2} = \frac{\Delta y}{\Delta x}.$$

Δy is the change in y -values;
 Δx is the change in x -values.

EXAMPLE 1 Finding the slope of a line

Find the slope of the line representing the linear function that satisfies $f(0) = 2.1$ and $f(3) = 0$.

SOLUTION Notes**EXPLORE Figure 1.21**

1. Find values of m for which the line goes up as x increases.
2. Find values of m for which the line goes down as x increases.
3. Find values of m for which the line is horizontal.
4. Graph $f(x) = 1.5x - 3$ in the figure. Select show x and Δx and show grids.
 - a. Adjust Δx so that $\Delta x = 2$, then vary x . Find the corresponding Δy .
 - b. Select $\Delta x = 1$, then vary x . Find the corresponding Δy .
 - c. With $\Delta x = 1$, observe Δy for various values of m and b . Complete the following: If the input increases by one, the output changes by _____.
 - d. While changing Δx , observe Δy for various values of m and b . Complete the following: If the input increases by Δx , the output changes by _____.
5. Explain what is meant by the statement: Linear functions exhibit additive growth.

Difference Quotient

The ratio $\frac{f(x + \Delta x) - f(x)}{\Delta x}$ is a **difference quotient**. It is the familiar slope formula from algebra, $m = \frac{\text{rise}}{\text{run}}$, expressed in function notation. In the case where f is a linear function, the difference quotient is constant for all x and nonzero Δx , and its value m is the slope of the line representing the linear function $f(x) = mx + b$.

Linear Function Forms

Two common forms for the equation of a line are the slope-intercept form and the point-slope form. Each of these has a counterpart using function notation:

	Equation of a Line	Linear Function	
slope-intercept form	$y = m x + b$	$f(x) = m x + b$	slope-intercept form
point-slope form	$y - y_1 = m (x - x_1)$	$f(x) = m (x - h) + k$	transformation form

EXAMPLE 2 Finding a linear function

- Find a transformation form of the linear function f where $f(-2) = 1$ and $f(3) = 4$.
- Find the slope-intercept form of the linear function g whose graph passes through the points $(0, 5)$ and $(4, 0)$.

SOLUTION Notes



EXAMPLE 3 Finding a function from its graph

Find the function f represented by the line in Figure 1.22.

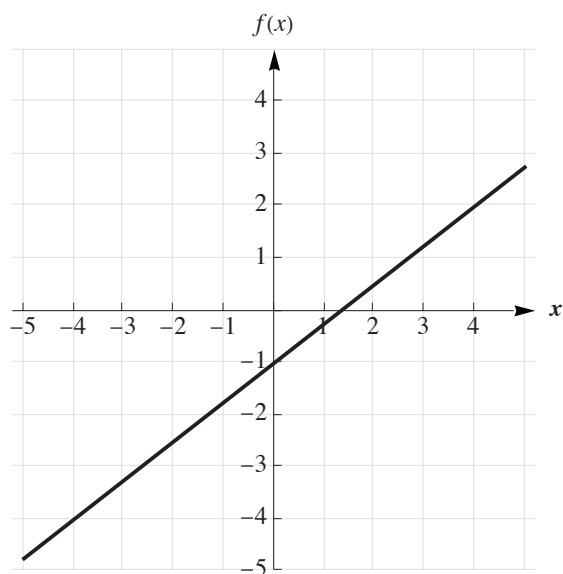
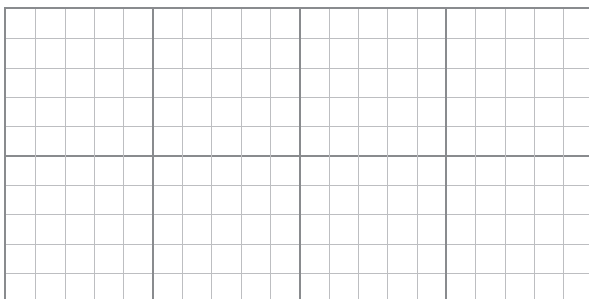


Figure 1.22

EXAMPLE 4 Water in a tank

A tank containing 850 gallons of water is emptied by pumping 2.5 gallons per minute out of the tank.

- Find a function w giving the volume of water in the tank after t minutes of pumping.
- Identify the vertical-axis intercept and explain its significance in the context of the water in the tank.
- Find the time it takes to empty the tank.
- Find the domain and range of w .
- Graph w .

SOLUTION Notes**Graphs**

Two distinct linear functions are said to be **parallel** when their slopes are the same and **perpendicular** when their graphs intersect at a right angle. There are infinitely many functions perpendicular to a given linear function—the lines representing these infinitely many functions are parallel to each other.

