Chapter 4: Exponential and Logarithmic Equations

Section 4.1: Composite Functions

Exploration 1*: Form a Composite Function

1. Suppose you have a job that pays $10 per hour. Write a function, g that can be used to determine your gross pay (your pay before taxes are taken out) per hour, h, that you worked.
   $g(h) =$

2. Now let's write a formula for how much money you'll actually take home of that paycheck. Let's assume your employer withholds 20% of your gross pay for taxes. Write a function, n, that determines your net pay based off of your gross income, g.
   $n(g) =$

3. How much money would you net if you worked for 20 hours?

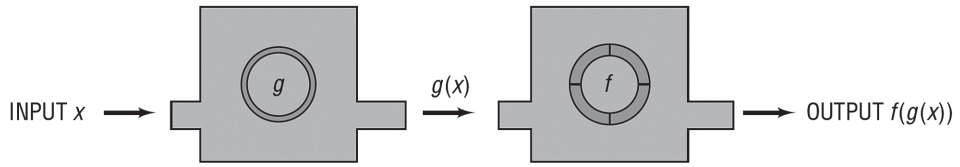
4. Instead of having to use two different functions to find out your net pay, as you most likely did in (3), let's combine our functions from (1) and (2) and write them as one function. This is called ***composing functions.***

   Write a function that relates the number of hours worked, h, to your net pay, n.

Definition: Given two functions f and g, the **composite function,** denoted by _____ (read as "f composed with g") is defined by_____.

Note: $f \circ g$ does *not* mean f multiplied by $g(x)$. It means input the function g into the function f: $f \circ g = f\left(g\left(x\right)\right) \neq f(x) \bullet g(x)$

The domain of $f \circ g$ is the set of all numbers x in the domain of g such that $g(x)$ is in the domain of f. In other words, $f \circ g$ is defined whenever both $g(x)$ and $f\left(g\left(x\right)\right)$ are defined.



Example 1*: Form a Composite Function; Evaluate a Composite Function
Suppose that $f(x) = 2x^2 + 3$ and $g(x) = 4x^3 + 1$. Find:
(a) $(f \circ g)(1)$ (b) $(g \circ f)(1)$ (c) $(f \circ f)(-2)$ (d) $(g \circ g)(-1)$

Example 2: Evaluate a Composite Function
If $f(x)$ and $g(x)$ are polynomial functions, use the table of values for $f(x)$ and $g(x)$ to complete the table of values for $(f \circ g)(x)$.

x	$g(x)$
-2	4
-1	1
0	0
1	1
2	4

x	$f(x)$
0	3
1	4
2	5
3	6
4	7

x	$(f \circ g)(x)$
-2	
-1	
0	
1	
3	

Example 3: Find the Domain of a Composite Function
Suppose that $f(x) = 2x^2 + 3$ and $g(x) = 4x^3 + 1$. Find the following and their domains:
(a) $f \circ g$ (b) $g \circ f$

Example 4*: Find the Domain of a Composite Function

Find the domain of $(f \circ g)(x)$ if $f(x) = \dfrac{1}{x+4}$ and $g(x) = \dfrac{4}{x-2}$.

Example 5: Find a Composite Function and Its Domain

Suppose that $f(x) = \dfrac{1}{x}$ and $g(x) = \sqrt{x-1}$. Find the following and their domains:

(a) $f \circ g$

(b) $f \circ f$

Example 6*: Showing Two Composite Functions Are Equal

If $f(x) = 2x$ and $g(x) = \dfrac{1}{2}x$, show that $(f \circ g)(x) = (g \circ f)(x) = x$ for every x in the domain of $f \circ g$ and $g \circ f$.

Example 7: Showing Two Composite Functions Are Equal

If $f(x)=\frac{1}{2}(x-1)$ and $g(x)=2x+1,$ show that $(f\circ g)(x)=(g\circ f)(x)=x$ for every x in the domain of $f\circ g$ and $g\circ f$.

