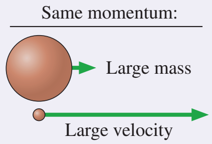
11 Impulse and Momentum

An exploding firework is a dramatic event. Nonetheless, the explosion obeys some simple laws of physics.

IN THIS CHAPTER, you will learn to use the concepts of impulse and momentum.

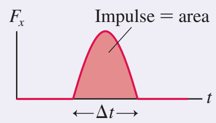
What is momentum?

An object's momentum is the product of its mass and velocity. An object can have a large momentum by having a large mass or a large velocity. Momentum is a vector, and it is especially important to pay attention to the *signs* of the components of momentum.



What is impulse?

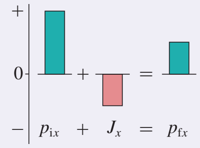
A force of short duration is an impulsive force. The impulse J_x that this force delivers to an object is the area under the force-versus-time graph. For time-dependent forces, impulse and momentum are often more useful than Newton's laws.



How are impulse and momentum related?

Working with momentum is similar to working with energy. It's important to clearly define the system. The momentum principle says that a system's momentum changes when an impulse is delivered:

$$\Delta p_x = J_x$$

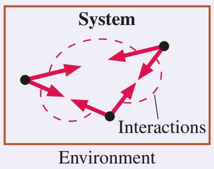


A momentum bar chart, similar to an energy bar chart, shows this principle graphically.

« LOOKING BACK Section 9.1 Energy overview

Is momentum conserved?

The total momentum of an isolated system is conserved. The particles of an isolated system interact with each other but not with the environment. Regardless of how intense the interactions are, the final momentum equals the initial momentum.

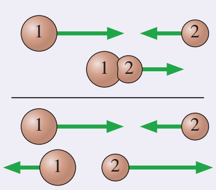


« LOOKING BACK Section 10.4 Energy conservation

How does momentum apply to collisions?

One important application of momentum conservation is the study of collisions.

- In a totally inelastic collision, the objects stick together. Momentum is conserved.
- In a perfectly elastic collision, the objects bounce apart. Both momentum and energy are conserved.

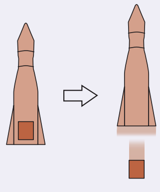


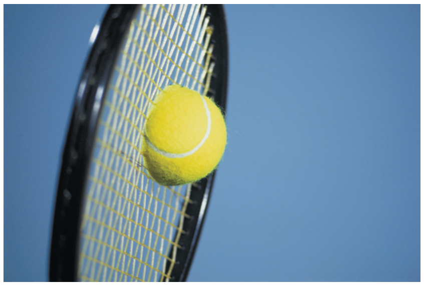
Where else is momentum used?

This chapter looks at two other important applications of momentum conservation:

- An explosion is a short interaction that drives two or more objects apart.

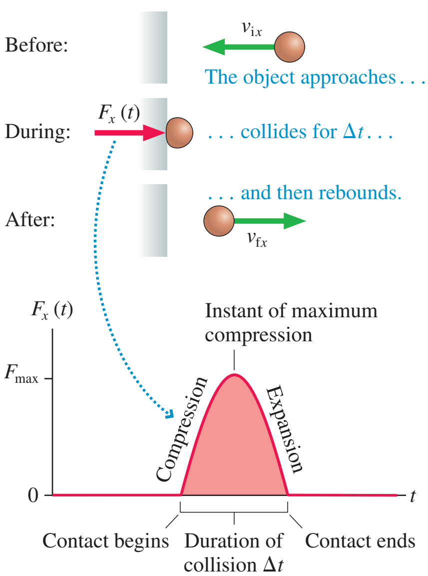
- In rocket propulsion the object's mass is changing continuously.





A tennis ball collides with a racket. Notice that the left side of the ball is flattened.

FIGURE 11.1 A collision.



11.1 Momentum and Impulse

A **collision** is a short-duration interaction between two objects. The collision between a tennis ball and a racket, or a baseball and a bat, may seem instantaneous to your eye, but that is a limitation of your perception. A high-speed photograph reveals that the side of the ball is significantly flattened during the collision. It takes time to compress the ball, and more time for the ball to re-expand as it leaves the racket or bat.

The duration of a collision depends on the materials from which the objects are made, but 1 to 10 ms (0.001 to 0.010 s) is fairly typical. This is the time during which the two objects are in contact with each other. The harder the objects, the shorter the contact time. A collision between two steel balls lasts less than 200 microseconds.

FIGURE 11.1 shows an object colliding with a wall. The object approaches with an initial horizontal velocity v_{ix}, experiences a force of duration Δt, and leaves with final velocity v_{fx}. Notice that the object, as in the photo above, *deforms* during the collision. A particle cannot be deformed, so we cannot model colliding objects as particles. Instead, we model a colliding object as an *elastic object* that compresses and then expands, much like a spring. Indeed, that's exactly what happens during a collision at the microscopic level: Molecular bonds compress, store elastic potential energy, then transform some or all of that potential energy back into the kinetic energy of the rebounding object. We'll examine the energy issues of collisions later in this chapter.

The force of a collision is usually very large in comparison to other forces exerted on the object. A large force exerted for a small interval of time is called an **impulsive force.** The graph of Figure 11.1 shows how a typical impulsive force behaves, rapidly growing to a maximum at the instant of maximum compression, then decreasing back to zero. The force is zero before contact begins and after contact ends. Because an impulsive force is a function of time, we will write it as $F_x(t)$.

NOTE Both v_x and F_x are components of vectors and thus have *signs* indicating which way the vectors point.

We can use Newton's second law to find how the object's velocity changes as a result of the collision. Acceleration in one dimension is $a_x = dv_x/dt$, so the second law is

$$ma_x = m\frac{dv_x}{dt} = F_x(t)$$

After multiplying both sides by dt, we can write the second law as

$$m\,dv_x = F_x(t)\,dt \tag{11.1}$$

The force is nonzero only during an interval of time from t_i to $t_f = t_i + \Delta t$, so let's integrate Equation 11.1 over this interval. The velocity changes from v_{ix} to v_{fx} during the collision; thus

$$m\int_{v_i}^{v_f} dv_x = mv_{fx} - mv_{ix} = \int_{t_i}^{t_f} F_x(t)\,dt \tag{11.2}$$

We need some new tools to help us make sense of Equation 11.2.

Momentum

The product of a particle's mass and velocity is called the *momentum* of the particle:

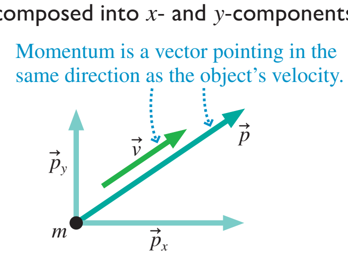
$$\textbf{momentum} = \vec{p} \equiv m\vec{v} \tag{11.3}$$

Momentum, like velocity, is a vector. The units of momentum are kg m/s. The plural of "momentum" is "momenta," from its Latin origin.

The momentum vector $\vec{p}$ is parallel to the velocity vector $\vec{v}$. **FIGURE 11.2** shows that $\vec{p}$, like any vector, can be decomposed into x- and y-components. Equation 11.3, which is a vector equation, is a shorthand way to write the simultaneous equations

$$p_x = mv_x$$
$$p_y = mv_y$$

FIGURE 11.2 The momentum $\vec{p}$ can be decomposed into x- and y-components.



NOTE One of the most common errors in momentum problems is a failure to use the appropriate signs. The momentum component p_x has the same sign as v_x. Momentum is *negative* for a particle moving to the left (on the x-axis) or down (on the y-axis).

An object can have a large momentum by having either a small mass but a large velocity or a small velocity but a large mass. For example, a 5.5 kg (12 lb) bowling ball rolling at a modest 2 m/s has momentum of magnitude $p = (5.5\text{ kg})(2\text{ m/s}) = 11\text{ kg m/s}$. This is almost exactly the same momentum as a 9 g bullet fired from a high-speed rifle at 1200 m/s.

Newton actually formulated his second law in terms of momentum rather than acceleration:

$$\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt} = \frac{d(m\vec{v})}{dt} = \frac{d\vec{p}}{dt} \tag{11.4}$$

This statement of the second law, saying that **force is the rate of change of momentum,** is more general than our earlier version $\vec{F} = m\vec{a}$. It allows for the possibility that the mass of the object might change, such as a rocket that is losing mass as it burns fuel.

Returning to Equation 11.2, you can see that mv_{ix} and mv_{fx} are p_{ix} and p_{fx}, the x-component of the particle's momentum before and after the collision. Further, $p_{fx} - p_{ix}$ is Δp_x, the *change* in the particle's momentum. In terms of momentum, Equation 11.2 is

$$\Delta p_x = p_{fx} - p_{ix} = \int_{t_i}^{t_f} F_x(t)\, dt \tag{11.5}$$

Now we need to examine the right-hand side of Equation 11.5.

Impulse

Equation 11.5 tells us that the particle's change in momentum is related to the time integral of the force. Let's define a quantity J_x called the *impulse* to be

$$\textbf{impulse} = J_x \equiv \int_{t_i}^{t_f} F_x(t)\, dt = \text{area under the } F_x(t) \text{ curve between } t_i \text{ and } t_f \tag{11.6}$$

Strictly speaking, impulse has units of N s, but you should be able to show that N s are equivalent to kg m/s, the units of momentum.

The interpretation of the integral in Equation 11.6 as an area under a curve is especially important. **FIGURE 11.3a** portrays the impulse graphically. Because the force changes in a complicated way during a collision, it is often useful to describe the collision in terms of an *average* force F_{avg}. As **FIGURE 11.3b** shows, F_{avg} is the height of a rectangle that has the same area, and thus the same impulse, as the real force curve. The impulse exerted during the collision is

$$J_x = F_{avg}\, \Delta t \tag{11.7}$$

Equation 11.2, which we found by integrating Newton's second law, can now be rewritten in terms of impulse and momentum as

$$\Delta p_x = J_x \qquad \text{(momentum principle)} \tag{11.8}$$

This result, called the **momentum principle,** says that **an impulse delivered to an object causes the object's momentum to change.** The momentum p_{fx} "after" an interaction, such as a collision or an explosion, is equal to the momentum p_{ix} "before" the interaction *plus* the impulse that arises from the interaction:

$$p_{fx} = p_{ix} + J_x \tag{11.9}$$

FIGURE 11.3 Looking at the impulse graphically.

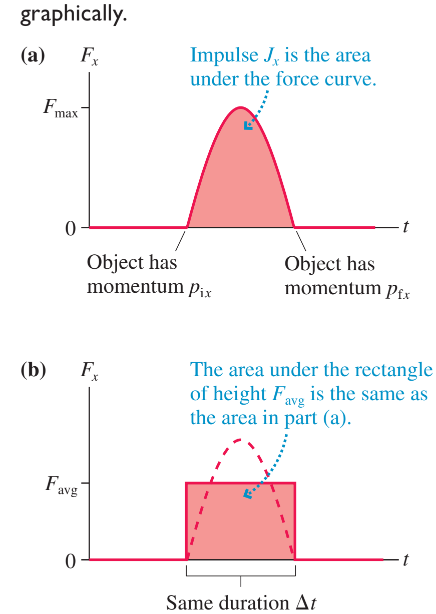


FIGURE 11.4 The momentum principle helps us understand a rubber ball bouncing off a wall.

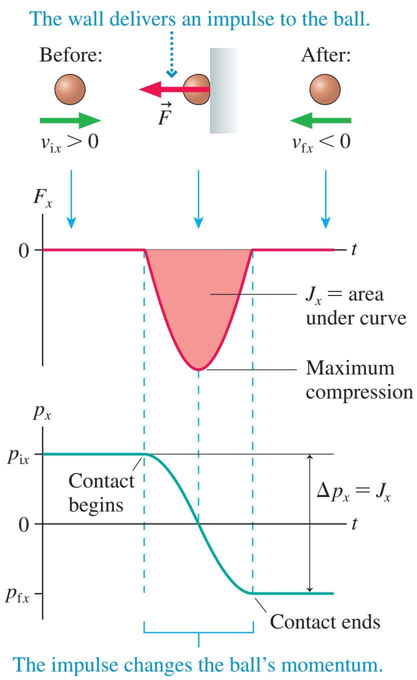


FIGURE 11.4 illustrates the momentum principle for a rubber ball bouncing off a wall. Notice the signs; they are very important. The ball is initially traveling toward the right, so v_{ix} and p_{ix} are positive. After the bounce, v_{fx} and p_{fx} are negative. The force *on the ball* is toward the left, so F_x is also negative. The graphs show how the force and the momentum change with time.

Although the interaction is very complex, the impulse—the area under the force graph—is all we need to know to find the ball's velocity as it rebounds from the wall. The final momentum is

$$p_{fx} = p_{ix} + J_x = p_{ix} + \text{area under the force curve} \tag{11.10}$$

and the final velocity is $v_{fx} = p_{fx}/m$. In this example, the area has a negative value.

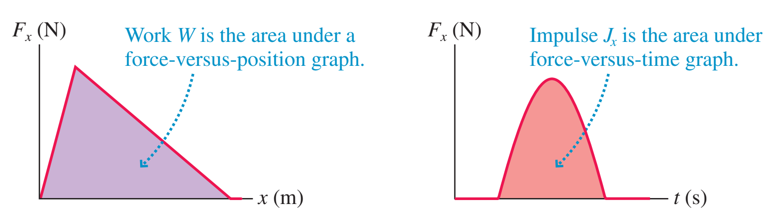
An Analogy with the Energy Principle

You've probably noticed that there is a similarity between the momentum principle and the energy principle of Chapters 9 and 10. For a system of one object acted on by a force:

$$\begin{aligned} &\text{energy principle:} \qquad \Delta K = W = \int_{x_i}^{x_f} F_x\,dx \\ &\text{momentum principle:} \qquad \Delta p_x = J_x = \int_{t_i}^{t_f} F_x\,dt \end{aligned} \tag{11.11}$$

In both cases, a force acting on an object changes the state of the system. If the force acts over the spatial interval from x_i to x_f, it does *work* that changes the object's kinetic energy. If the force acts over a time interval from t_i to t_f, it delivers an *impulse* that changes the object's momentum. **FIGURE 11.5** shows that the geometric interpretation of work as the area under the F-versus-x graph parallels an interpretation of impulse as the area under the F-versus-t graph.

FIGURE 11.5 Impulse and work are both the area under a force curve, but it's very important to know what the horizontal axis is.



This does not mean that a force *either* creates an impulse *or* does work but does not do both. Quite the contrary. A force acting on a particle *both* creates an impulse *and* does work, changing both the momentum and the kinetic energy of the particle. Whether you use the energy principle or the momentum principle depends on the question you are trying to answer.

In fact, we can express the kinetic energy in terms of momentum as

$$K = \tfrac{1}{2}mv^2 = \frac{(mv)^2}{2m} = \frac{p^2}{2m} \tag{11.12}$$

You cannot change a particle's kinetic energy without also changing its momentum.

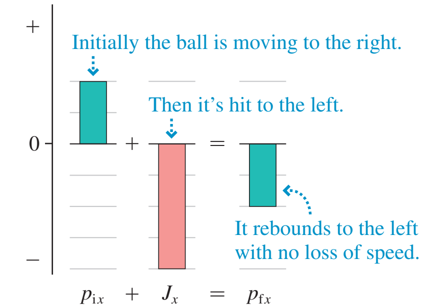
Momentum Bar Charts

The momentum principle tells us that **impulse transfers momentum to an object.** If an object has 2 kg m/s of momentum, a 1 kg m/s impulse delivered to the object increases its momentum to 3 kg m/s. That is, $p_{fx} = p_{ix} + J_x$.

Just as we did with energy, we can represent this "momentum accounting" with a **momentum bar chart.** For example, the bar chart of **FIGURE 11.6** represents the ball colliding with a wall in Figure 11.4. Momentum bar charts are a tool for visualizing an interaction.

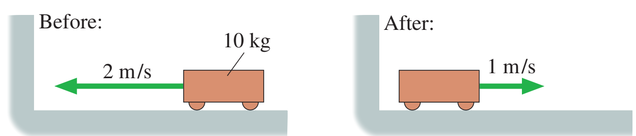
NOTE The vertical scale of a momentum bar chart has no numbers; it can be adjusted to match any problem. However, be sure that all bars in a given problem use a consistent scale.

FIGURE 11.6 A momentum bar chart.



STOP TO THINK 11.1 The cart's change of momentum is

a. −30 kg m/s
b. −20 kg m/s
c. 0 kg m/s
d. 10 kg m/s
e. 20 kg m/s
f. 30 kg m/s



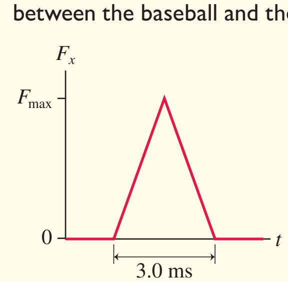
Solving Impulse and Momentum Problems

Impulse and momentum problems, like energy problems, relate the situation before an interaction to the situation afterward. Consequently, the *before-and-after pictorial representation* remains our primary visualization tool. Let's look at an example.

EXAMPLE 11.1 Hitting a baseball

A 150 g baseball is thrown with a speed of 20 m/s. It is hit straight back toward the pitcher at a speed of 40 m/s. The interaction force between the ball and the bat is shown in **FIGURE 11.7**. What *maximum* force F_{max} does the bat exert on the ball? What is the *average* force of the bat on the ball?

FIGURE 11.7 The interaction force between the baseball and the bat.



MODEL Model the baseball as an elastic object and the interaction as a collision.