EXPLORE Figure 1.24

- Select $h = 3$, $k = -2$, and show $f_{\perp}$. Find m and $m_{\perp}$ so that the graph of $f_{\perp}$ contains the origin.
- Select $h = 3$, $k = 2$, $\Delta x = 1$, show $f_{\perp}$, and show grids. Vary m . Describe the relationship between the *rise* and the *run* on the graph of f and the *rise* and the *run* on the graph of $f_{\perp}$.

DEFINITION Perpendicular Linear Functions

Two linear functions are **perpendicular** if and only if the slopes of their graphs are negative reciprocals of each other. Equivalently, $f(x) = m x + b$ and $g(x) = M x + B$, where $m \neq 0$ and $M \neq 0$, are perpendicular if and only if $m M = -1$.

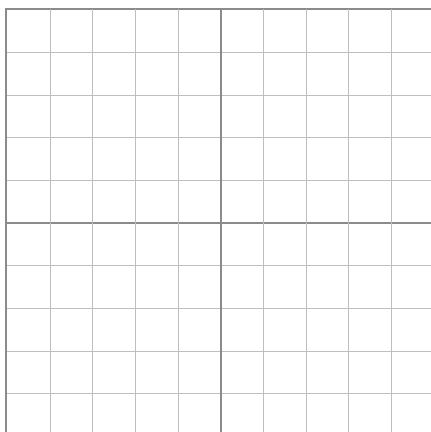
- What must be true about $f(x) = m x + b$ and $g(x) = M x + B$ if f and g are parallel?

9. What must be true about $f(x) = mx + b$ and $g(x) = Mx + B$ if f and g are perpendicular?

EXAMPLE 5 Finding a perpendicular function

Find a linear function perpendicular to $f(x) = \frac{5}{2}x + 1$ that passes through the point $(-4, 3)$.

SOLUTION Notes

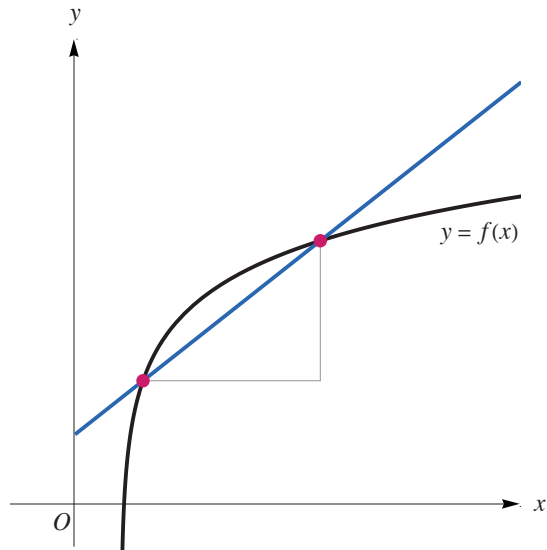


Average Rate of Change

Finding the ratio of the difference in outputs to the difference in inputs can be done for nonlinear functions as well. The ratio, or difference quotient, $\frac{f(x + \Delta x) - f(x)}{\Delta x}$ is called the **average rate of change** and is the slope of the line passing through the points $(x, f(x))$ and $(x + \Delta x, f(x + \Delta x))$. This line, as well as any line passing through two points on the graph of a nonlinear function, is known as a **secant line**.

EXPLORE Figure 1.25

10. Label x , $x + \Delta x$, $f(x)$, and $f(x + \Delta x)$ on the horizontal and vertical axes in the figure below.
11. Add to the figure below the labels Δx and $f(x + \Delta x) - f(x)$.



12. Explain the graphical significance of the quotient $\frac{f(x + \Delta x) - f(x)}{\Delta x}$.
13. Explain why Δx cannot be equal to 0.
14. Explain how the average rate of change, $\frac{f(x + \Delta x) - f(x)}{\Delta x}$, is related to the difference quotient, $\frac{f(x + h) - f(x)}{h}$, introduced in Section 1.1.

EXAMPLE 6 Road trip

You have to drive from Casper, Wyoming, to Fort Collins, Colorado, on Interstate 25. The two cities are 222 miles apart. After traveling for 2.5 hours, you stop in Cheyenne for an hour to have dinner. You then continue on to Fort Collins, arriving a bit more than 4 hours after leaving Casper. A plot of the distance traveled, $d(t)$, is shown in Figure 1.26. The average rate of change in this context is an average speed:

$$\frac{d(t + \Delta t) - d(t)}{\Delta t} = \frac{\text{change in distance}}{\text{change in time}} = \text{average speed}.$$

- Find the average speed for the drive from Casper to Cheyenne.
- Find the average speed for the drive from Casper to Fort Collins.
- Find the average speed for the time period from when you arrived in Cheyenne until the end of the trip.