Example 8*: Find the Components of a Composite Function

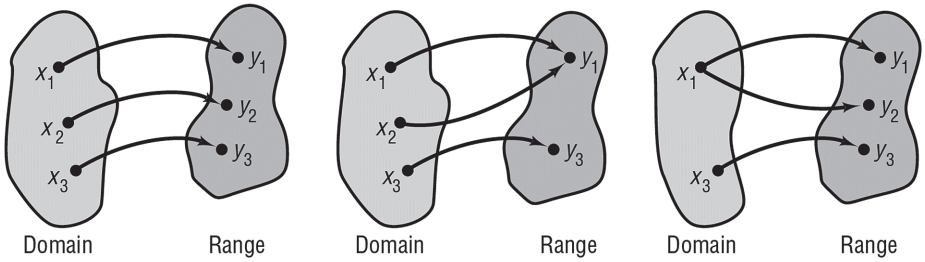
Find functions f and g such that $f\circ g=H$ if $H(x)=(2x+3)^4$

Example 9: Find the Components of a Composite Function

Find functions f and g such that $f\circ g=H$ if $H(x)=\frac{1}{2x^2-3}$

Chapter 4: Exponential and Logarithmic Functions
Section 4.2: One-to-One Functions; Inverse Functions

Definition: A function is **one – to – one** if any two different inputs in the domain correspond to _____. That is, if x_1 and x_2 are two different inputs of a function f, is one – to – one if _____.



(a) One-to-one function: Each x in the domain has one and only one image in the range.

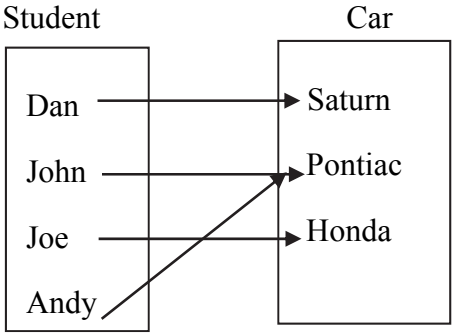
(b) Not a one-to-one function: y_1 is the image of both x_1 and x_2.

(c) Not a function: x_1 has two images, y_1 and y_2.

Example 1*: Determine Whether a Function is One – to – One

Determine whether the following functions are one – to – one. Explain why or why not.

(a)



(b)

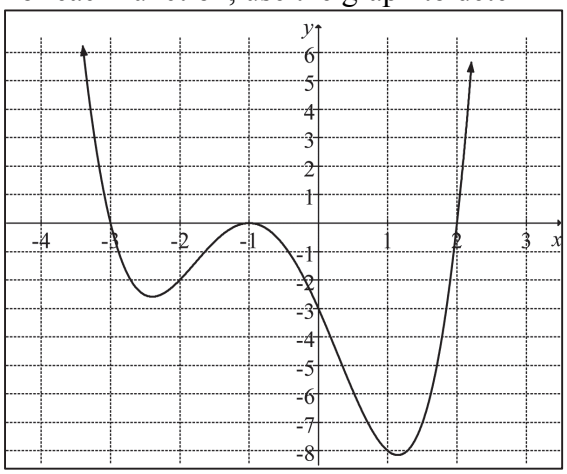
{(1,5), (2,8), (3,11), (4,14)}

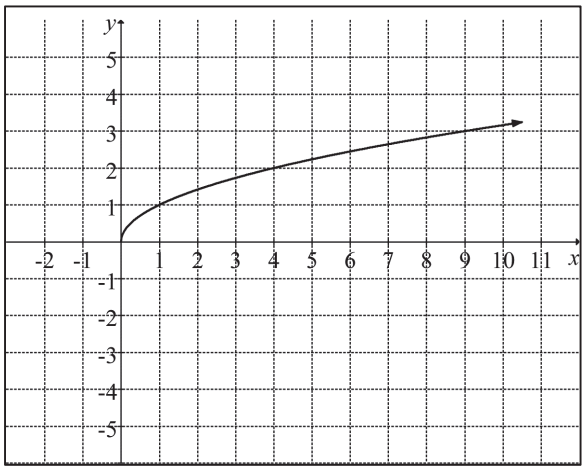
The Horizontal Line Test Theorem: If every horizontal line intersects the graph of a function f in at most _____, then f is one – to – one.