VISUALIZE **FIGURE 11.8** is a before-and-after pictorial representation. Because F_x is positive (a force to the right), we know the ball was initially moving toward the left and is hit back toward the right. Thus we converted the statements about *speeds* into information about *velocities,* with v_{ix} negative.

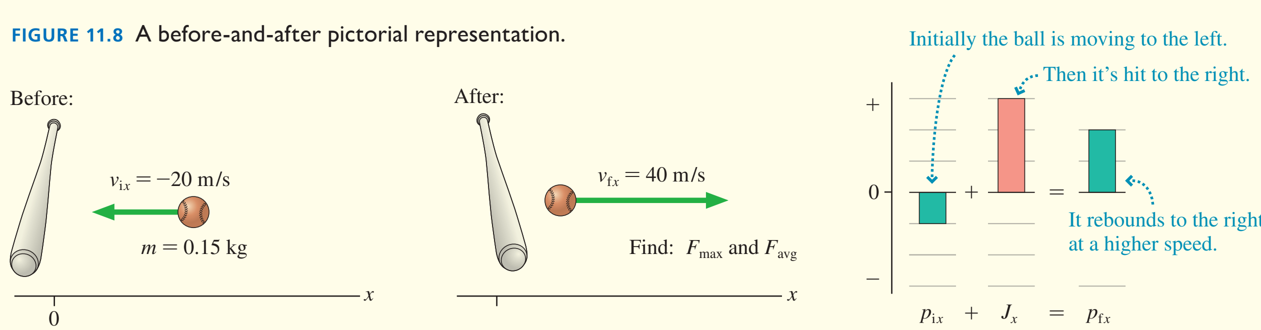
SOLVE The momentum principle is

$$\Delta p_x = J_x = \text{area under the force curve}$$

We know the velocities before and after the collision, so we can calculate the ball's momenta:

$$p_{ix} = mv_{ix} = (0.15\text{ kg})(-20\text{ m/s}) = -3.0\text{ kg m/s}$$
$$p_{fx} = mv_{fx} = (0.15\text{ kg})(40\text{ m/s}) = 6.0\text{ kg m/s}$$

FIGURE 11.8 A before-and-after pictorial representation.



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Thus the *change* in momentum is

$$\Delta p_x = p_{fx} - p_{ix} = 9.0 \text{ kg m/s}$$

The force curve is a triangle with height F_{max} and width 3.0 ms. The area under the curve is

$$J_x = \text{area} = \tfrac{1}{2}(F_{max})(0.0030 \text{ s}) = (F_{max})(0.0015 \text{ s})$$

Using this information in the momentum principle, we have

$$9.0 \text{ kg m/s} = (F_{max})(0.0015 \text{ s})$$

Thus the *maximum* force is

$$F_{max} = \frac{9.0 \text{ kg m/s}}{0.0015 \text{ s}} = 6000 \text{ N}$$

The *average* force, which depends on the collision duration $\Delta t = 0.0030$ s, has the smaller value:

$$F_{avg} = \frac{J_x}{\Delta t} = \frac{\Delta p_x}{\Delta t} = \frac{9.0 \text{ kg m/s}}{0.0030 \text{ s}} = 3000 \text{ N}$$

ASSESS F_{max} is a large force, but quite typical of the impulsive forces during collisions. The main thing to focus on is our new perspective: An impulse changes the momentum of an object.

Other forces often act on an object during a collision or other brief interaction. In Example 11.1, for instance, the baseball is also acted on by gravity. Usually these other forces are *much* smaller than the interaction forces. The 1.5 N weight of the ball is vastly less than the 6000 N force of the bat on the ball. We can reasonably neglect these small forces *during* the brief time of the impulsive force by using what is called the **impulse approximation.**

When we use the impulse approximation, p_{ix} and p_{fx} (and v_{ix} and v_{fx}) are the momenta (and velocities) *immediately* before and *immediately* after the collision. For example, the velocities in Example 11.1 are those of the ball just before and after it collides with the bat. We could then do a follow-up problem, including gravity and drag, to find the ball's speed a second later as the second baseman catches it. We'll look at some two-part examples later in the chapter.

STOP TO THINK 11.2 A 10 g rubber ball and a 10 g clay ball are thrown at a wall with equal speeds. The rubber ball bounces, the clay ball sticks. Which ball delivers a larger impulse to the wall?

a. The clay ball delivers a larger impulse because it sticks.
b. The rubber ball delivers a larger impulse because it bounces.
c. They deliver equal impulses because they have equal momenta.
d. Neither delivers an impulse to the wall because the wall doesn't move.

11.2 Conservation of Momentum

The momentum principle was derived from Newton's second law and is really just an alternative way of looking at single-particle dynamics. To discover the real power of momentum for problem solving, we need also to invoke Newton's third law, which will lead us to one of the most important principles in physics: conservation of momentum.

FIGURE 11.9 A collision between two objects.

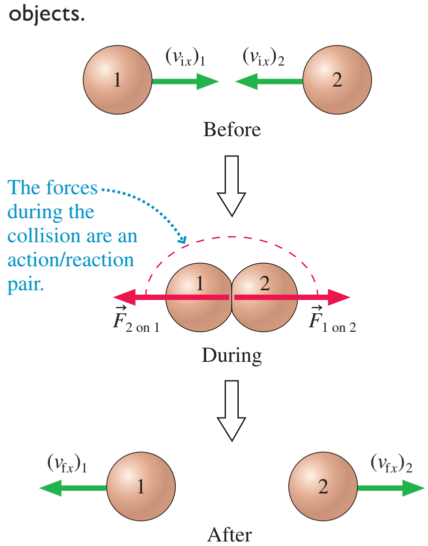


FIGURE 11.9 shows two objects with initial velocities $(v_{ix})_1$ and $(v_{ix})_2$. The objects collide, then bounce apart with final velocities $(v_{fx})_1$ and $(v_{fx})_2$. The forces during the collision, as the objects are interacting, are the action/reaction pair $\vec{F}_{1 \text{ on } 2}$ and $\vec{F}_{2 \text{ on } 1}$. For now, we'll continue to assume that the motion is one dimensional along the x-axis.

NOTE The notation, with all the subscripts, may seem excessive. But there are two objects, and each has an initial and a final velocity, so we need to distinguish among four different velocities.

Newton's second law for each object *during* the collision is

$$\frac{d(p_x)_1}{dt} = (F_x)_{2 \text{ on } 1}$$
$$\frac{d(p_x)_2}{dt} = (F_x)_{1 \text{ on } 2} = -(F_x)_{2 \text{ on } 1} \tag{11.13}$$

We made explicit use of Newton's third law in the second equation.

Although Equations 11.13 are for two different objects, suppose—just to see what happens—we were to *add* these two equations. If we do, we find that

$$\frac{d(p_x)_1}{dt} + \frac{d(p_x)_2}{dt} = \frac{d}{dt}\Big[(p_x)_1 + (p_x)_2\Big] = (F_x)_{2\text{ on }1} + (-(F_x)_{2\text{ on }1}) = 0 \qquad (11.14)$$

If the time derivative of the quantity $(p_x)_1 + (p_x)_2$ is zero, it must be the case that

$$(p_x)_1 + (p_x)_2 = \text{constant} \qquad (11.15)$$

Equation 11.15 is a conservation law! If $(p_x)_1 + (p_x)_2$ is a constant, then the sum of the momenta *after* the collision equals the sum of the momenta *before* the collision. That is,

$$(p_{\text{fx}})_1 + (p_{\text{fx}})_2 = (p_{\text{ix}})_1 + (p_{\text{ix}})_2 \qquad (11.16)$$

Furthermore, this equality is independent of the interaction force. We don't need to know *anything* about $\vec{F}_{1\text{ on }2}$ and $\vec{F}_{2\text{ on }1}$ to make use of Equation 11.16.

As an example, **FIGURE 11.10** is a before-and-after pictorial representation of two equal-mass train cars colliding and coupling. Equation 11.16 relates the momenta of the cars after the collision to their momenta before the collision:

$$m_1(v_{\text{fx}})_1 + m_2(v_{\text{fx}})_2 = m_1(v_{\text{ix}})_1 + m_2(v_{\text{ix}})_2$$

Initially, car 1 is moving with velocity $(v_{\text{ix}})_1 = v_\text{i}$ while car 2 is at rest. Afterward, they roll together with the common final velocity v_f. Furthermore, $m_1 = m_2 = m$. With this information, the momentum equation is

$$mv_\text{f} + mv_\text{f} = mv_\text{i} + 0$$

The mass cancels, and we find that the train cars' final velocity is $v_\text{f} = \frac{1}{2}v_\text{i}$. That is, we can predict that the final speed is exactly half the initial speed of car 1 without knowing anything about the complex interaction between the two cars as they collide.

FIGURE 11.10 Two colliding train cars.

Systems of Particles

Equation 11.16 illustrates the idea of a conservation law for momentum, but it was derived for the specific case of two particles colliding in one dimension. Our goal is to develop a more general law of conservation of momentum, a law that will be valid in three dimensions and that will work for any type of interaction. The next few paragraphs are fairly mathematical, so you might want to begin by looking ahead to Equations 11.24 and the statement of the law of conservation of momentum to see where we're heading.

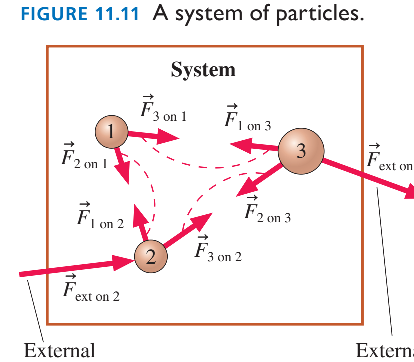
Our study of energy in the last two chapters has emphasized the importance of having a clearly defined system. The same is true for momentum. Consider a system consisting of N particles. **FIGURE 11.11** shows a simple case where $N = 3$, but N could be anything. The particles might be large entities (cars, baseballs, etc.), or they might be the microscopic atoms in a gas. We can identify each particle by an identification number k. Every particle in the system *interacts* with every other particle via action/reaction pairs of forces $\vec{F}_{j\text{ on }k}$ and $\vec{F}_{k\text{ on }j}$. In addition, every particle is subjected to possible *external forces* $\vec{F}_{\text{ext on }k}$ from agents outside the system.

If particle k has velocity $\vec{v}_k$, its momentum is $\vec{p}_k = m_k\vec{v}_k$. We define the **total momentum** $\vec{P}$ of the system as the vector sum

$$\vec{P} = \text{total momentum} = \vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \cdots + \vec{p}_N = \sum_{k=1}^{N} \vec{p}_k \qquad (11.17)$$

That is the total momentum *of the system* is the vector sum of the individual momenta.

FIGURE 11.11 A system of particles.



The total momentum of the rocket + gases system is conserved, so the rocket accelerates forward as the gases are expelled backward.

The time derivative of $\vec{P}$ tells us how the total momentum of the system changes with time:

$$\frac{d\vec{P}}{dt} = \sum_k \frac{d\vec{p}_k}{dt} = \sum_k \vec{F}_k \tag{11.18}$$

where we used Newton's second law for each particle in the form $\vec{F}_k = d\vec{p}_k/dt$, which was Equation 11.4.

The net force acting on particle k can be divided into *external forces,* from outside the system, and *interaction forces* due to all the other particles in the system:

$$\vec{F}_k = \sum_{j \neq k} \vec{F}_{j \text{ on } k} + \vec{F}_{\text{ext on } k} \tag{11.19}$$

The restriction $j \neq k$ expresses the fact that particle k does not exert a force on itself. Using this in Equation 11.18 gives the rate of change of the total momentum $\vec{P}$ of the system:

$$\frac{d\vec{P}}{dt} = \sum_k \sum_{j \neq k} \vec{F}_{j \text{ on } k} + \sum_k \vec{F}_{\text{ext on } k} \tag{11.20}$$

The double sum on $\vec{F}_{j \text{ on } k}$ adds *every* interaction force within the system. But the interaction forces come in action/reaction pairs, with $\vec{F}_{k \text{ on } j} = -\vec{F}_{j \text{ on } k}$, so $\vec{F}_{k \text{ on } j} + \vec{F}_{j \text{ on } k} = \vec{0}$. Consequently, **the sum of all the interaction forces is zero.** As a result, Equation 11.20 becomes

$$\frac{d\vec{P}}{dt} = \sum_k \vec{F}_{\text{ext on } k} = \vec{F}_{\text{net}} \tag{11.21}$$

where $\vec{F}_{\text{net}}$ is the net force exerted on the system by agents outside the system. But this is just Newton's second law written for the system as a whole! That is, **the rate of change of the total momentum of the system is equal to the net force applied to the system.**

Equation 11.21 has two very important implications. First, we can analyze the motion of the system as a whole without needing to consider interaction forces between the particles that make up the system. In fact, we have been using this idea all along as an *assumption* of the particle model. When we treat cars and rocks and baseballs as particles, we assume that the internal forces between the atoms—the forces that hold the object together—do not affect the motion of the object as a whole. Now we have *justified* that assumption.

Isolated Systems

The second implication of Equation 11.21, and the more important one from the perspective of this chapter, applies to an isolated system. In Chapter 10, we defined an *isolated system* as one that is not influenced or altered by external forces from the environment. For momentum, that means a system on which the *net* external force is zero: $\vec{F}_{\text{net}} = \vec{0}$. That is, an isolated system is one on which there are *no* external forces or for which the external forces are balanced and add to zero.

For an isolated system, Equation 11.21 is simply

$$\frac{d\vec{P}}{dt} = \vec{0} \qquad \text{(isolated system)} \tag{11.22}$$

In other words, **the *total* momentum of an isolated system does not change.** The total momentum $\vec{P}$ remains constant, *regardless* of whatever interactions are going on *inside* the system. The importance of this result is sufficient to elevate it to a law of nature, alongside Newton's laws.

Law of conservation of momentum The total momentum $\vec{P}$ of an isolated system is a constant. Interactions within the system do not change the system's total momentum. Mathematically, the law of conservation of momentum is

$$\vec{P}_f = \vec{P}_i \tag{11.23}$$

The total momentum *after* an interaction is equal to the total momentum *before* the interaction. Because Equation 11.23 is a vector equation, the equality is true for each of the components of the momentum vector. That is,

$$\begin{aligned}(p_{fx})_1 + (p_{fx})_2 + (p_{fx})_3 + \cdots &= (p_{ix})_1 + (p_{ix})_2 + (p_{ix})_3 + \cdots \\ (p_{fy})_1 + (p_{fy})_2 + (p_{fy})_3 + \cdots &= (p_{iy})_1 + (p_{iy})_2 + (p_{iy})_3 + \cdots\end{aligned} \tag{11.24}$$

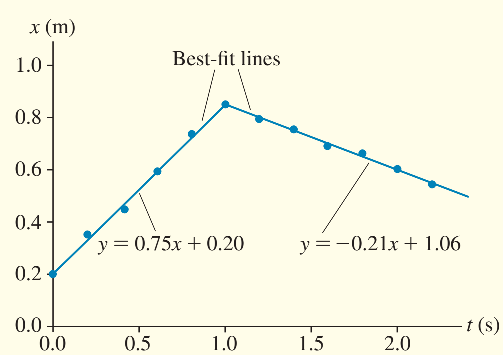
The x-equation is an extension of Equation 11.16 to N interacting particles.

NOTE It is worth emphasizing the critical role of Newton's third law. The law of conservation of momentum is a direct consequence of the fact that interactions within an isolated system are action/reaction pairs.

EXAMPLE 11.2 A glider collision

A 250 g air-track glider is pushed across a level track toward a 500 g glider that is at rest. FIGURE 11.12 shows a position-versus-time graph of the 250 g glider as recorded by a motion detector. Best-fit lines have been found. What is the speed of the 500 g glider after the collision?

FIGURE 11.12 Position graph of the 250 g glider.



MODEL Let the system be the two gliders. The gliders interact with each other, but the external forces (normal force and gravity) balance to make $\vec{F}_{net} = \vec{0}$. Thus the gliders form an isolated system and their total momentum is conserved.

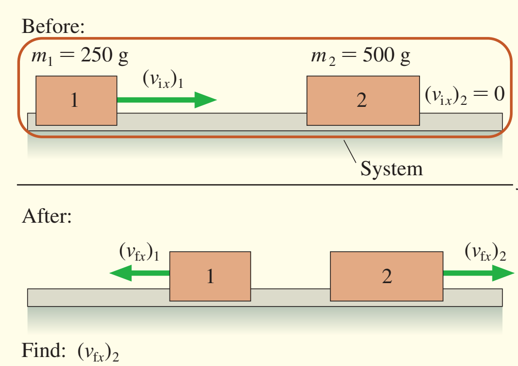
VISUALIZE FIGURE 11.13 is a before-and-after pictorial representation. The graph of Figure 11.12 tells us that the 250 g glider initially moves to the right, collides at $t = 1.0$ s, then rebounds to the left (decreasing x). Note that the best-fit lines are written as a generic $y = \ldots$, which is what you would see in data-analysis software.

SOLVE Conservation of momentum for this one-dimensional problem requires that the final momentum equal the initial momentum: $P_{fx} = P_{ix}$. In terms of the individual components, conservation of momentum is

$$(p_{fx})_1 + (p_{fx})_2 = (p_{ix})_1 + (p_{ix})_2$$

Each momentum is mv_x, so conservation of momentum in terms of velocities is

FIGURE 11.13 Before-and-after representation of a collision.



$$m_1(v_{fx})_1 + m_2(v_{fx})_2 = m_1(v_{ix})_1 + m_2(v_{ix})_2 = m_1(v_{ix})_1$$

where, in the last step, we used $(v_{ix})_2 = 0$ for the 500 g glider. Solving for the heavier glider's final velocity gives

$$(v_{fx})_2 = \frac{m_1}{m_2}\left[(v_{ix})_1 - (v_{fx})_1\right]$$

From Chapter 2 kinematics, the velocities of the 250 g glider before and after the collision are the slopes of the position-versus-time graph. Referring to Figure 11.12, we see that $(v_{ix})_1 = 0.75$ m/s and $(v_{fx})_1 = -0.21$ m/s. The latter is negative because the rebound motion is to the left. Thus

$$(v_{fx})_2 = \frac{250\text{ g}}{500\text{ g}}\left[0.75\text{ m/s} - (-0.21\text{ m/s})\right] = 0.48\text{ m/s}$$

The 500 g glider moves away from the collision at 0.48 m/s.