SOLUTION Notes

Exercises**Getting Started**

- Describe the graphs of functions of the form $f(x) = m x + b$ where $m < 0$ and $b = 0$. Sketch an example.
- Describe the graphs of functions of the form $f(x) = m x + b$ where $m = 0$ and $b < 0$. Sketch an example.
- Write the slope-intercept form of a linear function.
- Write the transformation form of a linear function.
- Find the slope of any line that is perpendicular to the graph of $f(x) = -\frac{2}{5}x + 1$.
- Given two linear functions, how can you determine whether or not they are perpendicular?
- What is a secant line?
- Use Figure 1.25 to explain why the slope of a secant line is $\frac{f(x + \Delta x) - f(x)}{\Delta x}$.
- Explain why the average rate of change $\frac{f(x + \Delta x) - f(x)}{\Delta x}$ is constant for a linear function.
- The maximum heart rate for a healthy adult during physical activity can be estimated by subtracting the individual's age from 220. Find a function m giving the estimated maximum heart rate of an individual t years old. (*Source:* Centers for Disease Control and Prevention, U.S. Department of Health & Human Services)

Practice

11–14. Slope of a line Find the slope of the line representing the linear function whose graph passes through each pair of points.

- (0, 6) and (2, 2)
- (4, 0) and (8, 2)
- (4, 6) and (2, 20)
- (-5, 2) and (10, 32)

15–20. Transformation form Find a transformation form of the linear function f whose graph passes through the following pairs of points.

15. $(-2, -1)$ and $(10, 12)$

16. $(4, -3)$ and $(-5, 2)$

17. $\left(\frac{1}{2}, \frac{3}{4}\right)$ and $\left(3, \frac{9}{2}\right)$

18. $\left(\frac{4}{3}, \frac{1}{5}\right)$ and $\left(-\frac{5}{3}, 2\right)$

19. $(0.38, 1.94)$ and $(0.75, 3.45)$

20. $(21.4, 32.5)$ and $(59.6, 71.2)$

21–24. Slope-intercept form Find the slope-intercept form of the linear function g whose graph passes through the following pairs of points.

21. $(-3, 0)$ and $(0, 10)$

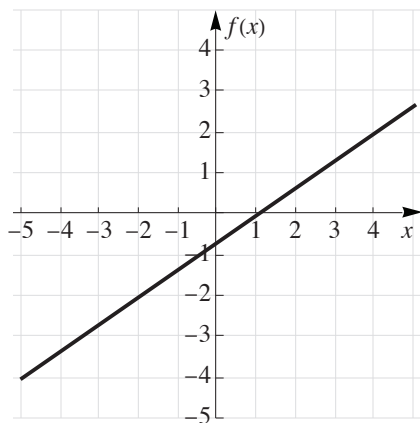
22. $(0, -2)$ and $(1, 5)$

23. $(2, 3)$ and $(5, 1)$

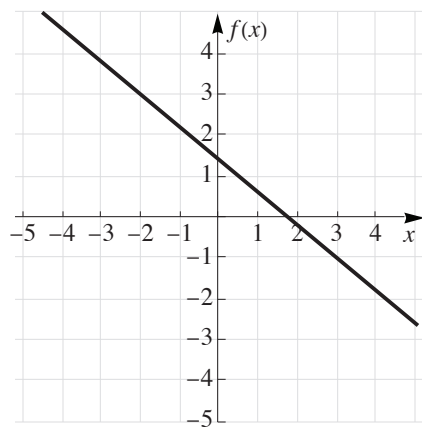
24. $(-4, 3)$ and $(1, -6)$

25–30. Function from a graph Find the function represented by each of the following lines.

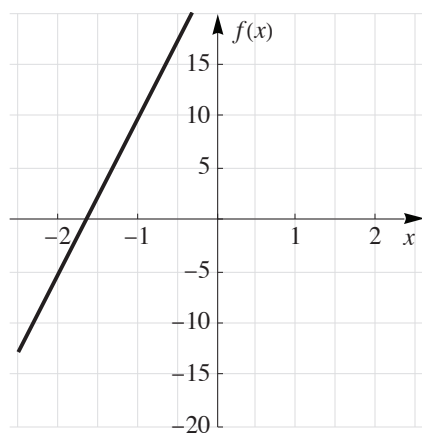
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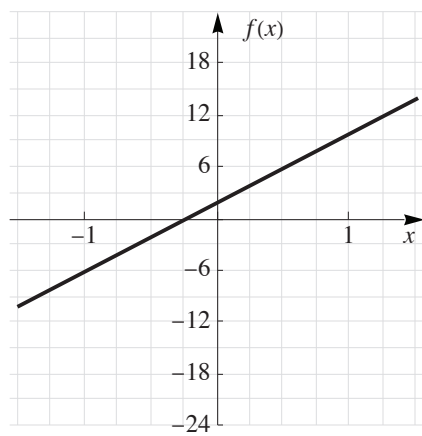
26.