Why does this test work? You may want to refer to the definition of one – to – one functions.

Example 2: Determine whether a Function is One – to – One Using the Horizontal Line Test

For each function, use the graph to determine whether the function is one – to – one.





Theorem: A function that is increasing on an interval *I* is a one – to – one function on *I*.
A function that is decreasing on an interval *I* is a one – to – one function on *I*.

Why is this theorem true?

Exploration 1: Inverse Functions – Reverse the Process

You might have experienced converting between degrees Fahrenheit and degrees Celsius when measuring a temperature. The standard formula for determining temperature in degrees Fahrenheit, when given the temperature in degrees Celsius, is $F = \frac{9}{5}C + 32$. We can use this formula to define a function named *g*, namely $F = g(C) = \frac{9}{5}C + 32$, where C is the number of degrees Celsius and $g(C)$ is a number of degrees Fahrenheit. The function *g* defines a process for converting degrees Celsius to degrees Fahrenheit.

1. What is the value of $g(100)$? What does it represent?

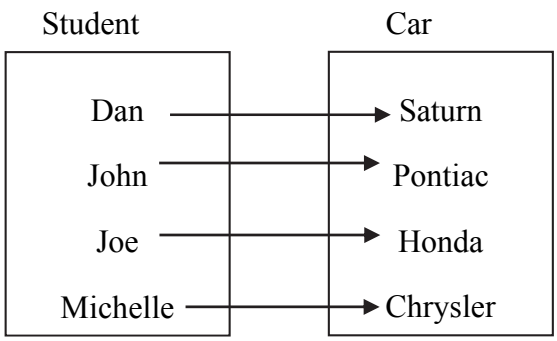
2. Solve the equation $g(C) = 112$ and describe the meaning of your answer.

3. What happens if you want to input degree Fahrenheit and output degree Celsius?
   Reverse the process of the formula $F = \frac{9}{5}C + 32$ by solving for C.

Definition: Suppose that f is a one – to – one function. Then, to each x in the domain of f, there is _____ y in the range (because f is a function); and to each y in the range of f there is exactly one x in the domain (because f is one – to – one). The correspondence from the range of f back to the _____ of f is called the **inverse function of *f*.** We use the symbol f^{-1} to denote the inverse of f. **Note:** $f^{-1} \neq \frac{1}{f}$

In other words, two ***functions*** are said to be inverses of each other if they are the reverse process of each other. Notice in the exploration, the formula found in part (c) was the **reverse process of *g*.** Instead of inputting Celsius and outputting Fahrenheit, the new function inputs Fahrenheit and outputs Celsius.

Example 3*: Determine the Inverse of a Function
Find the inverse of the following function. Let the domain of the function represent certain students, and let the range represent the make of that student's car. State the domain and the range of the inverse function.

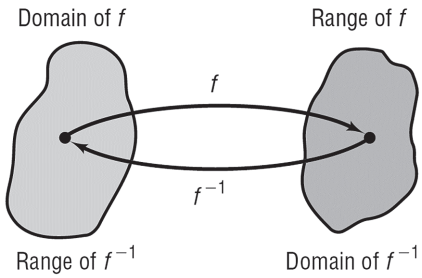


Example 4*: Determine the Inverse of a Function
Find the inverse of the following one – to – one function. Then state the domain and range of the function and its inverse.

$$\{(1,5), (2,8), (3,11), (4,14)\}$$

Domain and Range of Inverse Functions: Since the inverse function, f^{-1}, is a reverse mapping of the function f:

Domain of f = _____ of f^{-1} and Range of f = _____ of f^{-1}



Fact: What f does, f^{-1} undoes and vice versa. Therefore,

$f^{-1}\left(f\left(x\right)\right) =$ ____ where x is in the domain of f

$f\left(f^{-1}\left(x\right)\right) =$ ____ where x is in the domain of f^{-1}

We can use this fact to verify if two functions are inverses of each other.