ASSESS The 500 g glider has twice the mass of the glider that was pushed, so a somewhat smaller speed seems reasonable. Paying attention to the *signs*—which are positive and which negative—was very important for reaching a correct answer. We didn't convert the masses to kilograms because only the mass *ratio* of 0.50 was needed.

A Strategy for Conservation of Momentum Problems

PROBLEM-SOLVING STRATEGY 11.1 MP

Conservation of momentum

MODEL Clearly define the *system*.

- If possible, choose a system that is isolated ($\vec{F}_{net} = \vec{0}$) or within which the interactions are sufficiently short and intense that you can ignore external forces for the duration of the interaction (the impulse approximation). Momentum is conserved.
- If it's not possible to choose an isolated system, try to divide the problem into parts such that momentum is conserved during one segment of the motion. Other segments of the motion can be analyzed using Newton's laws or conservation of energy.

VISUALIZE Draw a before-and-after pictorial representation. Define symbols that will be used in the problem, list known values, and identify what you're trying to find.

SOLVE The mathematical representation is based on the law of conservation of momentum: $\vec{P}_f = \vec{P}_i$. In component form, this is

$$(p_{fx})_1 + (p_{fx})_2 + (p_{fx})_3 + \cdots = (p_{ix})_1 + (p_{ix})_2 + (p_{ix})_3 + \cdots$$
$$(p_{fy})_1 + (p_{fy})_2 + (p_{fy})_3 + \cdots = (p_{iy})_1 + (p_{iy})_2 + (p_{iy})_3 + \cdots$$

ASSESS Check that your result has correct units and significant figures, is reasonable, and answers the question.

Exercise 17

EXAMPLE 11.3 | **Rolling away**

Bob sees a stationary cart 8.0 m in front of him. He decides to run to the cart as fast as he can, jump on, and roll down the street. Bob has a mass of 75 kg and the cart's mass is 25 kg. If Bob accelerates at a steady 1.0 m/s^2, what is the cart's speed just after Bob jumps on?

MODEL This is a two-part problem. First Bob accelerates across the ground. Then Bob lands on and sticks to the cart, a "collision" between Bob and the cart. The interaction forces between Bob and the cart (i.e., friction) act only over the fraction of a second it takes Bob's feet to become stuck to the cart. Using the impulse approximation allows the system Bob + cart to be treated as an isolated system during the brief interval of the "collision," and thus the total momentum of Bob + cart is conserved during this interaction. But the system Bob + cart is *not* an isolated system for the entire problem because Bob's initial acceleration has nothing to do with the cart.

VISUALIZE Our strategy is to divide the problem into an *acceleration* part, which we can analyze using kinematics, and a *collision* part, which we can analyze with momentum conservation. The pictorial representation of **FIGURE 11.14** includes information about both parts. Notice that Bob's velocity $(v_{1x})_B$ at the end of his run is his "before" velocity for the collision.

SOLVE The first part of the mathematical representation is kinematics. We don't know how long Bob accelerates, but we do know his acceleration and the distance. Thus

$$(v_{1x})_B{}^2 = (v_{0x})_B{}^2 + 2a_x\,\Delta x = 2a_x x_1$$

His velocity after accelerating for 8.0 m is

$$(v_{1x})_B = \sqrt{2a_x x_1} = 4.0 \text{ m/s}$$

The second part of the problem, the collision, uses conservation of momentum: $P_{2x} = P_{1x}$. Equation 11.24 is

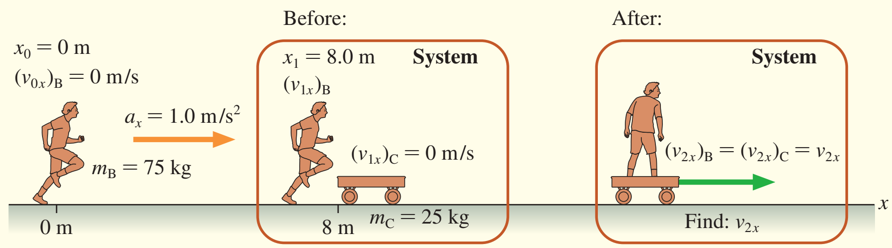
$$m_B(v_{2x})_B + m_C(v_{2x})_C = m_B(v_{1x})_B + m_C(v_{1x})_C = m_B(v_{1x})_B$$

where we've used $(v_{1x})_C = 0$ m/s because the cart starts at rest. In this problem, Bob and the cart move together at the end with a common velocity, so we can replace both $(v_{2x})_B$ and $(v_{2x})_C$ with simply v_{2x}. Solving for v_{2x}, we find

$$v_{2x} = \frac{m_B}{m_B + m_C}(v_{1x})_B = \frac{75 \text{ kg}}{100 \text{ kg}} \times 4.0 \text{ m/s} = 3.0 \text{ m/s}$$

The cart's speed is 3.0 m/s immediately after Bob jumps on.

FIGURE 11.14 Pictorial representation of Bob and the cart.



Notice how easy this was! No forces, no acceleration constraints, no simultaneous equations. Why didn't we think of this before? Conservation laws are indeed powerful, but they can answer only certain questions. Had we wanted to know how far Bob slid across the cart before sticking to it, how long the slide took, or what the cart's acceleration was during the collision, we would not have been able to answer such questions on the basis of the conservation law. There is a price to pay for finding a simple connection between before and after, and that price is the loss of information about the details of the interaction. If we are satisfied with knowing only about before and after, then conservation laws are a simple and straightforward way to proceed. But many problems *do* require us to understand the interaction, and for these there is no avoiding Newton's laws.

It Depends on the System

The first step in the problem-solving strategy asks you to clearly define the system. **The goal is to choose a system whose momentum will be conserved.** Even then, it is the *total* momentum of the system that is conserved, not the momenta of the individual objects within the system.

As an example, consider what happens if you drop a rubber ball and let it bounce off a hard floor. Is momentum conserved? You might be tempted to answer yes because the ball's rebound speed is very nearly equal to its impact speed. But there are two errors in this reasoning.

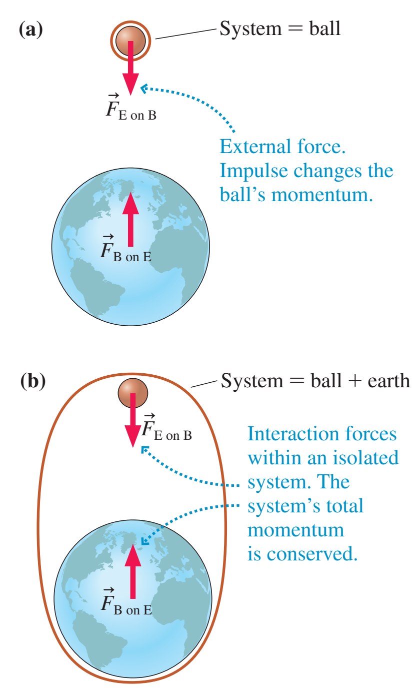
First, momentum depends on *velocity,* not speed. The ball's velocity and momentum change sign during the collision. Even if their magnitudes are equal, the ball's momentum after the collision is *not* equal to its momentum before the collision.

But more important, we haven't defined the system. The momentum of what? Whether or not momentum is conserved depends on the system. **FIGURE 11.15** shows two different choices of systems. In Figure 11.15a, where the ball itself is chosen as the system, the gravitational force of the earth on the ball is an external force. This force causes the ball to accelerate toward the earth, changing the ball's momentum. When the ball hits, the force of the floor on the ball is also an external force. The impulse of $\vec{F}_{\text{floor on ball}}$ changes the ball's momentum from "down" to "up" as the ball bounces. The momentum of this system is most definitely *not* conserved.

Figure 11.15b shows a different choice. Here the system is ball + earth. Now the gravitational forces and the impulsive forces of the collision are interactions *within* the system. This is an isolated system, so the *total* momentum $\vec{P} = \vec{p}_{\text{ball}} + \vec{p}_{\text{earth}}$ is conserved.

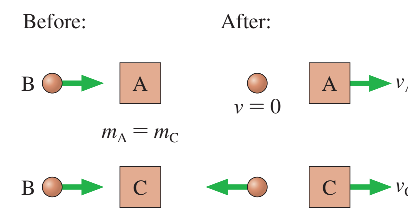
In fact, the total momentum (in this reference frame) is $\vec{P} = \vec{0}$ because both the ball and the earth are initially at rest. The ball accelerates toward the earth after you release it, while the earth—due to Newton's third law—accelerates toward the ball in such a way that their individual momenta are always equal but opposite.

FIGURE 11.15 Whether or not momentum is conserved as a ball falls to earth depends on your choice of the system.



Why don't we notice the earth "leaping up" toward us each time we drop something? Because of the earth's enormous mass relative to everyday objects—roughly 10^{25} times larger. Momentum is the product of mass and velocity, so the earth would need an "upward" speed of only about 10^{-25} m/s to match the momentum of a typical falling object. At that speed, it would take 300 million years for the earth to move the diameter of an atom! The earth does, indeed, have a momentum equal and opposite to that of the ball, but we'll never notice it.

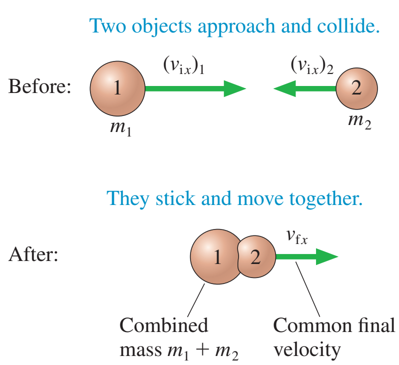
STOP TO THINK 11.3 Objects A and C are made of different materials, with different "springiness," but they have the same mass and are initially at rest. When ball B collides with object A, the ball ends up at rest. When ball B is thrown with the same speed and collides with object C, the ball rebounds to the left. Compare the velocities of A and C after the collisions. Is v_A greater than, equal to, or less than v_C?



11.3 Collisions

Collisions can have different possible outcomes. A rubber ball dropped on the floor bounces, but a ball of clay sticks to the floor without bouncing. A golf club hitting a golf ball causes the ball to rebound away from the club, but a bullet striking a block of wood embeds itself in the block.

FIGURE 11.16 An inelastic collision.



Inelastic Collisions

A collision in which the two objects stick together and move with a common final velocity is called a **perfectly inelastic collision.** The clay hitting the floor and the bullet embedding itself in the wood are examples of perfectly inelastic collisions. Other examples include railroad cars coupling together upon impact and darts hitting a dart board. As **FIGURE 11.16** shows, the key to analyzing a perfectly inelastic collision is the fact that **the two objects have a common final velocity.**

A system consisting of the two colliding objects is isolated, so its total momentum is conserved. However, mechanical energy is *not* conserved because some of the initial kinetic energy is transformed into thermal energy during the collision.

EXAMPLE 11.4 **An inelastic glider collision**

In a laboratory experiment, a 200 g air-track glider and a 400 g air-track glider are pushed toward each other from opposite ends of the track. The gliders have Velcro tabs on the front and will stick together when they collide. The 200 g glider is pushed with an initial speed of 3.0 m/s. The collision causes it to reverse direction at 0.40 m/s. What was the initial speed of the 400 g glider?

MODEL Define the system to be the two gliders. This is an isolated system, so its total momentum is conserved in the collision. The gliders stick together, so this is a perfectly inelastic collision.

VISUALIZE **FIGURE 11.17** shows a pictorial representation. We've chosen to let the 200 g glider (glider 1) start out moving to the right, so $(v_{ix})_1$ is a positive 3.0 m/s. The gliders move to the left after the collision, so their common final velocity is $v_{fx} = -0.40$ m/s.

SOLVE The law of conservation of momentum, $P_{fx} = P_{ix}$, is

$$(m_1 + m_2)v_{fx} = m_1(v_{ix})_1 + m_2(v_{ix})_2$$

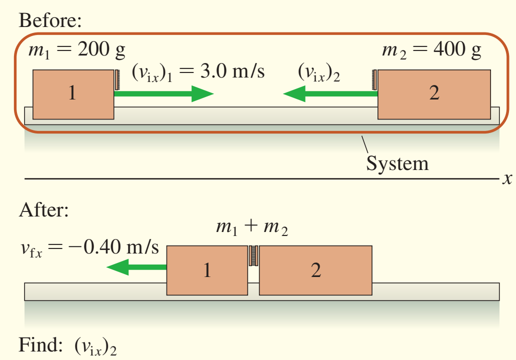
where we made use of the fact that the combined mass $m_1 + m_2$ moves together after the collision. We can easily solve for the initial velocity of the 400 g glider:

$$(v_{ix})_2 = \frac{(m_1 + m_2)v_{fx} - m_1(v_{ix})_1}{m_2}$$

$$= \frac{(0.60\text{ kg})(-0.40\text{ m/s}) - (0.20\text{ kg})(3.0\text{ m/s})}{0.40\text{ kg}} = -2.1\text{ m/s}$$

The negative sign indicates that the 400 g glider started out moving to the left. The initial *speed* of the glider, which we were asked to find, is 2.1 m/s.

FIGURE 11.17 The before-and-after pictorial representation of an inelastic collision.



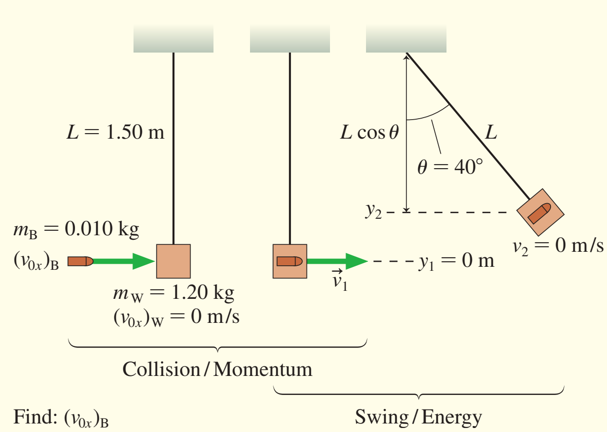
EXAMPLE 11.5 A ballistic pendulum

A 10 g bullet is fired into a 1200 g wood block hanging from a 150-cm-long string. The bullet embeds itself into the block, and the block then swings out to an angle of 40°. What was the speed of the bullet? (This is called a *ballistic pendulum.*)

MODEL This is a two-part problem. Part one, the impact of the bullet on the block, is an inelastic collision. For this part, we define the system to be bullet + block. Momentum is conserved, but mechanical energy is not because some of the energy is transformed into thermal energy. For part two, the subsequent swing, mechanical energy is conserved for the system bullet + block + earth (there's no friction). The *total* momentum is conserved, including the momentum of the earth, but that's not helpful. The momentum of the block with the bullet—which is all that we can calculate—is not conserved because the block is acted on by the external forces of tension and gravity.

VISUALIZE **FIGURE 11.18** is a pictorial representation in which we've identified before-and-after quantities for both the collision and the swing.

FIGURE 11.18 A ballistic pendulum is used to measure the speed of a bullet.



SOLVE The momentum conservation equation $P_f = P_i$ applied to the inelastic collision gives

$$(m_W + m_B)v_{1x} = m_W(v_{0x})_W + m_B(v_{0x})_B$$

The wood block is initially at rest, with $(v_{0x})_W = 0$, so the bullet's velocity is

$$(v_{0x})_B = \frac{m_W + m_B}{m_B} v_{1x}$$

where v_{1x} is the velocity of the block + bullet *immediately* after the collision, as the pendulum begins to swing. If we can determine v_{1x} from an analysis of the swing, then we will be able to calculate the speed of the bullet. Turning our attention to the swing, the energy conservation equation $K_f + U_{Gf} = K_i + U_{Gi}$ is

$$\tfrac{1}{2}(m_W + m_B)v_2^2 + (m_W + m_B)gy_2 = \tfrac{1}{2}(m_W + m_B)v_1^2 + (m_W + m_B)gy_1$$

We used the *total* mass $(m_W + m_B)$ of the block and embedded bullet, but notice that it cancels out. We also dropped the x-subscript on v_1 because for energy calculations we need only speed, not velocity. The speed is zero at the top of the swing $(v_2 = 0)$, and we've defined the y-axis such that $y_1 = 0$ m. Thus

$$v_1 = \sqrt{2gy_2}$$

The initial speed is found simply from the maximum height of the swing. You can see from the geometry of Figure 11.18 that

$$y_2 = L - L\cos\theta = L(1 - \cos\theta) = 0.351\text{ m}$$

With this, the initial velocity of the pendulum, immediately after the collision, is

$$v_{1x} = v_1 = \sqrt{2gy_2} = \sqrt{2(9.80\text{ m/s}^2)(0.351\text{ m})} = 2.62\text{ m/s}$$

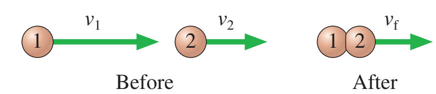
Having found v_{1x} from an energy analysis of the swing, we can now calculate that the speed of the bullet was

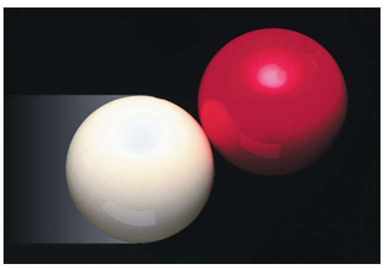
$$(v_{0x})_B = \frac{m_W + m_B}{m_B} v_{1x} = \frac{1.210\text{ kg}}{0.010\text{ kg}} \times 2.62\text{ m/s} = 320\text{ m/s}$$

ASSESS It would have been very difficult to solve this problem using Newton's laws, but it yielded to a straightforward analysis based on the concepts of momentum and energy.