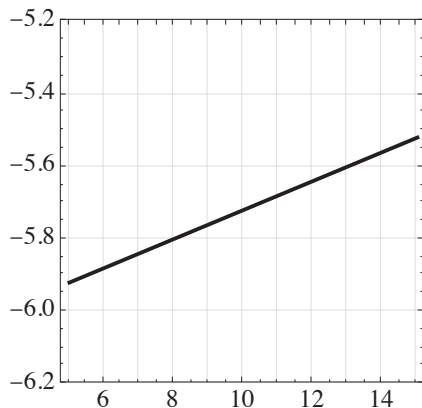
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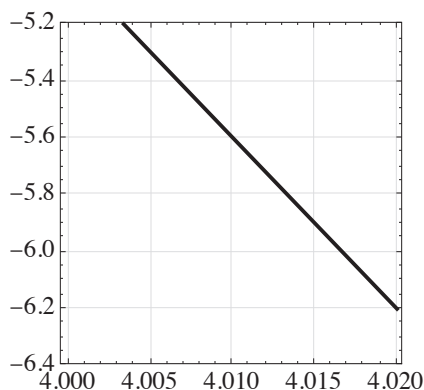
28.



29.



30.



- 31. Filling a hot tub** A hot tub can contain at most 785 gallons of water. A hose delivering 4.2 gallons per minute is used to fill an empty hot tub.
- Find a function w giving the volume of water in the hot tub after t minutes.
 - Find the domain and range of w .
 - Graph w .
- 32. Emptying a hot tub** A hot tub containing 785 gallons of water is emptied using a pump rated at 3.5 gallons per minute.
- Find a function w giving the volume of water remaining in the hot tub after t minutes of pumping.
 - Find the domain and range of w .
 - Graph w .
 - Identify the horizontal-axis intercept and explain its significance in the context of the water in the hot tub.
- 33. Fueling a yacht** A 70-foot yacht can carry 2500 gallons of diesel onboard and the fuel tanks must be filled before the yacht heads out to sea. The tanks contain 400 gallons of fuel when the yacht arrives at the refueling dock. Diesel is pumped into the tanks at 40 gallons per minute.
- Find a function d giving the total volume of diesel in the tanks after t minutes of pumping fuel.
 - Find the time it takes to fill the tanks.
 - Find the domain and range of d .
 - Graph d .
 - Let $p(t)$ be the total price of the diesel pumped after t minutes. Find $p(t)$ assuming the price of diesel at the fuel dock is \$4.89 per gallon.

- 34. Fueling a semi** An 18-wheeler carries 150 gallons of diesel and the fuel tanks must be filled before the driver leaves on a cross-country trip. The tanks contain 20 gallons of fuel when the semi arrives at a gas station. Diesel is pumped into the tanks at 18 gallons per minute.
- Find a function d giving the volume of diesel in the tanks after t minutes of pumping fuel.
 - Find the time it takes to fill the truck's diesel tanks.
 - Find the domain and range of d .
 - Graph d .
 - Let $p(t)$ be the total price of the diesel pumped after t minutes. Find $p(t)$ assuming the price of diesel at the station is \$4.39 per gallon.

35–40. Perpendicular functions Find a linear function $f_{\perp}$ perpendicular to f and passing through the given point.

35. $f(x) = 3x + 2$; $(0, -5)$

36. $f(x) = 2 - 4x$; $(0, -3)$

37. $f(x) = -\frac{3}{2}x - \frac{4}{5}$; $(2, 4)$

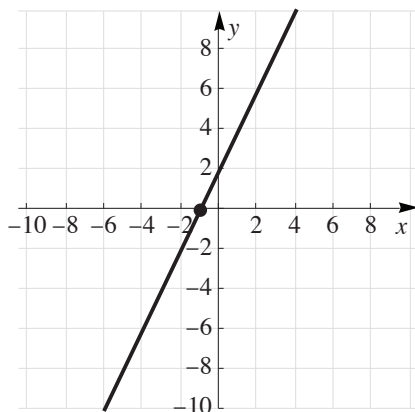
38. $f(x) = \frac{7}{3}x + 1$; $(-6, 2)$