Example 5*: Determine the Inverse of a Function ; Verifying Inverse Functions

Verify that the inverse of $g\left(x\right) = x^3 + 2$ is $g^{-1}\left(x\right) = \sqrt[3]{x-2}$ by showing that $g\left(g^{-1}\left(x\right)\right) = x$ for all x in the domain of g and that $g^{-1}\left(g\left(x\right)\right) = x$ for all x in the domain of g^{-1}.

Exploration 2: Graphs of Inverse Functions

1. Using a graphing utility, graph the following functions on the same screen

$$y = x,\ y = x^3, \text{ and } y = \sqrt[3]{x}$$

2. What do you notice about the graphs of $y = x^3$, its inverse $y = \sqrt[3]{x}$, and the line $y = x$?

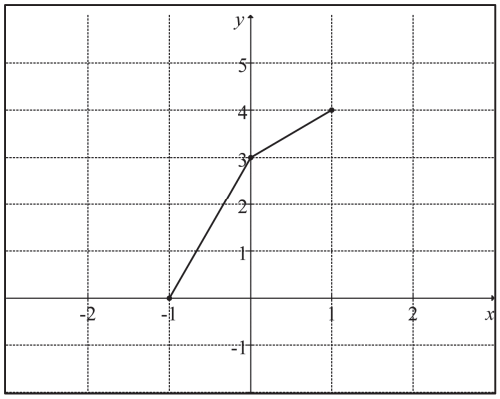
3. Repeat this experiment by graphing the following functions on the same screen:

$$y = x,\ y = 2x + 3, \text{ and } y = \frac{1}{2}(x-3)$$

4. What do you notice about the graphs of $y = 2x + 3$, its inverse $y = \frac{1}{2}(x-3)$, and the line $y = x$?

Theorem: The graph of a one – to – one function *f* and the graph of its inverse f^{-1} are symmetric with respect to the line _____.

Example 6*: Obtain the Graph of the Inverse Function
The graph shown is that of a one – to – one function. Draw the graph of its inverse.



Procedure for Finding the Inverse of a One – to – One Function

Step 1: In $y = f(x)$, interchange the variables *x* and *y* to obtain _____. This equation defines the inverse function f^{-1} implicitly.

Step 2: If possible, solve the implicit equation for *y* in terms of *x* to obtain the explicit form of f^{-1}: _____.

Step 3: Check the result by showing that _____ and _____.

Example 7: Find the Inverse Function from an Equation
Find the inverse of $f(x) = -\frac{1}{3}x + 1$

Example 8*: Find the Inverse Function from an Equation
The function $f(x) = \frac{2x-1}{x+1}$, $x \neq -1$ is one – to – one. Find its inverse and state the domain and range of both *f* and its inverse function.

Chapter 4: Exponential and Logarithmic Functions
Section 4.3: Exponential Functions

Before delving into exponential functions, let's make sure we can use our calculators to evaluate exponential expressions.

Most calculators have either an $\boxed{x^y}$ key or a carot key $\boxed{\wedge}$ for working with exponents. To evaluate expressions of the form a^x, enter the base *a*, then press the $\boxed{x^y}$ key (or $\boxed{\wedge}$), enter the exponent *x*, and press $\boxed{=}$, (or $\boxed{\text{enter}}$).

Example 1*: Evaluating Exponential Functions: Using the Calculator
Using a calculator, evaluate:

(a) $2^{1.4}$ (b) $2^{1.41}$ (c) $2^{1.414}$ (d) $2^{1.4142}$ (e) $2^{\sqrt{2}}$

Laws of Exponents

If *s, t, a,* and *b* are real numbers $a > 0$ and $b > 0$ then,

1) $a^s \cdot a^t =$ _____ 2) $(a^s)^t =$ _____ 3) $(ab)^s =$ _____

4) $1^s =$ _____ 5) $a^{-s} =$ ____ = ____ 6) $(a)^0 =$ _____

Exploration 1: Exponential Functions
There is a classic riddle involving a person's choice of pay for a job. The person is given two choices:
Option A: \$1 million dollars right now that covers the whole month, or
Option B: a payment of 1¢ just for taking the job that doubles each day starting on day 1 until the end of the month.

Which would you choose?