STOP TO THINK 11.4 The two particles are both moving to the right. Particle 1 catches up with particle 2 and collides with it. The particles stick together and continue on with velocity v_f. Which of these statements is true?

a. v_f is greater than v_1.
b. $v_f = v_1$
c. v_f is greater than v_2 but less than v_1.
d. $v_f = v_2$
e. v_f is less than v_2.
f. Can't tell without knowing the masses.





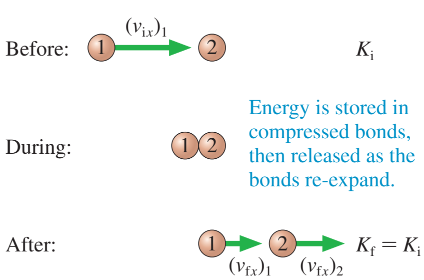
A perfectly elastic collision conserves both momentum and mechanical energy.

Elastic Collisions

In an inelastic collision, some of the mechanical energy is dissipated inside the objects as thermal energy and not all of the kinetic energy is recovered. We're now interested in "perfect bounce" collisions in which kinetic energy is stored as elastic potential energy in compressed molecular bonds, and then *all* of the stored energy is transformed back into the post-collision kinetic energy of the objects. A collision in which mechanical energy is conserved is called a **perfectly elastic collision.** A perfectly elastic collision is an idealization, like a frictionless surface, but collisions between two very hard objects, such as two billiard balls or two steel balls, come close to being perfectly elastic.

FIGURE 11.19 shows a head-on, perfectly elastic collision of a ball of mass m_1, having initial velocity $(v_{ix})_1$, with a ball of mass m_2 that is initially at rest. The balls' velocities after the collision are $(v_{fx})_1$ and $(v_{fx})_2$. These are velocities, not speeds, and have signs. Ball 1, in particular, might bounce backward and have a negative value for $(v_{fx})_1$.

FIGURE 11.19 A perfectly elastic collision.



The collision must obey two conservation laws: conservation of momentum (obeyed in any collision) and conservation of mechanical energy (because the collision is perfectly elastic). Although the energy is transformed into potential energy during the collision, the mechanical energy before and after the collision is purely kinetic energy. Thus

$$\text{momentum conservation:} \quad m_1(v_{fx})_1 + m_2(v_{fx})_2 = m_1(v_{ix})_1 \tag{11.25}$$

$$\text{energy conservation:} \quad \tfrac{1}{2}m_1(v_{fx})_1^{\,2} + \tfrac{1}{2}m_2(v_{fx})_2^{\,2} = \tfrac{1}{2}m_1(v_{ix})_1^{\,2} \tag{11.26}$$

Momentum conservation alone is not sufficient to analyze the collision because there are two unknowns: the two final velocities. Energy conservation provides the additional information that we need. Isolating $(v_{fx})_1$ in Equation 11.25 gives

$$(v_{fx})_1 = (v_{ix})_1 - \frac{m_2}{m_1}(v_{fx})_2 \tag{11.27}$$

We substitute this into Equation 11.26:

$$\tfrac{1}{2}m_1\left[(v_{ix})_1 - \frac{m_2}{m_1}(v_{fx})_2\right]^2 + \tfrac{1}{2}m_2(v_{fx})_2^{\,2} = \tfrac{1}{2}m_1(v_{ix})_1^{\,2}$$

With a bit of algebra, this can be rearranged to give

$$(v_{fx})_2\left[\left(1 + \frac{m_2}{m_1}\right)(v_{fx})_2 - 2(v_{ix})_1\right] = 0 \tag{11.28}$$

One possible solution to this equation is seen to be $(v_{fx})_2 = 0$. However, this solution is of no interest; it is the case where ball 1 misses ball 2. The other solution is

$$(v_{fx})_2 = \frac{2m_1}{m_1 + m_2}(v_{ix})_1$$

which, finally, can be substituted back into Equation 11.27 to yield $(v_{fx})_1$. The complete solution is

$$(v_{fx})_1 = \frac{m_1 - m_2}{m_1 + m_2}(v_{ix})_1 \qquad (v_{fx})_2 = \frac{2m_1}{m_1 + m_2}(v_{ix})_1 \tag{11.29}$$

(perfectly elastic collision with ball 2 initially at rest)

Equations 11.29 allow us to compute the final velocity of each ball. These equations are a little difficult to interpret, so let us look at the three special cases shown in **FIGURE 11.20**.

Case a: $m_1 = m_2$. This is the case of one billiard ball striking another of equal mass. For this case, Equations 11.29 give

$$v_{f1} = 0 \qquad v_{f2} = v_{i1}$$

Case b: $m_1 \gg m_2$. This is the case of a bowling ball running into a Ping-Pong ball. We do not want an exact solution here, but an approximate solution for the limiting case that $m_1 \to \infty$. Equations 11.29 in this limit give

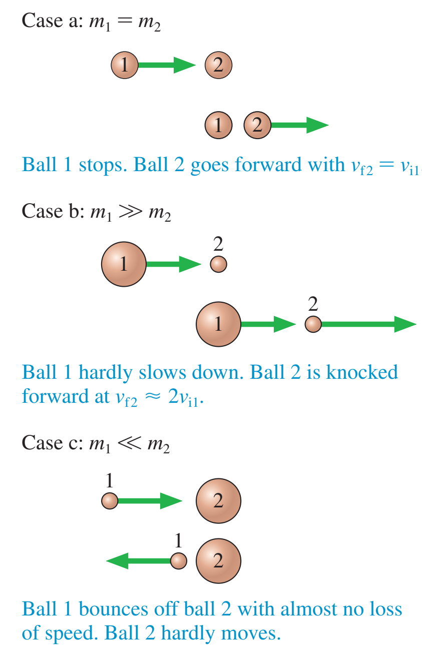
$$v_{f1} \approx v_{i1} \qquad v_{f2} \approx 2v_{i1}$$

Case c: $m_1 \ll m_2$. Now we have the reverse case of a Ping-Pong ball colliding with a bowling ball. Here we are interested in the limit $m_1 \to 0$, in which case Equations 11.29 become

$$v_{f1} \approx -v_{i1} \qquad v_{f2} \approx 0$$

These cases agree well with our expectations and give us confidence that Equations 11.29 accurately describe a perfectly elastic collision.

FIGURE 11.20 Three special elastic collisions.

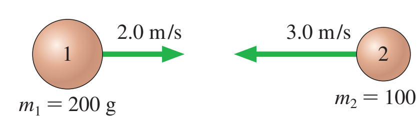


Using Reference Frames

Equations 11.29 assumed that ball 2 was at rest prior to the collision. Suppose, however, you need to analyze the perfectly elastic collision that is just about to take place in **FIGURE 11.21**. What are the direction and speed of each ball after the collision? You could solve the simultaneous momentum and energy equations, but the mathematics becomes quite messy when both balls have an initial velocity. Fortunately, there's an easier way.

You already know the answer—Equations 11.29—when ball 2 is initially at rest. And in Chapter 4 you learned the Galilean transformation of velocity. This transformation relates an object's velocity as measured in one reference frame to its velocity in a different reference frame that moves with respect to the first. The Galilean transformation provides an elegant and straightforward way to analyze the collision of Figure 11.21.

FIGURE 11.21 A perfectly elastic collision in which both balls have an initial velocity.



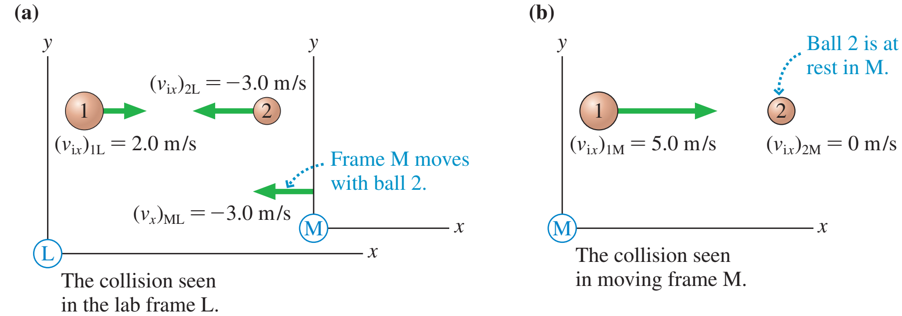
TACTICS BOX 11.1 MP

Analyzing elastic collisions

1. Use the Galilean transformation to transform the initial velocities of balls 1 and 2 from the "lab frame" to a reference frame in which ball 2 is at rest.
2. Use Equations 11.29 to determine the outcome of the collision in the frame where ball 2 is initially at rest.
3. Transform the final velocities back to the "lab frame."

FIGURE 11.22a on the next page shows the situation, just before the collision, in the lab frame L. Ball 1 has initial velocity $(v_{ix})_{1L} = 2.0$ m/s. Recall from Chapter 4 that the subscript notation means "velocity of ball 1 relative to the lab frame L." Because ball 2 is moving to the left, it has $(v_{ix})_{2L} = -3.0$ m/s. We would like to observe the collision from a reference frame in which ball 2 is at rest. That will be true if we choose a moving reference frame M that travels alongside ball 2 with the same velocity: $(v_x)_{ML} = -3.0$ m/s.

FIGURE 11.22 The collision seen in two reference frames: the lab frame L and a moving frame M in which ball 2 is initially at rest.



We first need to transform the balls' velocities from the lab frame to the moving reference frame. From Chapter 4, the Galilean transformation of velocity for an object O is

$$(v_x)_{OM} = (v_x)_{OL} + (v_x)_{LM} \tag{11.30}$$

That is, O's velocity in reference frame M is its velocity in reference frame L plus the velocity of frame L relative to frame M. Because reference frame M is moving to the left relative to L with $(v_x)_{ML} = -3.0$ m/s, reference frame L is moving to the right relative to M with $(v_x)_{LM} = +3.0$ m/s. Applying the transformation to the two initial velocities gives

$$\begin{aligned}(v_{ix})_{1M} &= (v_{ix})_{1L} + (v_x)_{LM} = 2.0 \text{ m/s} + 3.0 \text{ m/s} = 5.0 \text{ m/s}\\(v_{ix})_{2M} &= (v_{ix})_{2L} + (v_x)_{LM} = -3.0 \text{ m/s} + 3.0 \text{ m/s} = 0 \text{ m/s}\end{aligned} \tag{11.31}$$

$(v_{ix})_{2M} = 0$ m/s, as expected, because we chose a moving reference frame in which ball 2 would be at rest.

FIGURE 11.22b now shows a situation—with ball 2 initially at rest—in which we can use Equations 11.29 to find the post-collision velocities in frame M:

$$\begin{aligned}(v_{fx})_{1M} &= \frac{m_1 - m_2}{m_1 + m_2}(v_{ix})_{1M} = 1.7 \text{ m/s}\\(v_{fx})_{2M} &= \frac{2m_1}{m_1 + m_2}(v_{ix})_{1M} = 6.7 \text{ m/s}\end{aligned} \tag{11.32}$$

Reference frame M hasn't changed—it's still moving to the left in the lab frame at 3.0 m/s—but the collision has changed both balls' velocities in frame M.

To finish, we need to transform the post-collision velocities in frame M back to the lab frame L. We can do so with another application of the Galilean transformation:

$$\begin{aligned}(v_{fx})_{1L} &= (v_{fx})_{1M} + (v_x)_{ML} = 1.7 \text{ m/s} + (-3.0 \text{ m/s}) = -1.3 \text{ m/s}\\(v_{fx})_{2L} &= (v_{fx})_{2M} + (v_x)_{ML} = 6.7 \text{ m/s} + (-3.0 \text{ m/s}) = 3.7 \text{ m/s}\end{aligned} \tag{11.33}$$

FIGURE 11.23 The post-collision velocities in the lab frame.

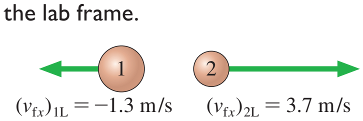


FIGURE 11.23 shows the outcome of the collision in the lab frame. It's not hard to confirm that these final velocities do, indeed, conserve both momentum and energy.

Two Collision Models

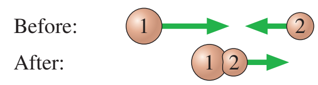
No collision is perfectly elastic, although collisions between two very hard objects (metal spheres) or between two springs (such as a collision on an air track) come close. Collisions can be perfectly inelastic, although many real-world inelastic collisions exhibit a small residual bounce. Thus perfectly elastic and perfectly inelastic collisions are *models* of collisions in which we simplify reality in order to gain understanding without getting bogged down in the messy details of real collisions.

MODEL 11.1

Collisions

For two colliding objects.

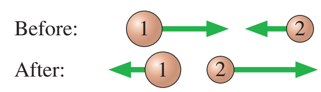
- Represent the objects as elastic objects moving in a straight line.
- In a **perfectly inelastic collision,** the objects stick and move together. Kinetic energy is transformed into thermal energy.



  Mathematically:

$$(m_1 + m_2)v_{fx} = m_1(v_{ix})_1 + m_2(v_{ix})_2$$

- In a **perfectly elastic collision,** the objects bounce apart with no loss of energy.



  Mathematically:
  - If object 2 is initially at rest, then

$$(v_{fx})_1 = \frac{m_1 - m_2}{m_1 + m_2}(v_{ix})_1 \qquad (v_{ix})_2 = \frac{2m_1}{m_1 + m_2}(v_{ix})_1$$

  - If both objects are moving, use the Galilean transformation to transform the velocities to a reference frame in which object 2 is at rest.
- Limitations: Model fails if the collision is not head-on or cannot reasonably be approximated as a “thud” or as a “perfect bounce.”

Exercise 22

11.4 Explosions

An **explosion,** where the particles of the system move apart from each other after a brief, intense interaction, is the opposite of a collision. The explosive forces, which could be from an expanding spring or from expanding hot gases, are *internal* forces. If the system is isolated, its total momentum during the explosion will be conserved.

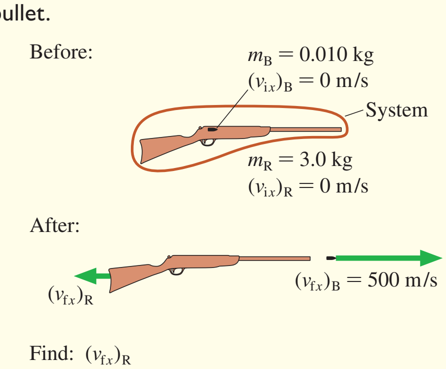
EXAMPLE 11.6 | **Recoil**

A 10 g bullet is fired from a 3.0 kg rifle with a speed of 500 m/s. What is the recoil speed of the rifle?

MODEL The rifle causes a small mass of gunpowder to explode, and the expanding gas then exerts forces on *both* the bullet and the rifle. Let’s define the system to be bullet + gas + rifle. The forces due to the expanding gas during the explosion are internal forces, within the system. Any friction forces between the bullet and the rifle as the bullet travels down the barrel are also internal forces. Gravity is balanced by the upward force of the person holding the rifle, so $\vec{F}_{net} = \vec{0}$. This is an isolated system and the law of conservation of momentum applies.

VISUALIZE **FIGURE 11.24** shows a pictorial representation before and after the bullet is fired.

FIGURE 11.24 Before-and-after pictorial representation of a rifle firing a bullet.



Continued

SOLVE The x-component of the total momentum is $P_x = (p_x)_B + (p_x)_R + (p_x)_{gas}$. Everything is at rest before the trigger is pulled, so the initial momentum is zero. After the trigger is pulled, the momentum of the expanding gas is the sum of the momenta of all the molecules in the gas. For every molecule moving in the forward direction with velocity v and momentum mv there is, on average, another molecule moving in the opposite direction with velocity $-v$ and thus momentum $-mv$. When the values are summed over the enormous number of molecules in the gas, we will be left with $p_{gas} \approx 0$. In addition, the mass of the gas is much less than that of the rifle or bullet. For both reasons, we can reasonably neglect the momentum of the gas. The law of conservation of momentum is thus

$$P_{fx} = m_B(v_{fx})_B + m_R(v_{fx})_R = P_{ix} = 0$$

Solving for the rifle's velocity, we find

$$(v_{fx})_R = -\frac{m_B}{m_R}(v_{fx})_B = -\frac{0.010 \text{ kg}}{3.0 \text{ kg}} \times 500 \text{ m/s} = -1.7 \text{ m/s}$$

The minus sign indicates that the rifle's recoil is to the left. The recoil *speed* is 1.7 m/s.

We would not know where to begin to solve a problem such as this using Newton's laws. But Example 11.6 is a simple problem when approached from the before-and-after perspective of a conservation law. The selection of bullet + gas + rifle as "the system" was the critical step. For momentum conservation to be a useful principle, we had to select a system in which the complicated forces due to expanding gas and friction were all internal forces. The rifle by itself is *not* an isolated system, so its momentum is *not* conserved.