39. $f(x) = 1.34x + 19.76$; $(52.3, 87.1)$

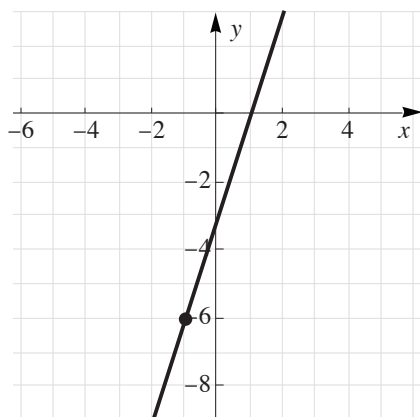
40. $f(x) = 101.28 - 0.29x$; $(29.4, 153.2)$

41–42. Graphing perpendicular functions Find the linear function $f_{\perp}$ perpendicular to f and passing through the plotted point, and graph $f_{\perp}$.

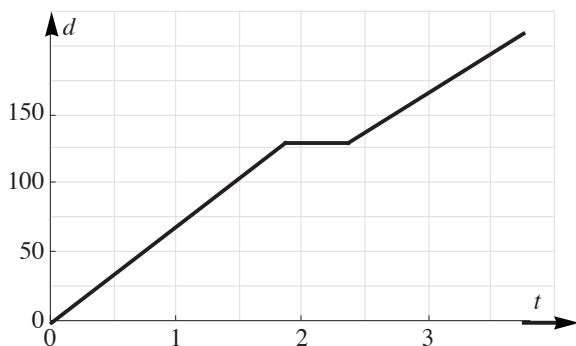
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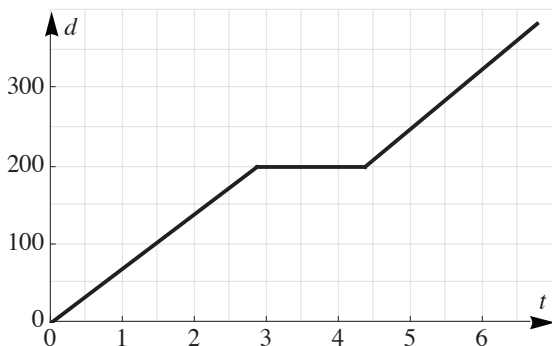
42.



43. You have to drive 213 miles from Nashville, Tennessee, to Memphis on Interstate 40. After traveling for 130 miles at 70 miles per hour, you stop in Jackson for a half-hour to eat lunch. You then continue on to Memphis, arriving 3 hours and 45 minutes after leaving Nashville.



- Find the average speed for the drive from Jackson to Memphis.
 - Find the average speed for the drive from Nashville to Memphis.
 - Find the average speed for the time period from when you left Nashville until you left Jackson.
44. You have to drive 384 miles from Los Angeles, California, to Sacramento on Interstate 5. After traveling for 200 miles at 70 miles per hour, you stop at a rest area for 1.5 hours. You then continue on to Sacramento, arriving 6 hours and 45 minutes after leaving Los Angeles.

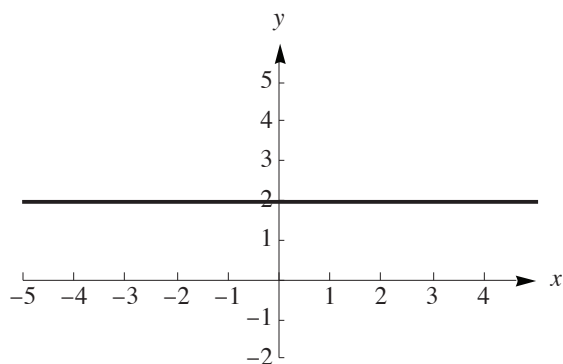


- Find the average speed for the drive from the rest area to Sacramento.
 - Find the average speed for the drive from Los Angeles to Sacramento.
 - Find the average speed for the time period from when you left Los Angeles until you left the rest area.
45. **Explain why or why not** Determine whether the following statements are true, and give an explanation or counterexample.

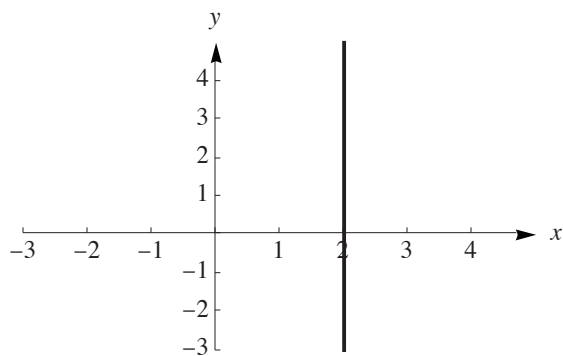
- a. The line $y = -3x + 6$ has an x -intercept $(4, 0)$.
- b. The line described by the equation $2y = 3x + 8$ has a y -intercept $(0, 8)$.
- c. The lines $y = 4x - b$ and $y = -\frac{1}{4}x + c$ are perpendicular for all values of b and c .
- d. The lines $y = 2$ and $x = -2$ are perpendicular.