We can easily tell how much money we would have at the end of the month with Option A, $1,000,000, but we don't know the total for Option B. Let's explore this by creating a table of values that models this situation (we will assume there are 30 days in a month).

Days	Pay ($)
0	0.01
1	0.02
2	
3	
4	
5	
6	
7	
8	
9	

Days	Pay ($)
10	
11	
12	
13	
14	
15	
16	
17	
18	
19	

Days	Pay ($)
20	
21	
22	
23	
24	
25	
26	
27	
28	
29	
30	

How much money would you have at the end of the month if you chose Option B?

Create a formula to calculate the pay, P, in dollars as a function of the number of days, d, after the first day for option B. Note: P represents the payment on the particular day, d, not the total overall payment. To get you started we've shown how the first couple of days in the chart are calculated. See if you can follow the pattern to create your formula.

$$P(0) = 0.01$$
$$P(1) = 2P(0) = 2(0.01)$$
$$P(2) = 2P(1) = 2\big(2(0.01)\big)$$
$$P(3) = 2P(2) = 2\Big(2\big(2(0.01)\big)\Big)$$
$$P(4) = 2P(3) = 2\Big(2\Big(2\big(2(0.01)\big)\Big)\Big)$$

$P(d) =$ _____

Option B in the exploration is an example of an exponential growth function.

Definition: An **exponential function** is a function of the form _____ where a is a positive real number ($a > 0$) and $a \neq 1$ and $C \neq 0$ is a real number. The domain of f is _____.

The base a is the _____ **factor**, and because $f(0) = Ca^0 = C$, C is called the _____.

Example 1: Identify the Parts of an Exponential Function
Find the initial value, C, and growth or decay factor, a, of the following exponential functions.

a. $f(x) = 4(0.86)^x$ b. $d(t) = 100(1+.04)^t$ c. $n(r) = 3^r$

Think/Pair/Share: We will discuss this more in depth later –do you have any thoughts about what makes a number a growth factor verses a decay factor?

Exploration 2*: Evaluate Exponential Functions: Linear or Exponential?
Now that we have studied both linear and exponential functions, we should be able to look at data and determine whether it is either of these functions. But how? Let's explore this.

1. Evaluate $f(x) = 2^x$ and $g(x) = 3x + 2$ at $x = -2, -1, 0, 1, 2,$ and 3

x	$f(x) = 2^x$
-2	
-1	
0	
1	
2	
3	

x	$g(x) = 3x + 2$
-2	
-1	
0	
1	
2	
3	

2. Comment on the patterns that exist in the values of f and g.

Theorem: For an exponential function, $f(x) = a^x$, $a > 1, a \neq 1$, if x is any real number, then

$$\frac{f(x+1)}{f(x)} = \_\_\_\_\_ \quad \text{or} \quad f(x+1) = \_\_\_\_\_$$

Example 2: Identify Linear or Exponential Functions
Determine whether the given function is linear, exponential, or neither. For those that are linear, find a linear function that models the data. For those that are exponential, find an exponential function that models the data.

(a)

x	$y = f(x)$	Average Rate of Change	Ratio of consecutive outputs
-1	-4.5		
0	-3		
1	-1.5		
2	0		
3	1.5		

$f(x) =$ _____

(b) .

x	$y = g(x)$	Average Rate of Change	Ratio of consecutive outputs
-1	-4.5		
0	-3		
1	-1.5		
2	0		
3	1.5		

$g(x) =$ _____

(c)

x	$y = h(x)$	Average Rate of Change	Ratio of consecutive outputs
-1	20		
0	16		
1	12		
2	8		
3	4		

$h(x) =$ _____

(d) .