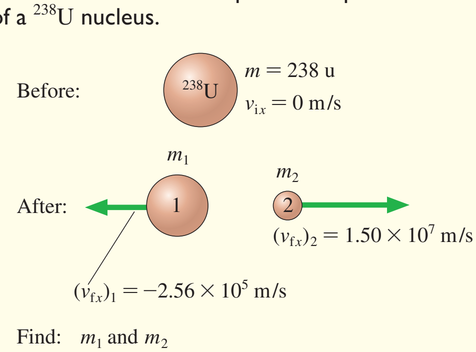
EXAMPLE 11.7 Radioactivity

A ^{238}U uranium nucleus is radioactive. It spontaneously disintegrates into a small fragment that is ejected with a measured speed of 1.50×10^7 m/s and a "daughter nucleus" that recoils with a measured speed of 2.56×10^5 m/s. What are the atomic masses of the ejected fragment and the daughter nucleus?

MODEL The notation ^{238}U indicates the isotope of uranium with an atomic mass of 238 u, where u is the abbreviation for the *atomic mass unit.* The nucleus contains 92 protons (uranium is atomic number 92) and 146 neutrons. The disintegration of a nucleus is, in essence, an explosion. Only *internal* nuclear forces are involved, so the total momentum is conserved in the decay.

VISUALIZE FIGURE 11.25 shows the pictorial representation. The mass of the daughter nucleus is m_1 and that of the ejected fragment is m_2. Notice that we converted the speed information to velocity information, giving $(v_{fx})_1$ and $(v_{fx})_2$ opposite signs.

FIGURE 11.25 Before-and-after pictorial representation of the decay of a ^{238}U nucleus.



SOLVE The nucleus was initially at rest, hence the total momentum is zero. The momentum after the decay is still zero if the two pieces fly apart in opposite directions with momenta equal in magnitude but opposite in sign. That is,

$$P_{fx} = m_1(v_{fx})_1 + m_2(v_{fx})_2 = P_{ix} = 0$$

Although we know both final velocities, this is not enough information to find the two unknown masses. However, we also have another conservation law, conservation of mass, that requires

$$m_1 + m_2 = 238 \text{ u}$$

Combining these two conservation laws gives

$$m_1(v_{fx})_1 + (238 \text{ u} - m_1)(v_{fx})_2 = 0$$

The mass of the daughter nucleus is

$$m_1 = \frac{(v_{fx})_2}{(v_{fx})_2 - (v_{fx})_1} \times 238 \text{ u}$$

$$= \frac{1.50 \times 10^7 \text{ m/s}}{(1.50 \times 10^7 - (-2.56 \times 10^5)) \text{ m/s}} \times 238 \text{ u} = 234 \text{ u}$$

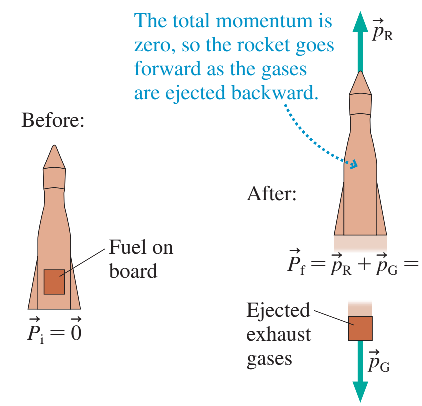
With m_1 known, the mass of the ejected fragment is $m_2 = 238 - m_1 = 4$ u.

ASSESS All we learn from a momentum analysis is the masses. Chemical analysis shows that the daughter nucleus is the element thorium, atomic number 90, with two fewer protons than uranium. The ejected fragment carried away two protons as part of its mass of 4 u, so it must be a particle with two protons and two neutrons. This is the nucleus of a helium atom, ^4He, which in nuclear physics is called an *alpha particle* α. Thus the radioactive decay of ^{238}U can be written as $^{238}U \rightarrow {}^{234}Th + \alpha$.

Much the same reasoning explains how a rocket or jet aircraft accelerates. **FIGURE 11.26** shows a rocket with a parcel of fuel on board. Burning converts the fuel to hot gases that are expelled from the rocket motor. If we choose rocket + gases to be the system, the burning and expulsion are both internal forces. There are no other forces, so the total momentum of the rocket + gases system must be conserved. The rocket gains forward velocity and momentum as the exhaust gases are shot out the back, but the *total* momentum of the system remains zero.

Section 11.6 looks at rocket propulsion in more detail, but even without the details you should be able to understand that jet and rocket propulsion is a consequence of momentum conservation.

FIGURE 11.26 Rocket propulsion is an example of conservation of momentum.



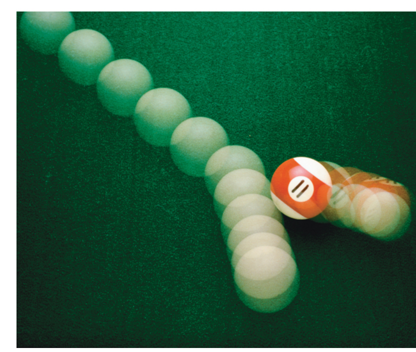
STOP TO THINK 11.5 An explosion in a rigid pipe shoots out three pieces. A 6 g piece comes out the right end. A 4 g piece comes out the left end with twice the speed of the 6 g piece. From which end, left or right, does the third piece emerge?

11.5 Momentum in Two Dimensions

The law of conservation of momentum $\vec{P}_f = \vec{P}_i$ is not restricted to motion along a line. Many interesting examples of collisions and explosions involve motion in a plane, and for these both the magnitude *and the direction* of the total momentum vector are unchanged. The total momentum is the vector sum of the individual momenta, so the total momentum is conserved only if each component is conserved:

$$\begin{aligned} (p_{fx})_1 + (p_{fx})_2 + (p_{fx})_3 + \cdots &= (p_{ix})_1 + (p_{ix})_2 + (p_{ix})_3 + \cdots \\ (p_{fy})_1 + (p_{fy})_2 + (p_{fy})_3 + \cdots &= (p_{iy})_1 + (p_{iy})_2 + (p_{iy})_3 + \cdots \end{aligned} \tag{11.34}$$

Let's look at some examples of momentum conservation in two dimensions.



Collisions and explosions often involve motion in two dimensions.

EXAMPLE 11.8 A peregrine falcon strike

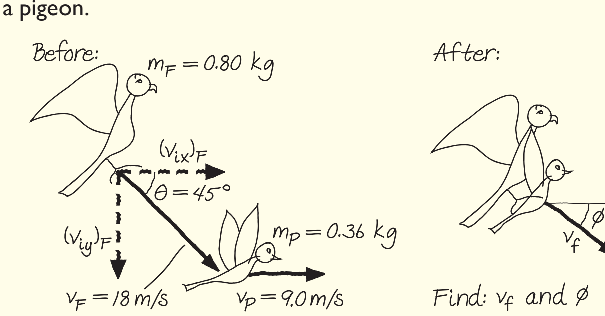
Peregrine falcons often grab their prey from above while both falcon and prey are in flight. A 0.80 kg falcon, flying at 18 m/s, swoops down at a 45° angle from behind a 0.36 kg pigeon flying horizontally at 9.0 m/s. What are the speed and direction of the falcon (now holding the pigeon) immediately after impact?

MODEL The two birds, modeled as particles, are the system. This is a perfectly inelastic collision because after the collision the falcon and pigeon move at a common final velocity. The birds are not a perfectly isolated system because of external forces of the air, but during the brief collision the external impulse delivered by the air resistance will be negligible. Within this approximation, the total momentum of the falcon + pigeon system is conserved during the collision.

VISUALIZE FIGURE 11.27 is a before-and-after pictorial representation. We've used angle ϕ to label the post-collision direction.

SOLVE The initial velocity components of the falcon are $(v_{ix})_F = v_F\cos\theta$ and $(v_{iy})_F = -v_F\sin\theta$. The pigeon's initial velocity is entirely along the x-axis. After the collision, when the falcon and pigeon have the common velocity $\vec{v}_f$, the components are $v_{fx} = v_f\cos\phi$ and $v_{fy} = -v_f\sin\phi$. Conservation of momentum in two dimensions requires conservation of both the x- and y-components of momentum. This gives two conservation equations:

FIGURE 11.27 Pictorial representation of a falcon catching a pigeon.



$$(m_F + m_P)v_{fx} = (m_F + m_P)v_f\cos\phi$$
$$= m_F(v_{ix})_F + m_P(v_{ix})_P = m_F v_F\cos\theta + m_P v_P$$
$$(m_F + m_P)v_{fy} = -(m_F + m_P)v_f\sin\phi$$
$$= m_F(v_{iy})_F + m_P(v_{iy})_P = -m_F v_F\sin\theta$$

Continued

The unknowns are v_f and ϕ. Dividing both equations by the total mass gives

$$v_f\cos\phi = \frac{m_F v_F \cos\theta + m_P v_P}{m_F + m_P} = 11.6 \text{ m/s}$$

$$v_f\sin\phi = \frac{m_F v_F \sin\theta}{m_F + m_P} = 8.78 \text{ m/s}$$

We can eliminate v_f by dividing the second equation by the first to give

$$\frac{v_f\sin\phi}{v_f\cos\phi} = \tan\phi = \frac{8.78 \text{ m/s}}{11.6 \text{ m/s}} = 0.757$$

$$\phi = \tan^{-1}(0.757) = 37°$$

Then $v_f = (11.6 \text{ m/s})/\cos(37°) = 15$ m/s. Immediately after impact, the falcon, with its meal, is traveling at 15 m/s at an angle 37° below the horizontal.

ASSESS It makes sense that the falcon would slow down after grabbing the slower-moving pigeon. And Figure 11.27 tells us that the total momentum is at an angle between 0° (the pigeon's momentum) and 45° (the falcon's momentum). Thus our answer seems reasonable.

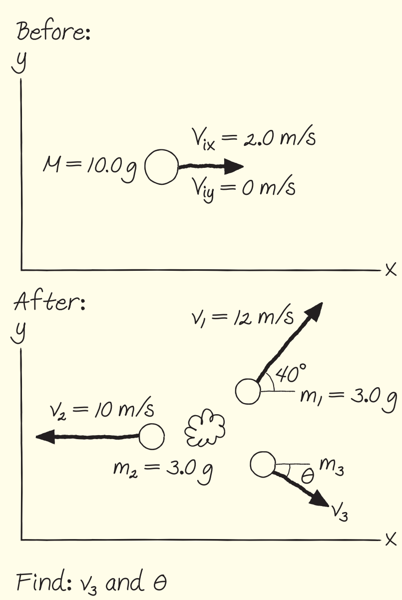
EXAMPLE 11.9 A three-piece explosion

A 10.0 g projectile is traveling east at 2.0 m/s when it suddenly explodes into three pieces. A 3.0 g fragment is shot due west at 10 m/s while another 3.0 g fragment travels 40° north of east at 12 m/s. What are the speed and direction of the third fragment?

MODEL Although many complex forces are involved in the explosion, they are all internal to the system. There are no external forces, so this is an isolated system and its total momentum is conserved.

VISUALIZE FIGURE 11.28 shows a before-and-after pictorial representation. We'll use uppercase M and V to distinguish the initial object from the three pieces into which it explodes.

FIGURE 11.28 Before-and-after pictorial representation of the three-piece explosion.



SOLVE The system is the initial object and the subsequent three pieces. Conservation of momentum requires

$$m_1(v_{fx})_1 + m_2(v_{fx})_2 + m_3(v_{fx})_3 = MV_{ix}$$

$$m_1(v_{fy})_1 + m_2(v_{fy})_2 + m_3(v_{fy})_3 = MV_{iy}$$

Conservation of mass implies that

$$m_3 = M - m_1 - m_2 = 4.0 \text{ g}$$

Neither the original object nor m_2 has any momentum along the y-axis. We can use Figure 11.28 to write out the x- and y-components of $\vec{v}_1$ and $\vec{v}_3$, leading to

$$m_1v_1\cos 40° - m_2v_2 + m_3v_3\cos\theta = MV$$

$$m_1v_1\sin 40° - m_3v_3\sin\theta = 0$$

where we used $(v_{fx})_2 = -v_2$ because m_2 is moving in the negative x-direction. Inserting known values in these equations gives us

$$-2.42 + 4v_3\cos\theta = 20$$

$$23.14 - 4v_3\sin\theta = 0$$

We can leave the masses in grams in this situation because the conversion factor to kilograms appears on both sides of the equation and thus cancels out. To solve, first use the second equation to write $v_3 = 5.79/\sin\theta$. Substitute this result into the first equation, noting that $\cos\theta/\sin\theta = 1/\tan\theta$, to get

$$-2.42 + 4\left(\frac{5.79}{\sin\theta}\right)\cos\theta = -2.42 + \frac{23.14}{\tan\theta} = 20$$

Now solve for θ:

$$\tan\theta = \frac{23.14}{20 + 2.42} = 1.03$$

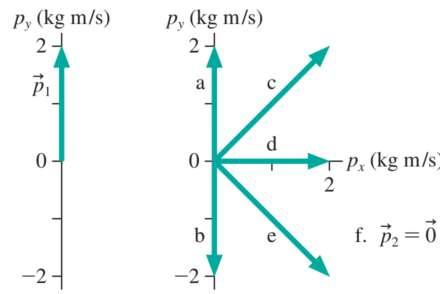
$$\theta = \tan^{-1}(1.03) = 45.8°$$

Finally, use this result in the earlier expression for v_3 to find

$$v_3 = \frac{5.79}{\sin 45.8°} = 8.1 \text{ m/s}$$

The third fragment, with a mass of 4.0 g, is shot 46° south of east at a speed of 8.1 m/s.

STOP TO THINK 11.6 An object traveling to the right with $\vec{p} = 2\hat{\imath}$ kg m/s suddenly explodes into two pieces. Piece 1 has the momentum $\vec{p}_1$ shown in the figure. What is the momentum $\vec{p}_2$ of the second piece?



11.6 ADVANCED TOPIC Rocket Propulsion

Newton's second law $\vec{F} = m\vec{a}$ applies to objects whose mass does not change. That's an excellent assumption for balls and bicycles, but what about something like a rocket that loses a significant amount of mass as its fuel is burned? Problems of varying mass are solved with momentum rather than acceleration. We'll look at one important example.

FIGURE 11.29 shows a rocket being propelled by the thrust of burning fuel but *not* influenced by gravity or drag. Perhaps it is a rocket in deep space where gravity is very weak in comparison to the rocket's thrust. This may not be highly realistic, but ignoring gravity allows us to understand the essentials of rocket propulsion without making the mathematics too complicated. Rocket propulsion with gravity is a Challenge Problem in the end-of-chapter problems.

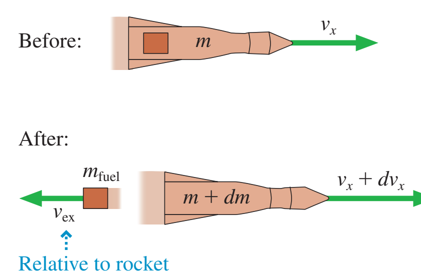


FIGURE 11.29 A before-and-after pictorial representation of a rocket burning a small amount of fuel.

The system rocket + exhaust gases is an isolated system, so its total momentum is conserved. The basic idea is simple: As exhaust gases are shot out the back, the rocket "recoils" in the opposite direction. Putting this idea on a mathematical footing is fairly straightforward—it's basically the same as analyzing an explosion—but we have to be extremely careful with signs.

We'll use a before-and-after approach, as we do with all momentum problems. The Before state is a rocket of mass m (including all onboard fuel) moving with velocity v_x and having initial momentum $P_{ix} = mv_x$. During a small interval of time dt, the rocket burns a small mass of fuel m_{fuel} and expels the resulting gases from the back of the rocket at an exhaust speed v_{ex} *relative to the rocket*. That is, a space cadet on the rocket sees the gases leaving the rocket at speed v_{ex} regardless of how fast the rocket is traveling through space.

After this little packet of burned fuel has been ejected, the rocket has new velocity $v_x + dv_x$ and new mass $m + dm$. Now you're probably thinking that this can't be right; the rocket *loses* mass rather than gaining mass. But that's *our* understanding of the physical situation. The mathematical analysis knows only that the mass changes, not whether it increases or decreases. Saying that the mass is $m + dm$ at time $t + dt$ is a formal statement that the mass has changed, and that's how analysis of change is done in calculus. The fact that the rocket's mass is decreasing means that dm has a negative value. That is, the minus goes with the value of dm, not with the statement that the mass has changed.

After the gas has been ejected, both the rocket and the gas have momentum. Conservation of momentum tells us that

$$P_{fx} = m_{\text{rocket}}(v_x)_{\text{rocket}} + m_{\text{fuel}}(v_x)_{\text{fuel}} = P_{ix} = mv_x \tag{11.35}$$

The mass of this little packet of burned fuel is the mass *lost* by the rocket: $m_{\text{fuel}} = -dm$. Mathematically, the minus sign tells us that the mass of the burned fuel (the gases) and the rocket mass are changing in opposite directions. Physically, we know that $dm < 0$, so the exhaust gases have a positive mass.

The gases are ejected toward the left at speed v_{ex} relative to the rocket. If the rocket's velocity is v_x, then the gas's velocity *through space* is $v_x - v_{ex}$. When we assemble all these pieces of information, the momentum conservation equation is

$$(m + dm)(v_x + dv_x) + (-dm)(v_x - v_{ex}) = mv_x \tag{11.36}$$

Multiplying this out gives

$$mv_x + v_x dm + m\,dv_x + dm\,dv_x - v_x dm + v_{ex}\,dm = mv_x \tag{11.37}$$

You can see that several terms cancel, leading to $m\,dv_x + v_{ex}\,dm + dm\,dv_x = 0$. We can drop the third term; it is the product of two infinitesimal terms and thus is negligible compared to the first two terms. With one final algebraic rearrangement, we're left with

$$dv_x = -v_{ex}\frac{dm}{m} \tag{11.38}$$

Remember that dm is negative—it's the mass *lost* by the rocket when a small amount of fuel is burned—and so dv_x is positive. Physically, Equation 11.38 is telling us the amount by which the rocket's velocity increases when it burns a small amount of fuel. Not surprisingly, a lighter rocket (smaller m) gains more velocity than a heavier rocket (larger m).