46. Horizontal and vertical lines Explain why the following lines do not represent linear functions.

a.



b.



47. Linear behavior Determine whether the values in the tables could be linear function values.

a.

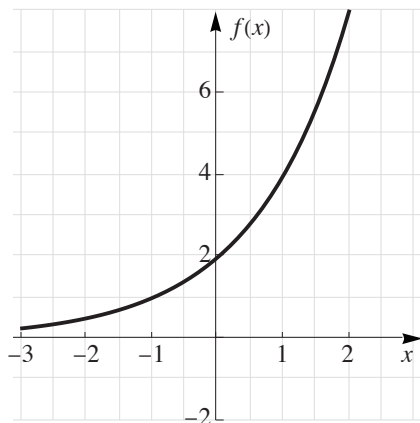
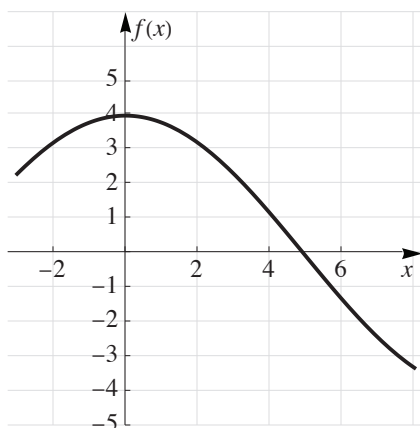
x	-2	0	2	4
$f(x)$	6	1	-4	-9

b.

x	-1	0	2	4
$f(x)$	1	5	25	125

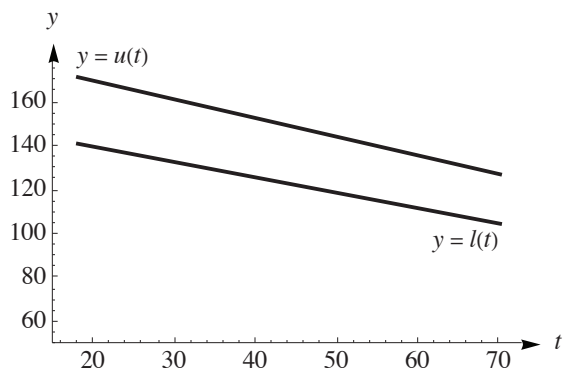
48–51. Average rate of change Find the average rate of change of the following functions on the given intervals.

48. $[-1, 1]$

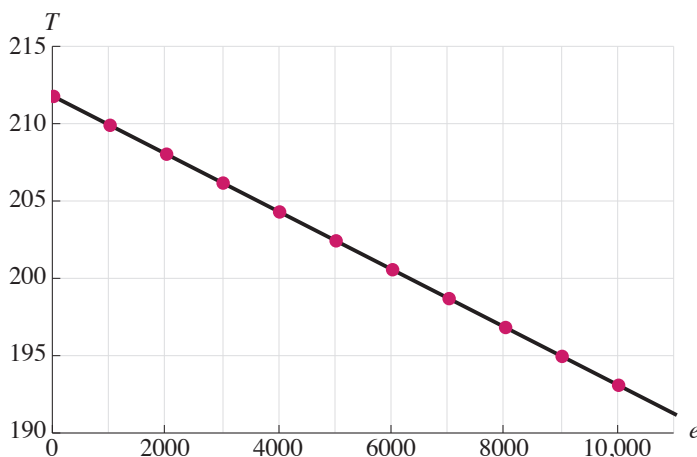
49. $[0, 5]$ 50. $f(x) = \frac{1}{x}; [1, 10]$ 51. $f(x) = \sqrt{x - 2}; [2, 6]$

52. **Vigorous physical activity** The heart rate of a healthy adult exercising vigorously should remain between 70% and 85% of their maximum heart rate. The maximum heart rate for an individual, measured in beats per minute, is estimated by subtracting his or her age from 220. (*Source*: Centers for Disease Control and Prevention, U.S. Department of Health & Human Services)

- Find a function l giving the lower limit for the heart rate of a t -year-old adult exercising vigorously.
- Find a function u giving the upper limit for the heart rate of a t -year-old adult exercising vigorously.
- Graph l and u for $18 \leq t \leq 70$.