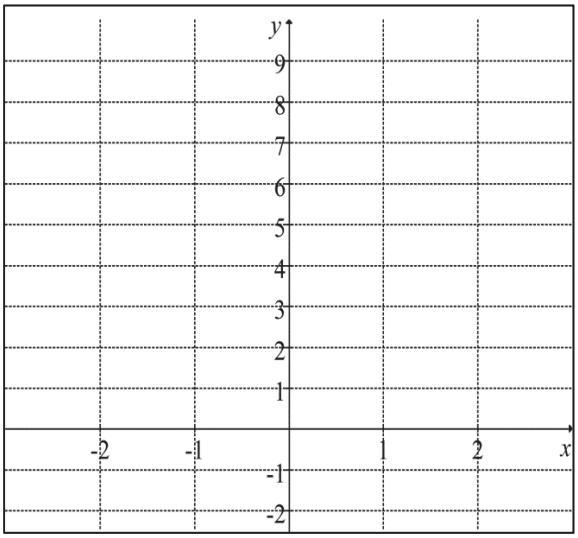
x	$y = j(x)$	Average Rate of Change	Ratio of consecutive outputs
-1	2		
0	3		
1	4.5		
2	6.75		
3	10.125		

$j(x) =$ _____

Exploration 3*: Graph Exponential Functions

1. Consider the functions $f(x) = 2^x$ and $g(x) = 3^x$. Graph these functions by filling in the table below. Label each of your graphs.

x	$y = f(x)$
-2	
-1	
0	
1	
2	



x	$y = g(x)$
-3	
-2	
-1	
0	
1	
2	

(a) What are the domain and range of each of these functions?

(b) Can $y = 0$? Why or why not?

(c) Does the function have any symmetry?

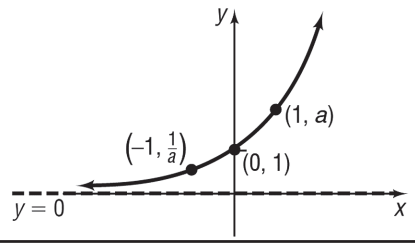
(d) What are the x and y – intercepts?

(e) Notice that both of these functions are increasing as x increases. What does this mean? Which function is increasing faster?

(f) Do these functions have a horizontal asymptote?

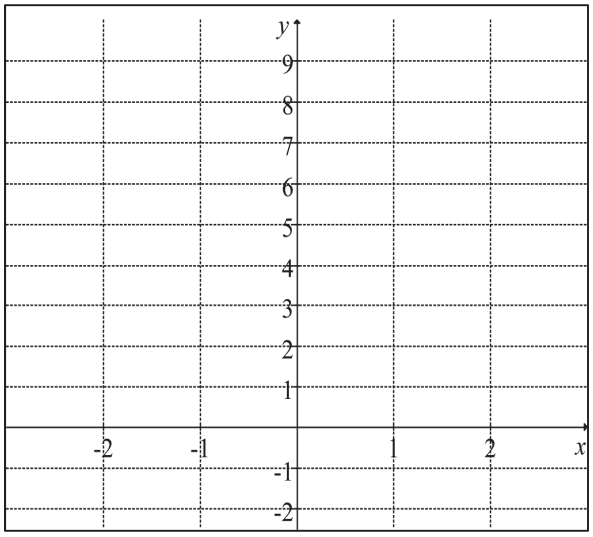
Properties of the Exponential Function $f(x) = a^x,\ a > 1$

1. The domain is the set of all real numbers or _____ using interval notation; the range is the set of positive real numbers or _____ using interval notation.
2. There are ____ x – intercepts; the y – intercept is ____.
3. The x – axis (_____) is a horizontal asymptote as _____ [_____].
4. $f(x) = a^x,\ a > 1$ is an _____ function and is _____.
5. The graph of f contains the points _____, _____, and _____.
6. The figure of f is smooth and continuous with no corners or gaps.



2. Now consider the functions $f(x) = \left(\frac{1}{2}\right)^x$ and $g(x) = \left(\frac{1}{3}\right)^x$. Graph these functions by filling in the table below. Label each of your graphs.

x	$y = f(x)$
-2	
-1	
0	
1	
2	



x	$y = g(x)$
-2	
-1	
0	
1	
2	

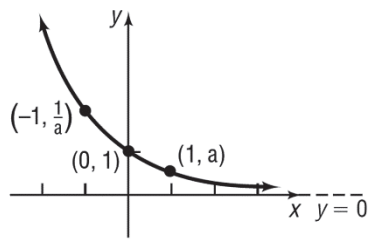
(a) Why do these functions decrease when the graphs in (1) increased (as x gets larger)?