There are a couple of ways to use Equation 11.38. First, divide both sides by the small interval of time dt in which the fuel is burned, which will make this a rate equation:

$$\frac{dv_x}{dt} = \frac{-v_{ex}\,dm/dt}{m} = \frac{v_{ex}R}{m} \tag{11.39}$$

where $R = |dm/dt|$ is the rate—in kg/s—at which fuel is burned. The fuel burn rate is reasonably constant for most rocket engines.

The left side of Equation 11.39 is the rocket's acceleration: $a_x = dv_x/dt$. Thus from Newton's second law, $a_x = F_x/m$, the numerator on the right side of Equation 11.39 must be a force. This is the *thrust* of the rocket engine:

$$F_{thrust} = v_{ex}R \tag{11.40}$$

So Equation 11.39 is just Newton's second law, $a = F_{thrust}/m$, for the instantaneous acceleration, which will change as the rocket's mass m changes. But now we know how the thrust force is related to physical properties of the rocket engine.

Returning to Equation 11.38, we can find out how the rocket's velocity changes as fuel is burned by integrating. Suppose the rocket starts from rest $(v_x = 0)$ with mass $m_0 = m_R + m_{F0}$, where m_R is the mass of the empty rocket and m_{F0} is the initial mass of the fuel. At a later time, when the mass has been reduced to m, the velocity is v.

Integrating between this Before and After, we find

$$\int_0^v dv_x = v = -v_{ex}\int_{m_0}^{m}\frac{dm}{m} = -v_{ex}\ln m\Big|_{m_0}^{m} \tag{11.41}$$

where $\ln m$ is the *natural logarithm* (logarithm with base e) of m. Evaluating this between the limits, and using the properties of logarithms, gives

$$-v_{ex}\ln m\Big|_{m_0}^{m} = -v_{ex}(\ln m - \ln m_0) = -v_{ex}\ln\left(\frac{m}{m_0}\right) = v_{ex}\ln\left(\frac{m_0}{m}\right) \tag{11.42}$$

Thus the rocket's velocity when its mass has decreased to m is

$$v = v_{ex}\ln\left(\frac{m_0}{m}\right) \tag{11.43}$$

Initially, when $m = m_0$, $v = 0$ because $\ln 1 = 0$. The maximum speed occurs when the fuel is completely gone and $m = m_R$. This is

$$v_{max} = v_{ex}\ln\left(\frac{m_R + m_{F0}}{m_R}\right) \tag{11.44}$$

Notice that the rocket speed can exceed v_{ex} if the fuel-mass-to-rocket-mass ratio is large enough. Also, because v_{max} is greatly improved by reducing m_R, we can see why rockets that need enough speed to go into orbit are usually *multistage rockets*, dropping off the mass of the lower stages when their fuel is depleted.

EXAMPLE 11.10 Firing a rocket

Sounding rockets are small rockets used to gather weather data and do atmospheric research. One of the most popular sounding rockets has been the fairly small (10-in-diameter, 16-ft-long) Black Brant III. It is loaded with 210 kg of fuel, has a launch mass of 290 kg, and generates 49 kN of thrust for 9.0 s. What would be the maximum speed of a Black Brant III if launched from rest in deep space?

MODEL We define the system to be the rocket and its exhaust gases. This is an isolated system, its total momentum is conserved, and the rocket's maximum speed is given by Equation 11.44.

SOLVE We're given $m_{F0} = 210$ kg. Knowing that the launch mass is 290 kg, we can deduce that the mass of the empty rocket is $m_R = 80$ kg. Because the rocket burns 210 kg of fuel in 9.0 s, the fuel burn rate is

$$R = \frac{210\text{ kg}}{9.0\text{ s}} = 23.3\text{ kg/s}$$

Knowing the burn rate and the thrust, we can use Equation 11.40 to calculate the exhaust velocity:

$$v_{ex} = \frac{F_{thrust}}{R} = \frac{49{,}000\text{ N}}{23.3\text{ kg/s}} = 2100\text{ m/s}$$

Thus the rocket's maximum speed in deep space would be

$$v_{max} = v_{ex}\ln\left(\frac{m_R + m_{F0}}{m_R}\right) = (2100\text{ m/s})\ln\left(\frac{290\text{ kg}}{80\text{ kg}}\right) = 2700\text{ m/s}$$

ASSESS An actual sounding rocket doesn't reach this speed because it's affected both by gravity and by drag. Even so, the rocket's acceleration is so large that gravity plays a fairly minor role. A Black Brant III launched into the earth's atmosphere achieves a maximum speed of 2100 m/s and, because it continues to coast upward long after the fuel is exhausted, reaches a maximum altitude of 175 km (105 mi).

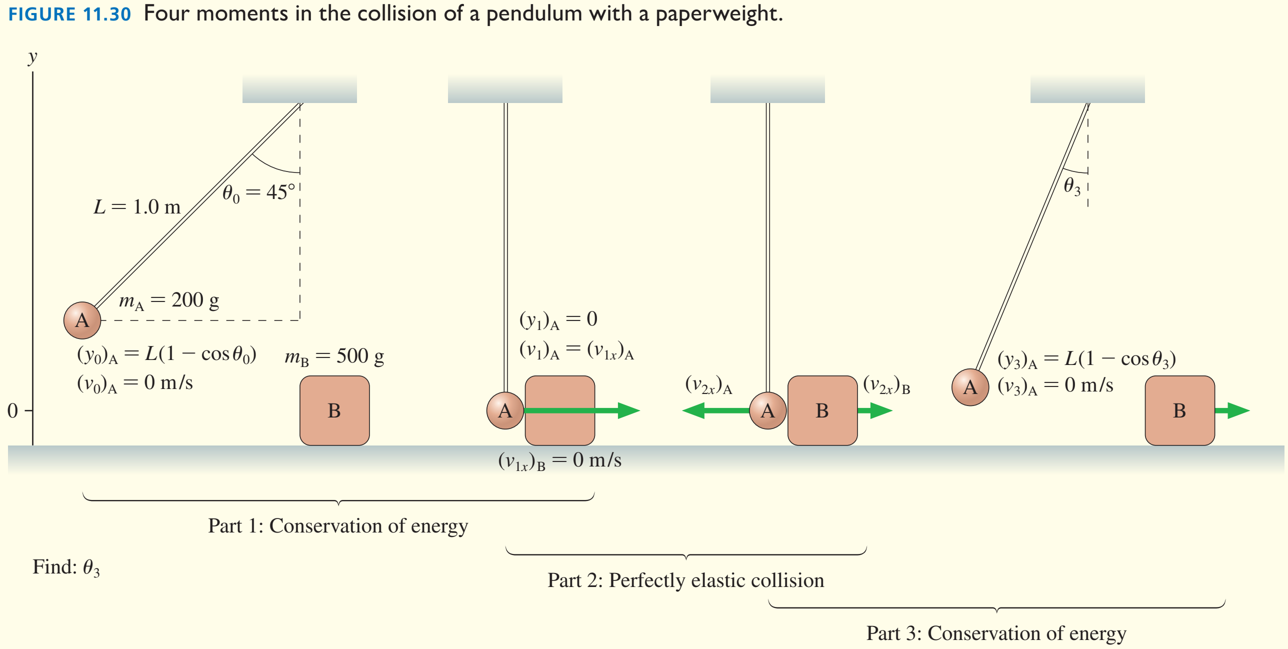
CHALLENGE EXAMPLE 11.11 A rebounding pendulum

A 200 g steel ball hangs on a 1.0-m-long string. The ball is pulled sideways so that the string is at a 45° angle, then released. At the very bottom of its swing the ball strikes a 500 g steel paperweight that is resting on a frictionless table. To what angle does the ball rebound?

MODEL We can divide this problem into three parts. First the ball swings down as a pendulum. Second, the ball and paperweight have a collision. Steel balls bounce off each other very well, so we will model the collision as perfectly elastic. Third, the ball, after it bounces off the paperweight, swings back up as a pendulum. For Parts 1 and 3, we define the system as the ball and the earth. This is an isolated, nondissipative system, so its mechanical energy is conserved. In Part 2, let the system consist of the ball and the paperweight, which have a perfectly elastic collision.

VISUALIZE FIGURE 11.30 shows four distinct moments of time: as the ball is released, an instant before the collision, an instant after the collision but before the ball and paperweight have had time to move, and as the ball reaches its highest point on the rebound. Call the ball A and the paperweight B, so $m_A = 0.20$ kg and $m_B = 0.50$ kg.

FIGURE 11.30 Four moments in the collision of a pendulum with a paperweight.



Continued

SOLVE Part 1: The first part involves the ball only. Its initial height is

$$(y_0)_A = L - L\cos\theta_0 = L(1 - \cos\theta_0) = 0.293 \text{ m}$$

We can use conservation of mechanical energy to find the ball's velocity at the bottom, just before impact on the paperweight:

$$\tfrac{1}{2}m_A(v_1)_A^2 + m_A g(y_1)_A = \tfrac{1}{2}m_A(v_0)_A^2 + m_A g(y_0)_A$$

We know $(v_0)_A = 0$. Solving for the velocity at the bottom, where $(y_1)_A = 0$, gives

$$(v_1)_A = \sqrt{2g(y_0)_A} = 2.40 \text{ m/s}$$

Part 2: The ball and paperweight undergo a perfectly elastic collision in which the paperweight is initially at rest. These are the conditions for which Equations 11.29 were derived. The velocities *immediately* after the collision, prior to any further motion, are

$$(v_{2x})_A = \frac{m_A - m_B}{m_A + m_B}(v_{1x})_A = -1.03 \text{ m/s}$$

$$(v_{2x})_B = \frac{2m_A}{m_A + m_B}(v_{1x})_A = +1.37 \text{ m/s}$$

The ball rebounds toward the left with a speed of 1.03 m/s while the paperweight moves to the right at 1.37 m/s. Kinetic energy has been conserved (you might want to check this), but it is now shared between the ball and the paperweight.

Part 3: Now the ball is a pendulum with an initial speed of 1.03 m/s. Mechanical energy is again conserved, so we can find its maximum height at the point where $(v_3)_A = 0$:

$$\tfrac{1}{2}m_A(v_3)_A^2 + m_A g(y_3)_A = \tfrac{1}{2}m_A(v_2)_A^2 + m_A g(y_2)_A$$

Solving for the maximum height gives

$$(y_3)_A = \frac{(v_2)_A^2}{2g} = 0.0541 \text{ m}$$

The height $(y_3)_A$ is related to angle θ_3 by $(y_3)_A = L(1 - \cos\theta_3)$. This can be solved to find the angle of rebound:

$$\theta_3 = \cos^{-1}\left(1 - \frac{(y_3)_A}{L}\right) = 19°$$

The paperweight speeds away at 1.37 m/s and the ball rebounds to an angle of 19°.

ASSESS The ball and the paperweight aren't hugely different in mass, so we expect the ball to transfer a significant fraction of its energy to the paperweight when they collide. Thus a rebound to roughly half the initial angle seems reasonable.

SUMMARY

The goals of Chapter 11 have been to learn to use the concepts of impulse and momentum.

GENERAL PRINCIPLES

Law of Conservation of Momentum

The total momentum $\vec{P} = \vec{p}_1 + \vec{p}_2 + \cdots$ of an isolated system is a constant. Thus

$$\vec{P}_f = \vec{P}_i$$

Newton's Second Law

In terms of momentum, Newton's second law is

$$\vec{F} = \frac{d\vec{p}}{dt}$$

Solving Momentum Conservation Problems

MODEL Choose an isolated system or a system that is isolated during at least part of the problem.

VISUALIZE Draw a pictorial representation of the system before and after the interaction.

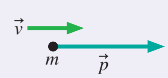
SOLVE Write the law of conservation of momentum in terms of vector components:

$$(p_{fx})_1 + (p_{fx})_2 + \cdots = (p_{ix})_1 + (p_{ix})_2 + \cdots$$
$$(p_{fy})_1 + (p_{fy})_2 + \cdots = (p_{iy})_1 + (p_{iy})_2 + \cdots$$

ASSESS Is the result reasonable?

IMPORTANT CONCEPTS

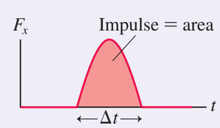
Momentum $\vec{p} = m\vec{v}$



Impulse $J_x = \int_{t_i}^{t_f} F_x(t)\,dt =$ area under force curve

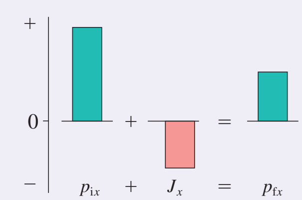
Impulse and momentum are related by the **momentum principle**

$$\Delta p_x = J_x$$



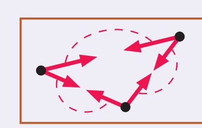
The impulse delivered to an object causes the object's momentum to change. This is an alternative statement of Newton's second law.

Momentum bar charts display the momentum principle $p_{fx} = p_{ix} + J_x$ in graphical form.



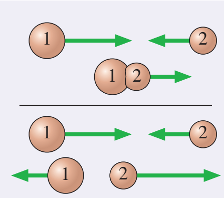
System A group of interacting particles.

Isolated system A system on which there are no external forces or the net external force is zero.

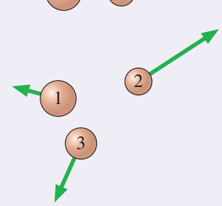


APPLICATIONS

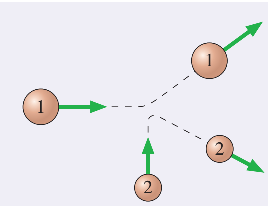
Collisions In a **perfectly inelastic collision,** two objects stick together and move with a common final velocity. In a **perfectly elastic collision,** they bounce apart and conserve mechanical energy as well as momentum.



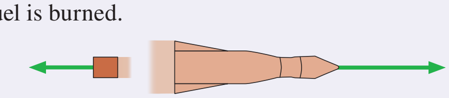
Explosions Two or more objects fly apart from each other. Their total momentum is conserved.



Two dimensions The same ideas apply in two dimensions. Both the x- and y-components of $\vec{P}$ must be conserved. This gives two simultaneous equations to solve.



Rockets The momentum of the exhaust-gas + rocket system is conserved. **Thrust** is the product of the exhaust speed and the rate at which fuel is burned.



TERMS AND NOTATION

collision	impulse, J_x	impulse approximation	perfectly inelastic collision
impulsive force	momentum principle	total momentum, $\vec{P}$	perfectly elastic collision
momentum, $\vec{p}$	momentum bar chart	law of conservation of momentum	explosion

CONCEPTUAL QUESTIONS

1. Rank in order, from largest to smallest, the momenta $(p_x)_a$ to $(p_x)_e$ of the objects in FIGURE Q11.1.

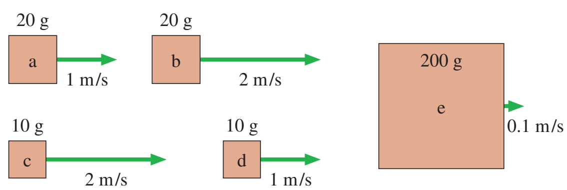


FIGURE Q11.1

2. A 2 kg object is moving to the right with a speed of 1 m/s when it experiences an impulse of 4 N s. What are the object's speed and direction after the impulse?
3. A 2 kg object is moving to the right with a speed of 1 m/s when it experiences an impulse of −4 N s. What are the object's speed and direction after the impulse?
4. A 0.2 kg plastic cart and a 20 kg lead cart can both roll without friction on a horizontal surface. Equal forces are used to push both carts forward for a time of 1 s, starting from rest. After the force is removed at $t = 1$ s, is the momentum of the plastic cart greater than, less than, or equal to the momentum of the lead cart? Explain.
5. A 0.2 kg plastic cart and a 20 kg lead cart can both roll without friction on a horizontal surface. Equal forces are used to push both carts forward for a distance of 1 m, starting from rest. After traveling 1 m, is the momentum of the plastic cart greater than, less than, or equal to the momentum of the lead cart? Explain.
6. Angie, Brad, and Carlos are discussing a physics problem in which two identical bullets are fired with equal speeds at equal-mass wood and steel blocks resting on a frictionless table. One bullet bounces off the steel block while the second becomes embedded in the wood block. "All the masses and speeds are the same," says Angie, "so I think the blocks will have equal speeds after the collisions." "But what about momentum?" asks Brad. "The bullet hitting the wood block transfers all its momentum and energy to the block, so the wood block should end up going faster than the steel block." "I think the bounce is an important factor," replies Carlos. "The steel block will be faster because the bullet bounces off it and goes back the other direction." Which of these three do you agree with, and why?
7. It feels better to catch a hard ball while wearing a padded glove than to catch it bare handed. Use the ideas of this chapter to explain why.
8. Automobiles are designed with "crumple zones" intended to collapse in a collision. Use the ideas of this chapter to explain why.
9. A golf club continues forward after hitting the golf ball. Is momentum conserved in the collision? Explain, making sure you are careful to identify "the system."
10. Suppose a rubber ball collides head-on with a more massive steel ball traveling in the opposite direction with equal speed. Which ball, if either, receives the larger impulse? Explain.
11. Two particles collide, one of which was initially moving and the other initially at rest.
    a. Is it possible for *both* particles to be at rest after the collision? Give an example in which this happens, or explain why it can't happen.
    b. Is it possible for *one* particle to be at rest after the collision? Give an example in which this happens, or explain why it can't happen.
12. Two ice skaters, Paula and Ricardo, push off from each other. Ricardo weighs more than Paula.
    a. Which skater, if either, has the greater momentum after the push-off? Explain.
    b. Which skater, if either, has the greater speed after the push-off? Explain.
13. Two balls of clay of known masses hang from the ceiling on massless strings of equal length. They barely touch when both hang at rest. One ball is pulled back until its string is at 45°, then released. It swings down, collides with the second ball, and they stick together. To determine the angle to which the balls swing on the opposite side, would you invoke (a) conservation of momentum, (b) conservation of mechanical energy, (c) both, (d) either but not both, or (e) these laws alone are not sufficient to find the angle? Explain.