- 53. Commission on sales** Suppose a salesperson working an 8-hour shift sold \$300 of merchandise and earned \$95 for the day. During a different 8-hour shift, the salesperson sold \$700 of merchandise and earned \$115 for the day. Assume the relationship between sales and daily wage is linear.
- Find the linear function $y = w(s)$ that gives the salesperson's daily wage for s dollars of merchandise sold.
 - Find the percent commission the salesperson earns on sales.
 - Find the minimum wage the salesperson can earn for an 8-hour shift.
- 54. Lease versus purchase** A car dealer offers you a purchase option and a lease option on a new car. You can buy the car outright for \$22,000 or you can lease it for a start-up fee of \$2000 plus monthly payments of \$300.
- Find the linear function $y = f(m)$ that gives the total amount you will have paid on the lease option after m months.
 - Graph the lease function in part (a) for $0 \leq m \leq 48$.
 - How much will you have you paid on the lease after 48 months?
 - At the end of a 48-month lease period, the car has a residual value of \$10,000, which is the amount that you could pay the dealer to buy the car. Given the result of part (c) and assuming no other costs, should you buy or lease the car? Explain.
- 55. Boiling point of water** At sea level, water boils at a temperature of 212°F . However, the boiling point of water decreases linearly with elevation according to the graph shown below. We let T represent the boiling point temperature in degrees Fahrenheit and e represent elevation above sea level in feet.

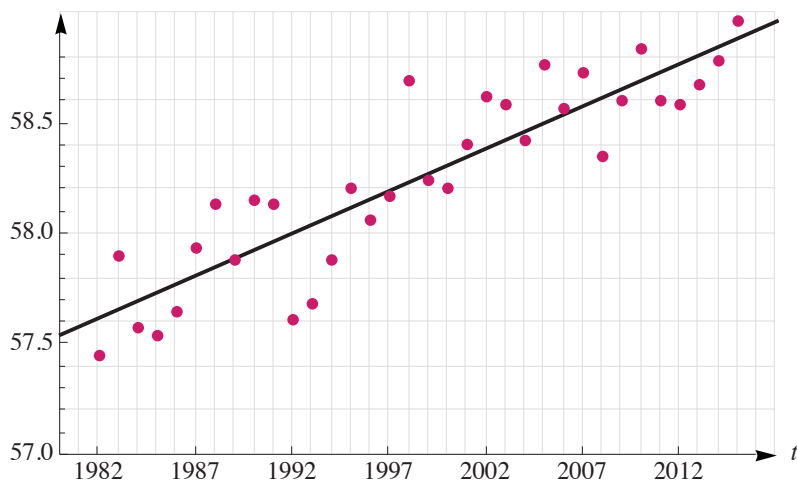


- Using the data points $(0, 212)$ and $(10,000, 193)$, find the slope of the line that fits the boiling point data.
- Find the equation of the line that fits the boiling point data.
- Assuming the same relationship between boiling point and elevation is valid at higher elevations, what is the boiling point of water at 20,000 feet?

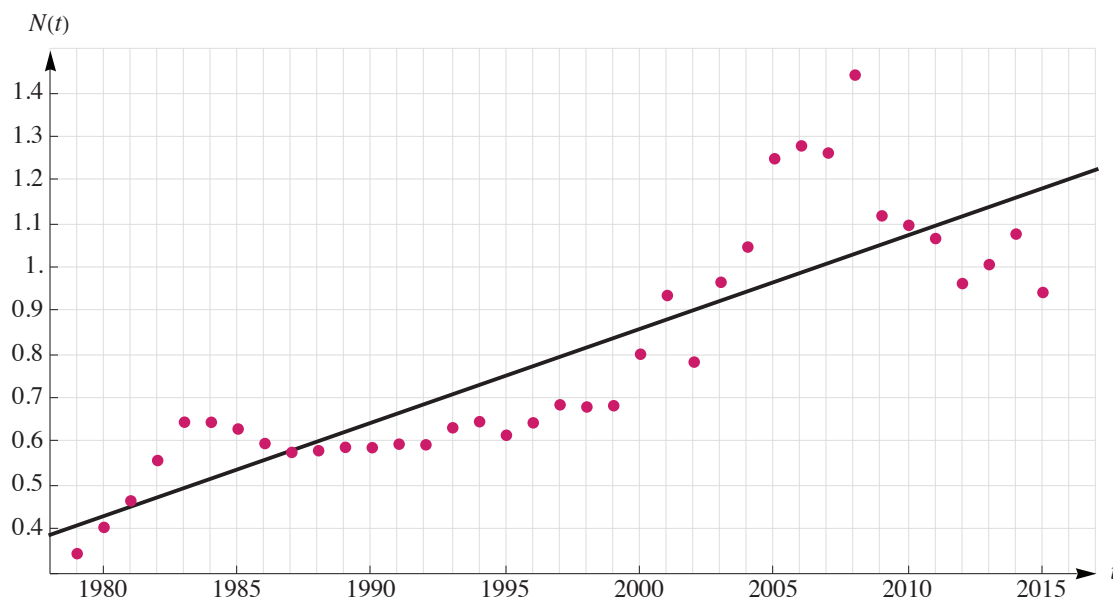
- d. Complete the following sentence: For every 1000 feet of elevation gain, the boiling point of water decreases by approximately _____ degrees.