(b) What are the domain and range of each of these functions?

(c) Do these functions have any asymptotes?

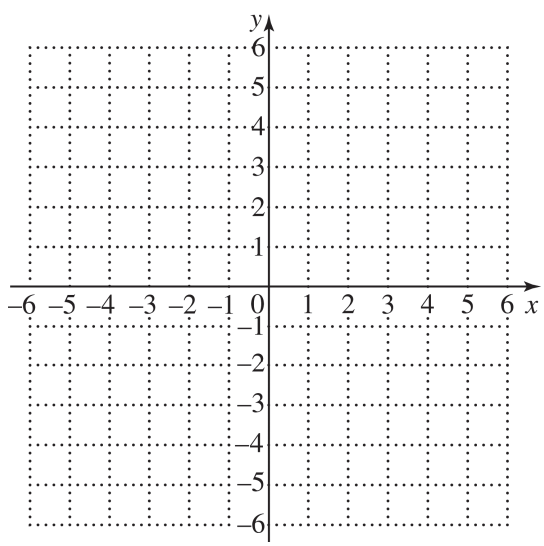
Properties of the Exponential Function $f(x) = a^x,\ 0 < a < 1$

1. The domain is the set of all real numbers or _____ using interval notation; the range is the set of positive real numbers or _____ using interval notation.
2. There are ____ x – intercepts; the y – intercept is ____.
3. The x – axis (_____) is a horizontal asymptote as _____ [_____].
4. $f(x) = a^x,\ 0 < a < 1$ is a _____ function and is _____.
5. The graph of f contains the points _____, _____, and _____.
6. The figure of f is smooth and continuous with no corners or gaps.



Example 3*: Graphing Exponential Functions Using Transformations

Graph $f(x) = 2 \cdot 3^{x+1} - 4$ and determine the domain, range, and horizontal asymptote of f. Make sure you graph and label the asymptote(s).



As we saw in our Exploration 1 exponential functions are often used in applications involving money. The act of doubling our money each day says that we are experiencing 100% growth each day. In many examples involving money, we experience growth on a cycle other than per day. Sometimes, our money may grow annually, quarterly, or monthly. These different cycles are called different compounding periods. As we compound more and more often, we say that we are compounding continuously. What is interesting is that as these compound periods approach ∞, we reach a limit. This limit is the number e.

Exploration 4*: Define the Number *e*

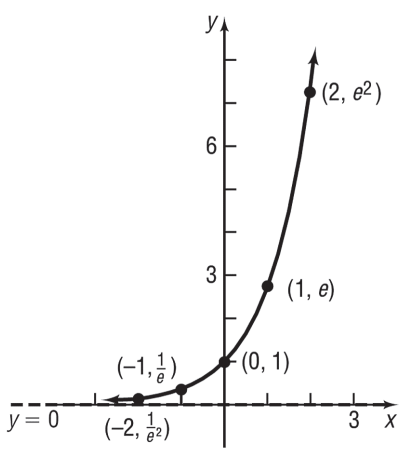
The number ***e*** is defined as $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$.

Let's explore this value by filling in this table using a graphing utility:

n	$f(n) = \left(1+\frac{1}{n}\right)^n$
10	
50	
100	
500	
1000	
10,000	
100,000	
1,000,000	

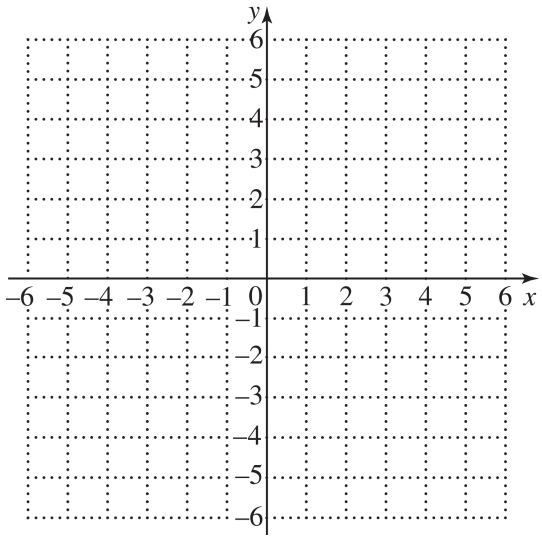
Based on the table, what is e approximately?