EXERCISES AND PROBLEMS

Problems labeled integrate material from earlier chapters.

Exercises

Section 11.1 Momentum and Impulse

1. | At what speed do a bicycle and its rider, with a combined mass of 100 kg, have the same momentum as a 1500 kg car traveling at 5.0 m/s?
2. | What is the magnitude of the momentum of
    a. A 3000 kg truck traveling at 15 m/s?
    b. A 200 g baseball thrown at 40 m/s?
3. || What impulse does the force shown in FIGURE EX11.3 exert on a 250 g particle?

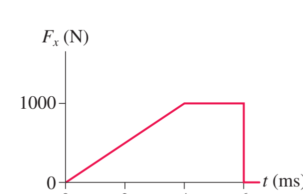


FIGURE EX11.3

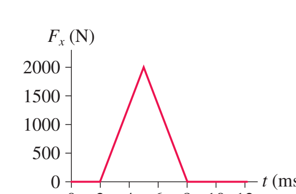


FIGURE EX11.4

4. || What is the impulse on a 3.0 kg particle that experiences the force shown in FIGURE EX11.4?

5. | In **FIGURE EX11.5**, what value of F_{max} gives an impulse of 6.0 N s?

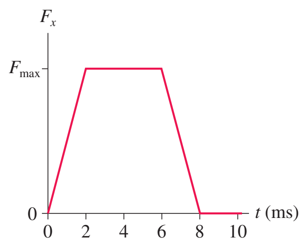


FIGURE EX11.5

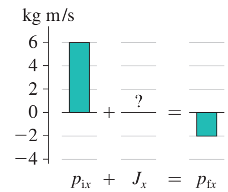


FIGURE EX11.6

6. | **FIGURE EX11.6** is an incomplete momentum bar chart for a collision that lasts 10 ms. What are the magnitude and direction of the average collision force exerted on the object?

7. | **FIGURE EX11.7** is an incomplete momentum bar chart for a 50 g particle that experiences an impulse lasting 10 ms. What were the speed and direction of the particle before the impulse?

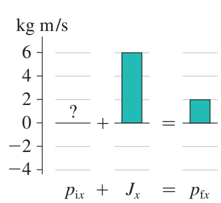


FIGURE EX11.7

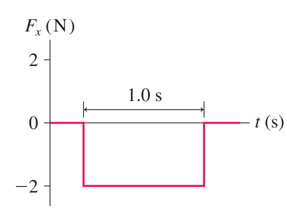


FIGURE EX11.8

8. | A 2.0 kg object is moving to the right with a speed of 1.0 m/s when it experiences the force shown in **FIGURE EX11.8**. What are the object's speed and direction after the force ends?

9. | A 2.0 kg object is moving to the right with a speed of 1.0 m/s when it experiences the force shown in **FIGURE EX11.9**. What are the object's speed and direction after the force ends?

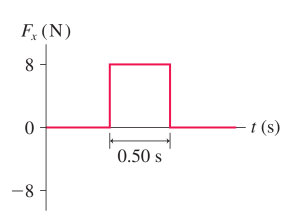


FIGURE EX11.9

10. | A sled slides along a horizontal surface on which the coefficient of kinetic friction is 0.25. Its velocity at point A is 8.0 m/s and at point B is 5.0 m/s. Use the momentum principle to find how long the sled takes to travel from A to B.

11. | Far in space, where gravity is negligible, a 425 kg rocket traveling at 75 m/s fires its engines. **FIGURE EX11.11** shows the thrust force as a function of time. The mass lost by the rocket during these 30 s is negligible.
   a. What impulse does the engine impart to the rocket?
   b. At what time does the rocket reach its maximum speed? What is the maximum speed?

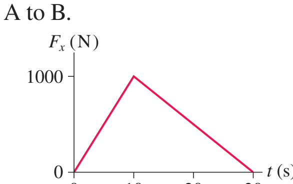


FIGURE EX11.11

12. | A 600 g air-track glider collides with a spring at one end of the track. **FIGURE EX11.12** shows the glider's velocity and the force exerted on the glider by the spring. How long is the glider in contact with the spring?

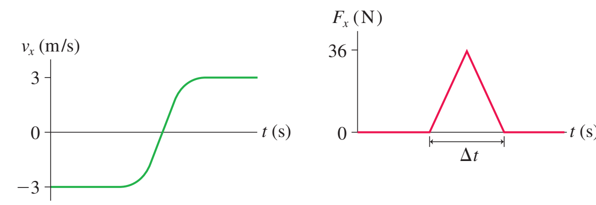


FIGURE EX11.12

13. | A 250 g ball collides with a wall. **FIGURE EX11.13** shows the ball's velocity and the force exerted on the ball by the wall. What is v_{fx}, the ball's rebound velocity?

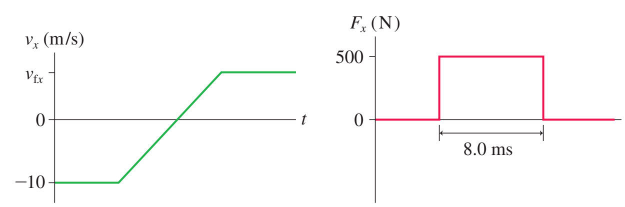


FIGURE EX11.13

Section 11.2 Conservation of Momentum

14. | A 5000 kg open train car is rolling on frictionless rails at 22 m/s when it starts pouring rain. A few minutes later, the car's speed is 20 m/s. What mass of water has collected in the car?

15. | A 10,000 kg railroad car is rolling at 2.0 m/s when a 4000 kg load of gravel is suddenly dropped in. What is the car's speed just after the gravel is loaded?

16. | A 10-m-long glider with a mass of 680 kg (including the passengers) is gliding horizontally through the air at 30 m/s when a 60 kg skydiver drops out by releasing his grip on the glider. What is the glider's velocity just after the skydiver lets go?

17. | Three identical train cars, coupled together, are rolling east at speed v_0. A fourth car traveling east at $2v_0$ catches up with the three and couples to make a four-car train. A moment later, the train cars hit a fifth car that was at rest on the tracks, and it couples to make a five-car train. What is the speed of the five-car train?

Section 11.3 Collisions

18. | A 300 g bird flying along at 6.0 m/s sees a 10 g insect heading straight toward it at a speed of 30 m/s. The bird opens its mouth wide and enjoys a nice lunch. What is the bird's speed immediately after swallowing?

19. | The parking brake on a 2000 kg Cadillac has failed, and it is rolling slowly, at 1.0 mph, toward a group of small children. Seeing the situation, you realize you have just enough time to drive your 1000 kg Volkswagen head-on into the Cadillac and save the children. With what speed should you impact the Cadillac to bring it to a halt?

20. | A 1500 kg car is rolling at 2.0 m/s. You would like to stop the car by firing a 10 kg blob of sticky clay at it. How fast should you fire the clay?

21. Fred (mass 60 kg) is running with the football at a speed of 6.0 m/s when he is met head-on by Brutus (mass 120 kg), who is moving at 4.0 m/s. Brutus grabs Fred in a tight grip, and they fall to the ground. Which way do they slide, and how far? The coefficient of kinetic friction between football uniforms and Astroturf is 0.30.

22. A 50 g marble moving at 2.0 m/s strikes a 20 g marble at rest. What is the speed of each marble immediately after the collision?

23. A proton is traveling to the right at 2.0×10^7 m/s. It has a head-on perfectly elastic collision with a carbon atom. The mass of the carbon atom is 12 times the mass of the proton. What are the speed and direction of each after the collision?

24. A 50 g ball of clay traveling at speed v_0 hits and sticks to a 1.0 kg brick sitting at rest on a frictionless surface.
   a. What is the speed of the brick after the collision?
   b. What percentage of the mechanical energy is lost in this collision?

25. A package of mass m is released from rest at a warehouse loading dock and slides down the 3.0-m-high, frictionless chute of FIGURE EX11.25 to a waiting truck. Unfortunately, the truck driver went on a break without having removed the previous package, of mass $2m$, from the bottom of the chute.
   a. Suppose the packages stick together. What is their common speed after the collision?
   b. Suppose the collision between the packages is perfectly elastic. To what height does the package of mass m rebound?

FIGURE EX11.25

Section 11.4 Explosions

26. A 50 kg archer, standing on frictionless ice, shoots a 100 g arrow at a speed of 100 m/s. What is the recoil speed of the archer?

27. A 70.0 kg football player is gliding across very smooth ice at 2.00 m/s. He throws a 0.450 kg football straight forward. What is the player's speed afterward if the ball is thrown at
   a. 15.0 m/s relative to the ground?
   b. 15.0 m/s relative to the player?

28. Dan is gliding on his skateboard at 4.0 m/s. He suddenly jumps backward off the skateboard, kicking the skateboard forward at 8.0 m/s. How fast is Dan going as his feet hit the ground? Dan's mass is 50 kg and the skateboard's mass is 5.0 kg.

29. Two ice skaters, with masses of 50 kg and 75 kg, are at the center of a 60-m-diameter circular rink. The skaters push off against each other and glide to opposite edges of the rink. If the heavier skater reaches the edge in 20 s, how long does the lighter skater take to reach the edge?

30. A ball of mass m and another ball of mass $3m$ are placed inside a smooth metal tube with a massless spring compressed between them. When the spring is released, the heavier ball flies out of one end of the tube with speed v_0. With what speed does the lighter ball emerge from the other end?

Section 11.5 Momentum in Two Dimensions

31. Two particles collide and bounce apart. FIGURE EX11.31 shows the initial momenta of both and the final momentum of particle 2. What is the final momentum of particle 1? Write your answer using unit vectors.

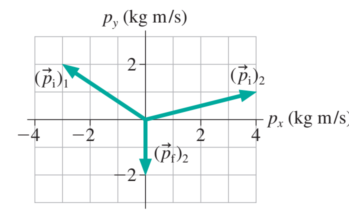


FIGURE EX11.31

32. An object at rest explodes into three fragments. FIGURE EX11.32 shows the momentum vectors of two of the fragments. What is the momentum of the third fragment? Write your answer using unit vectors.

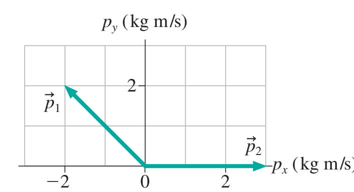


FIGURE EX11.32

33. A 20 g ball of clay traveling east at 3.0 m/s collides with a 30 g ball of clay traveling north at 2.0 m/s. What are the speed and the direction of the resulting 50 g ball of clay? Give your answer as an angle north of east.

34. At the center of a 50-m-diameter circular ice rink, a 75 kg skater traveling north at 2.5 m/s collides with and holds on to a 60 kg skater who had been heading west at 3.5 m/s.
   a. How long will it take them to glide to the edge of the rink?
   b. Where will they reach it? Give your answer as an angle north of west.

Section 11.6 Rocket Propulsion

35. A small rocket with 15 kN thrust burns 250 kg of fuel in 30 s. What is the exhaust speed of the hot gases?

36. A rocket in deep space has an empty mass of 150 kg and exhausts the hot gases of burned fuel at 2500 m/s. What mass of fuel is needed to reach a top speed of 4000 m/s?

37. A rocket in deep space has an exhaust-gas speed of 2000 m/s. When the rocket is fully loaded, the mass of the fuel is five times the mass of the empty rocket. What is the rocket's speed when half the fuel has been burned?

Problems

38. A tennis player swings her 1000 g racket with a speed of 10 m/s. She hits a 60 g tennis ball that was approaching her at a speed of 20 m/s. The ball rebounds at 40 m/s.
   a. How fast is her racket moving immediately after the impact? You can ignore the interaction of the racket with her hand for the brief duration of the collision.
   b. If the tennis ball and racket are in contact for 10 ms, what is the average force that the racket exerts on the ball? How does this compare to the gravitational force on the ball?

39. ‖ A 60 g tennis ball with an initial speed of 32 m/s hits a wall and rebounds with the same speed. **FIGURE P11.39** shows the force of the wall on the ball during the collision. What is the value of F_{max}, the maximum value of the contact force during the collision?

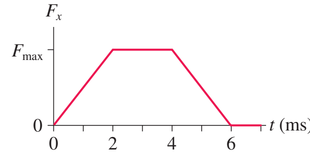


FIGURE P11.39

40. ‖ A 500 g cart is released from rest 1.00 m from the bottom of a frictionless, 30.0° ramp. The cart rolls down the ramp and bounces off a rubber block at the bottom. **FIGURE P11.40** shows the force during the collision. After the cart bounces, how far does it roll back up the ramp?

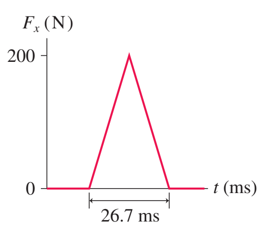


FIGURE P11.40

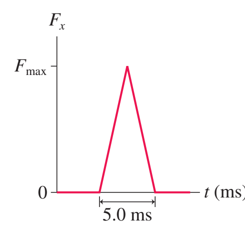


FIGURE P11.41

41. ‖‖ A 200 g ball is dropped from a height of 2.0 m, bounces on a hard floor, and rebounds to a height of 1.5 m. **FIGURE P11.41** shows the impulse received from the floor. What maximum force does the floor exert on the ball?

42. ‖ The flowers of the bunchberry plant open with astonishing
BIO force and speed, causing the pollen grains to be ejected out of the flower in a mere 0.30 ms at an acceleration of 2.5×10^4 m/s^2. If the acceleration is constant, what impulse is delivered to a pollen grain with a mass of 1.0×10^{-7} g?

43. | A particle of mass m is at rest at $t = 0$. Its momentum for $t > 0$
CALC is given by $p_x = 6t^2$ kg m/s, where t is in s. Find an expression for $F_x(t)$, the force exerted on the particle as a function of time.

44. ‖ Air-track gliders with masses 300 g, 400 g, and 200 g are lined up and held in place with lightweight springs compressed between them. All three are released at once. The 200 g glider flies off to the right while the 300 g glider goes left. Their position-versus-time graphs, as measured by motion detectors, are shown in **FIGURE P11.44**. What are the direction (right or left) and speed of the 400 g glider that was in the middle?

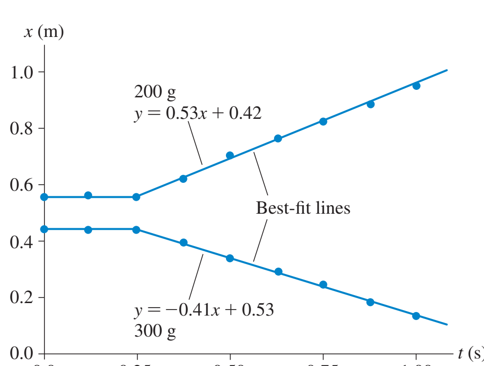


FIGURE P11.44

45. ‖ Most geologists believe that the dinosaurs became extinct 65 million years ago when a large comet or asteroid struck the earth, throwing up so much dust that the sun was blocked out for a period of many months. Suppose an asteroid with a diameter of 2.0 km and a mass of 1.0×10^{13} kg hits the earth (6.0×10^{24} kg) with an impact speed of 4.0×10^4 m/s.
   a. What is the earth's recoil speed after such a collision? (Use a reference frame in which the earth was initially at rest.)
   b. What percentage is this of the earth's speed around the sun? The earth orbits the sun at a distance of 1.5×10^{11} m.

46. ‖ Squids rely on jet propulsion to move around. A 1.50 kg squid
BIO (including the mass of water inside the squid) drifting at 0.40 m/s suddenly ejects 0.10 kg of water to get itself moving at 2.50 m/s. If drag is ignored over the small interval of time needed to expel the water (the impulse approximation), what is the water's ejection speed relative to the squid?

47. ‖ A firecracker in a coconut blows the coconut into three pieces. Two pieces of equal mass fly off south and west, perpendicular to each other, at speed v_0. The third piece has twice the mass as the other two. What are the speed and direction of the third piece? Give the direction as an angle east of north.

48. ‖ One billiard ball is shot east at 2.0 m/s. A second, identical billiard ball is shot west at 1.0 m/s. The balls have a glancing collision, not a head-on collision, deflecting the second ball by 90° and sending it north at 1.41 m/s. What are the speed and direction of the first ball after the collision? Give the direction as an angle south of east.

49. ‖ a. A bullet of mass m is fired into a block of mass M that is at rest. The block, with the bullet embedded, slides distance d across a horizontal surface. The coefficient of kinetic friction is μ_k. Find an expression for the bullet's speed v_{bullet}.
   b. What is the speed of a 10 g bullet that, when fired into a 10 kg stationary wood block, causes the block to slide 5.0 cm across a wood table?

50. ‖ You are part of a search-and-rescue mission that has been called out to look for a lost explorer. You've found the missing explorer, but, as **FIGURE P11.50** shows, you're separated from him by a 200-m-high cliff and a 30-m-wide raging river. To save his life, you need to get a 5.0 kg package of emergency supplies across the river. Unfortunately, you can't throw the package hard enough to make it across. Fortunately, you happen to have a 1.0 kg rocket intended for launching flares. Improvising quickly, you attach a sharpened stick to the front of the rocket, so that it will impale itself into the package of supplies, then fire the rocket at ground level toward the supplies. What minimum speed must the rocket have just before impact in order to save the explorer's life?