- 56. Printing and selling t-shirts** Many manufacturing processes involve a setup cost and a per-item cost. For example, suppose that printing a batch of t-shirts involves a one-time setup cost of \$120 after which the cost of the shirt and printing is \$7 per shirt. Suppose also that the shirts can be sold at a retail price of \$11.
- Find the linear function $C = f(n)$ that gives the cost in dollars of printing n shirts.
 - Find the linear function $R = g(n)$ that gives the revenue generated by selling n shirts.
 - The profit in making and selling n shirts is revenue minus costs, or $P(n) = R(n) - C(n)$. What is the profit if 40 shirts are printed and sold?
 - Graph the cost, revenue, and profit functions on the same set of axes.
 - What is the fewest number of shirts that can be printed and sold to make a (positive) profit?
- 57. Linear demand** The demand for an item is the amount of that item that can be sold at a given price. The owner of a small gas station examines his sales records over the past year and estimates that if gas is priced at \$3.80 per gallon, then the demand is 700 gallons of gas per day. If the price is \$4.00 per gallon, then the demand is 600 gallons per day. Let $D = f(p)$ be the number of gallons of gas that can be sold at a price p and assume that the demand function is linear.
- Find the equation of the line that passes through the points $(p, D) = (3.80, 700)$ and $(p, D) = (4.00, 600)$.
 - Graph the demand function in part (a).
 - How many gallons can be sold at a price of \$4.20? at a price of \$3.50?
 - According to this model, what is the price above which the demand is zero?
 - If the owner sets the price at p , then he expects to sell $D(p)$ gallons at p dollars per gallon, which gives him revenue of $R(p) = p D(p)$ dollars. Graph the revenue function.
 - At what price can the owner expect to maximize his revenue?
- 58. Mean surface temperature** The graph shows data for the annual mean surface temperature of Earth for the years from 1982 to 2015. The graph also shows the line that gives the best fit to the data. (Source: *GISS Surface Temperature Analysis*, National Aeronautics and Space Administration)

Surface temp ($^{\circ}\text{F}$)



- Find the slope of the best-fit line shown in the graph.
 - Explain what the slope of the line describes for this data.
 - Find the linear function T whose graph is the best-fit line.
- 59. Price of natural gas** The graph shows data for the average yearly price in dollars of 1 therm of natural gas for the years from 1978 to 2015 in the United States. The graph also shows the line that gives the best fit to the data. (Source: Bureau of Labor Statistics, U.S. Department of Labor)



- Find the average rate of change for the price of natural gas from the year 2003 to the year 2012.
- Find the average rate of change for the price of natural gas from 2008 to 2015.
- Find the slope of the best-fit line shown in the graph.
- Explain what the slope of the line describes for this data.
- Find the linear function N whose graph is the best-fit line.

Explorations and Challenges

- 60. Natural gas versus electricity** The energy in 1 therm of natural gas is equivalent to the energy in 29.3 kilowatt-hours of electricity. When natural gas is burned to generate heat for a home, not all of the energy in the natural gas is converted to heat energy to warm the home. The efficiency rating of a natural gas furnace is the percent of the energy in natural gas that is converted to heat energy and used to warm a home. The average price in the United States for 1 kilowatt-hour of electricity in 2015 was \$0.138 and the average price for 1 therm of natural gas in the same year was \$0.945.

 - Assume a furnace burns 1 therm of natural gas and then uses the converted heat energy to warm a home. Find a function f that takes as input the efficiency rating r of the furnace, and gives as output the cost of the electrical energy equal to the heat energy extracted from the natural gas.
 - Graph f for $0 < r < 1$.
 - For what efficiency ratings r is natural gas a more economical source of energy than electricity for heating a home?
- 61. Difference quotients** Consider the linear function $f(x) = mx + b$, where m and b are real numbers, $m \neq 0$. Show that $\frac{f(x+h) - f(x)}{h}$ is constant for all values of x and nonzero values of h . What is the value of the constant?
- 62. Lines and parabolas** Let $f(x) = x^2$ and let $P(1, 1)$ be a point on the graph of f .

 - Find the equation of the line L that passes through P with slope $m \neq 0$.
 - Graph f and the lines L with slopes $m = 1, 1.5, 2, 2.5$.
 - Which of the lines in part (b) intersect the graph of f exactly once?