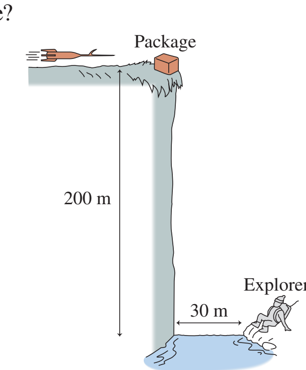


FIGURE P11.50

51. ‖ An object at rest on a flat, horizontal surface explodes into two fragments, one seven times as massive as the other. The heavier fragment slides 8.2 m before stopping. How far does the lighter fragment slide? Assume that both fragments have the same coefficient of kinetic friction.

52. || A 1500 kg weather rocket accelerates upward at 10 m/s^2. It explodes 2.0 s after liftoff and breaks into two fragments, one twice as massive as the other. Photos reveal that the lighter fragment traveled straight up and reached a maximum height of 530 m. What were the speed and direction of the heavier fragment just after the explosion?

53. || In a ballistics test, a 25 g bullet traveling horizontally at 1200 m/s goes through a 30-cm-thick 350 kg stationary target and emerges with a speed of 900 m/s. The target is free to slide on a smooth horizontal surface. What is the target's speed just after the bullet emerges?

54. || Two 500 g blocks of wood are 2.0 m apart on a frictionless table. A 10 g bullet is fired at 400 m/s toward the blocks. It passes all the way through the first block, then embeds itself in the second block. The speed of the first block immediately afterward is 6.0 m/s. What is the speed of the second block after the bullet stops in it?

55. ||| A 100 g granite cube slides down a 40° frictionless ramp. At the bottom, just as it exits onto a horizontal table, it collides with a 200 g steel cube at rest. How high above the table should the granite cube be released to give the steel cube a speed of 150 cm/s?

56. ||| You have been asked to design a "ballistic spring system" to measure the speed of bullets. A spring whose spring constant is k is suspended from the ceiling. A block of mass M hangs from the spring. A bullet of mass m is fired vertically upward into the bottom of the block and stops in the block. The spring's maximum compression d is measured.
   a. Find an expression for the bullet's speed v_B in terms of m, M, k, and d.
   b. What was the speed of a 10 g bullet if the block's mass is 2.0 kg and if the spring, with $k = 50$ N/m, was compressed by 45 cm?

57. || In **FIGURE P11.57**, a block of mass m slides along a frictionless track with speed v_m. It collides with a stationary block of mass M. Find an expression for the minimum value of v_m that will allow the second block to circle the loop-the-loop without falling off if the collision is (a) perfectly inelastic or (b) perfectly elastic.

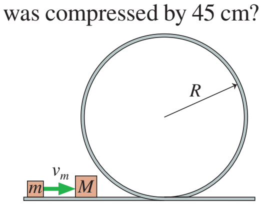


FIGURE P11.57

58. || The stoplight had just changed and a 2000 kg Cadillac had entered the intersection, heading north at 3.0 m/s, when it was struck by a 1000 kg eastbound Volkswagen. The cars stuck together and slid to a halt, leaving skid marks angled 35° north of east. How fast was the Volkswagen going just before the impact?

59. || Ann (mass 50 kg) is standing at the left end of a 15-m-long, 500 kg cart that has frictionless wheels and rolls on a frictionless track. Initially both Ann and the cart are at rest. Suddenly, Ann starts running along the cart at a speed of 5.0 m/s relative to the cart. How far will Ann have run *relative to the ground* when she reaches the right end of the cart?

60. ||| CALC Force $F_x = (10\text{ N})\sin(2\pi t/4.0\text{ s})$ is exerted on a 250 g particle during the interval $0\text{ s} \le t \le 2.0\text{ s}$. If the particle starts from rest, what is its speed at $t = 2.0$ s?

61. ||| CALC A 500 g particle has velocity $v_x = -5.0$ m/s at $t = -2$ s. Force $F_x = (4 - t^2)$ N, where t is in s, is exerted on the particle between $t = -2$ s and $t = 2$ s. This force increases from 0 N at $t = -2$ s to 4 N at $t = 0$ s and then back to 0 N at $t = 2$ s. What is the particle's velocity at $t = 2$ s?

62. || A 30 ton rail car and a 90 ton rail car, initially at rest, are connected together with a giant but massless compressed spring between them. When released, the 30 ton car is pushed away at a speed of 4.0 m/s relative to the 90 ton car. What is the speed of the 30 ton car relative to the ground?

63. ||| A 20 g ball is fired horizontally with speed v_0 toward a 100 g ball hanging motionless from a 1.0-m-long string. The balls undergo a head-on, perfectly elastic collision, after which the 100 g ball swings out to a maximum angle $\theta_{max} = 50°$. What was v_0?

64. || A 100 g ball moving to the right at 4.0 m/s catches up and collides with a 400 g ball that is moving to the right at 1.0 m/s. If the collision is perfectly elastic, what are the speed and direction of each ball after the collision?

65. || A 100 g ball moving to the right at 4.0 m/s collides head-on with a 200 g ball that is moving to the left at 3.0 m/s.
   a. If the collision is perfectly elastic, what are the speed and direction of each ball after the collision?
   b. If the collision is perfectly inelastic, what are the speed and direction of the combined balls after the collision?

66. || Old naval ships fired 10 kg cannon balls from a 200 kg cannon. It was very important to stop the recoil of the cannon, since otherwise the heavy cannon would go careening across the deck of the ship. In one design, a large spring with spring constant 20,000 N/m was placed behind the cannon. The other end of the spring braced against a post that was firmly anchored to the ship's frame. What was the speed of the cannon ball if the spring compressed 50 cm when the cannon was fired?

67. || A proton (mass 1 u) is shot toward an unknown target nucleus at a speed of 2.50×10^6 m/s. The proton rebounds with its speed reduced by 25% while the target nucleus acquires a speed of 3.12×10^5 m/s. What is the mass, in atomic mass units, of the target nucleus?

68. || The nucleus of the polonium isotope ^{214}Po (mass 214 u) is radioactive and decays by emitting an alpha particle (a helium nucleus with mass 4 u). Laboratory experiments measure the speed of the alpha particle to be 1.92×10^7 m/s. Assuming the polonium nucleus was initially at rest, what is the recoil speed of the nucleus that remains after the decay?

69. || A neutron is an electrically neutral subatomic particle with a mass just slightly greater than that of a proton. A free neutron is radioactive and decays after a few minutes into other subatomic particles. In one experiment, a neutron at rest was observed to decay into a proton (mass 1.67×10^{-27} kg) and an electron (mass 9.11×10^{-31} kg). The proton and electron were shot out back-to-back. The proton speed was measured to be 1.0×10^5 m/s and the electron speed was 3.0×10^7 m/s. No other decay products were detected.
   a. Did momentum seem to be conserved in the decay of this neutron?

   NOTE Experiments such as this were first performed in the 1930s and seemed to indicate a failure of the law of conservation of momentum. In 1933, Wolfgang Pauli postulated that the neutron might have a *third* decay product that is virtually impossible to detect. Even so, it can carry away just enough momentum to keep the total momentum conserved. This proposed particle was named the *neutrino,* meaning "little neutral one." Neutrinos were, indeed, discovered nearly 20 years later.

   b. If a neutrino was emitted in the above neutron decay, in which direction did it travel? Explain your reasoning.
   c. How much momentum did this neutrino "carry away" with it?

70. | A 20 g ball of clay traveling east at 2.0 m/s collides with a 30 g ball of clay traveling 30° south of west at 1.0 m/s. What are the speed and direction of the resulting 50 g blob of clay?

71. | FIGURE P11.71 shows a collision between three balls of clay. The three hit simultaneously and stick together. What are the speed and direction of the resulting blob of clay?

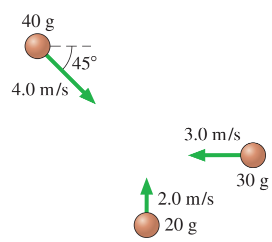


FIGURE P11.71

72. | A 2100 kg truck is traveling east through an intersection at 2.0 m/s when it is hit simultaneously from the side and the rear. (Some people have all the luck!) One car is a 1200 kg compact traveling north at 5.0 m/s. The other is a 1500 kg midsize traveling east at 10 m/s. The three vehicles become entangled and slide as one body. What are their speed and direction just after the collision?

73. | A rocket in deep space has an empty mass of 150 kg and exhausts the hot gases of burned fuel at 2500 m/s. It is loaded with 600 kg of fuel, which it burns in 30 s. What is the rocket's speed 10 s, 20 s, and 30 s after launch?

74. | a. To understand why rockets often have multiple stages, first consider a single-stage rocket with an empty mass of 200 kg, 800 kg of fuel, and a 2000 m/s exhaust speed. If fired in deep space, what is the rocket's maximum speed?
   b. Now divide the rocket into two stages, each with an empty mass of 100 kg, 400 kg of fuel, and a 2000 m/s exhaust speed. The first stage is released after it runs out of fuel. What is the top speed of the second stage? You'll need to consider how the equation for v_{max} should be altered when a rocket is not starting from rest.

In Problems 75 through 78 you are given the equation(s) used to solve a problem. For each of these, you are to

a. Write a realistic problem for which this is the correct equation(s).
b. Finish the solution of the problem, including a pictorial representation.

75. $(0.10\text{ kg})(40\text{ m/s}) - (0.10\text{ kg})(-30\text{ m/s}) = \frac{1}{2}(1400\text{ N})\Delta t$

76. $(600\text{ g})(4.0\text{ m/s}) = (400\text{ g})(3.0\text{ m/s}) + (200\text{ g})(v_{ix})_2$

77. $(3000\text{ kg})v_{fx} = (2000\text{ kg})(5.0\text{ m/s}) + (1000\text{ kg})(-4.0\text{ m/s})$

78. $(0.10\text{ kg} + 0.20\text{ kg})v_{1x} = (0.10\text{ kg})(3.0\text{ m/s})$

$$\tfrac{1}{2}(0.30\text{ kg})(0\text{ m/s})^2 + \tfrac{1}{2}(3.0\text{ N/m})(\Delta x_2)^2 = \tfrac{1}{2}(0.30\text{ kg})(v_{1x})^2 + \tfrac{1}{2}(3.0\text{ N/m})(0\text{ m})^2$$

Challenge Problems

79. ||| A 1000 kg cart is rolling to the right at 5.0 m/s. A 70 kg man is standing on the right end of the cart. What is the speed of the cart if the man suddenly starts running to the left with a speed of 10 m/s relative to the cart?

80. ||| A spaceship of mass 2.0×10^6 kg is cruising at a speed of 5.0×10^6 m/s when the antimatter reactor fails, blowing the ship into three pieces. One section, having a mass of 5.0×10^5 kg, is blown straight backward with a speed of 2.0×10^6 m/s. A second piece, with mass 8.0×10^5 kg, continues forward at 1.0×10^6 m/s. What are the direction and speed of the third piece?

81. ||| A 20 kg wood ball hangs from a 2.0-m-long wire. The maximum tension the wire can withstand without breaking is 400 N. A 1.0 kg projectile traveling horizontally hits and embeds itself in the wood ball. What is the greatest speed this projectile can have without causing the wire to break?

82. ||| A two-stage rocket is traveling at 1200 m/s with respect to the earth when the first stage runs out of fuel. Explosive bolts release the first stage and push it backward with a speed of 35 m/s relative to the second stage. The first stage is three times as massive as the second stage. What is the speed of the second stage after the separation?

83. ||| The air-track carts in FIGURE CP11.83 are sliding to the right at 1.0 m/s. The spring between them has a spring constant of 120 N/m and is compressed 4.0 cm. The carts slide past a flame that burns through the string holding them together. Afterward, what are the speed and direction of each cart?

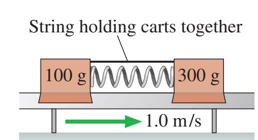


FIGURE P11.83

84. ||| CALC Section 11.6 found an equation for v_{max} of a rocket fired in deep space. What is v_{max} for a rocket fired vertically from the surface of an airless planet with free-fall acceleration g? Referring to Section 11.6, you can write an equation for ΔP_y, the change of momentum in the vertical direction, in terms of dm and dv_y. ΔP_y is no longer zero because now gravity delivers an impulse. Rewrite the momentum equation by including the impulse due to gravity during the time dt during which the mass changes by dm. Pay attention to signs! Your equation will have three differentials, but two are related through the fuel burn rate R. Use this relationship—again pay attention to signs; m is decreasing—to write your equation in terms of dm and dv_y. Then integrate to find a modified expression for v_{max} at the instant all the fuel has been burned.
   a. What is v_{max} for a vertical launch from an airless planet? Your answer will be in terms of m_R, the empty rocket mass; m_{F0}, the initial fuel mass; v_{ex}, the exhaust speed; R, the fuel burn rate; and g.
   b. A rocket with a total mass of 330,000 kg when fully loaded burns all 280,000 kg of fuel in 250 s. The engines generate 4.1 MN of thrust. What is this rocket's speed at the instant all the fuel has been burned if it is launched in deep space? If it is launched vertically from the earth?

PART II

KNOWLEDGE STRUCTURE

Conservation Laws

KEY FINDINGS What are the overarching findings of Part II?

- **Work** causes a system's energy to change. It is a *transfer* of energy to or from the environment.
- **Energy** can be *transformed* within a system, but the **total energy** of an isolated system does not change.
- An **impulse** causes a system's momentum to change.
- Momentum can be *exchanged* within a system, but the **total momentum** of an isolated system does not change.

LAWS What laws of physics govern energy and momentum?

Energy principle $\Delta E_{sys} = \Delta K + \Delta U + \Delta E_{th} = W_{ext}$ or $K_i + U_i + W_{ext} = K_f + U_f + \Delta E_{th}$

Conservation of energy For an isolated system ($W_{ext} = 0$), the total energy $E_{sys} = K + U + E_{th}$ is conserved. $\Delta E_{sys} = 0$. For an isolated, nondissipative system, the **mechanical energy** $E_{mech} = K + U$ is conserved.

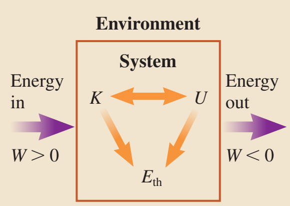
Momentum principle $\Delta \vec{p} = \vec{J}$

Conservation of momentum For an isolated system ($\vec{J} = \vec{0}$), the total momentum is conserved. $\Delta \vec{P} = \vec{0}$.

MODELS What are the most common models for using conservation laws?

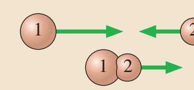
Basic energy model

- Energy is a property of the system.
- Energy is *transformed* within the system without loss.
- Energy is *transferred* to and from the system by forces that do *work*.
  - $W > 0$ for energy added.
  - $W < 0$ for energy removed.
  - $\Delta E_{sys} = \Delta K + \Delta U + \Delta E_{th} = W_{ext}$

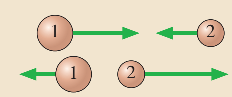


Collision model

- Perfectly inelastic collision: Objects stick together and move with a common final velocity.
  - Momentum is conserved.
  - $(m_1 + m_2)v_{fx} = m_1(v_{ix})_1 + m_2(v_{ix})_2$

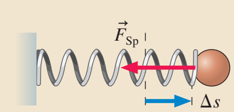


- Perfectly elastic collision: Objects bounce apart with no loss of energy.
  - Momentum and energy are both conserved.



Other models and approximations

- An isolated system does not interact with its environment.
  - For energy, an isolated system has no work done on it.
  - For momentum, an isolated system experiences no impulse.
- Thermal energy is the microscopic energy of moving atoms and stretched bonds.
- An ideal spring obeys **Hooke's law** for all displacements: $(F_{Sp})_s = -k\,\Delta s$.

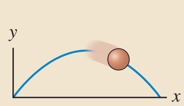


- The impulse approximation ignores forces that are small compared to impulsive forces during the brief time of a collision or explosion.

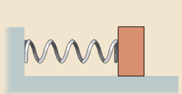
TOOLS What are the most important tools for using energy and momentum?

- Kinetic energy:
  $K = \frac{1}{2}mv^2$

  - Kinetic energy is an energy of *motion*.
- Gravitational potential energy:
  $U_G = mgy$



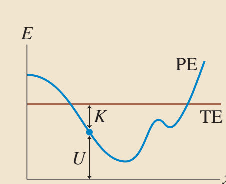
- Elastic potential energy:
  $U_{Sp} = \frac{1}{2}k(\Delta s)^2$



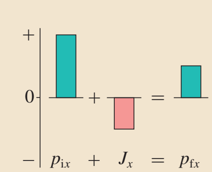
  - Potential energy is an energy of *position*.
- Momentum: $\vec{p} = m\vec{v}$.

- Force acting through a displacement does work.
  - Constant force:
    $W = \vec{F} \cdot \Delta\vec{r} = F(\Delta r)\cos\theta$
  - Variable force:
    $$W = \int_{s_i}^{s_f} F_s\,ds$$
    = area under the F_s-versus-s curve
- Force acting over time delivers an impulse:
  $$J_x = \int_{t_i}^{t_f} F_x\,dt$$
  = area under the F_x-versus-t curve

- Energy diagrams
  - Visualize speed changes and turning points. F_x = negative of the slope of the curve.



- Bar charts
  - Momentum and energy conservation.



- Before-and-after pictorial representation.
  - Important problem-solving tool.