Confirm the approximate value of e by typing in e into your calculator



We will do more applications with the number e in Section 4.7

Example 4*: Define the Number *e;* Graph *e* Using Transformations
Graph $f(x) = -e^{x-2}$ and determine the domain, range, and horizontal asymptote of *f*.



Solving Exponential Equations

Now that we know what exponential functions are let's learn about how we can solve exponential equations. For example, how would you solve the following:

$$5^{x+3} = \frac{1}{5}$$

What makes this equation different from equations we've seen before?

Solve Exponential Equations If $a^u = a^v$, then _____ .

This means that if you have the same bases on both sides of the equals sign, you set the exponents equal. The key here is to manipulate as needed so that the **base is the same**.

Example 5*: Solve Exponential Equations
Solve each equation.

(a)* $2^{3x-1} = 32$

(b) $5^x = 5^{-6}$

(c) $4^{2x-5} = \frac{1}{16}$

(d) $2^{2x-1} = 4$

(e) $5^{x^2+8} = 125^{2x}$

(f) $9^{2x} \cdot 27^{x^2} = 3^{-1}$

(g)* $e^{2x-1} = \frac{1}{e^{3x}} \cdot \left(e^{-x}\right)^4$

(h) $\left(\frac{1}{2}\right)^{x-5} = \left(8^x\right)\left(2^{x^2}\right)$

Chapter 4: Exponential and Logarithmic Functions
Section 4.4: Logarithmic Functions

Exploration 1: Logarithms
Before we define a logarithm, let's play around with them a little. See if you can follow the pattern below to be able to fill in the missing pieces to a – f.

$\log_3 9 = 2$	$\log_9 3 = \frac{1}{2}$
$\log_4 16 = 2$	$\log_3 27 = 3$

(a) $\log_2 8 =$ ___ (b) $\log_4 16 =$ ___

(c) $\log\_\_ 64 = 2$ (d) $\log\_\_ 64 = 3$

(e) $\log_2$____ $= 4$ (f) $\log_4 2 =$ ___

Logarithms - A logarithm is just a power
For example, $\log_2(32) = 5$ says "the logarithm to the base 2 of 32 is 5." It means 2 to the 5th power is 32. Notice that both in logarithms and exponents, the same number is called the base.

The **logarithmic function to the base a**, where $a > 0$ and $a \neq 1$, is denoted by $y = \log_a x$ (read as "y is the logarithm to the base a of x") and is defined by:

_____

The domain of the logarithmic function $y = \log_a x$ is _____.

Example 1*: Convert Exponential to Logarithmic Statements
Change each exponential equation to an equivalent equation involving a logarithm
(a) $5^8 = t$ (b) $x^{-2} = 12$ (c) $e^x = 10$

Example 2*: Convert Logarithmic to Exponential Statements
Change each logarithmic equation to an equivalent equation involving an exponent.

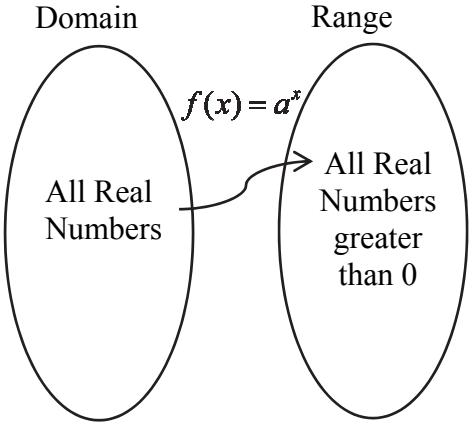
(a) $y = \log_2 21$ (b) $\log_z 12 = 6$ (c) $\log_2 10 = a$

Example 3*: Evaluate Logarithmic Expressions
Evaluate the following:

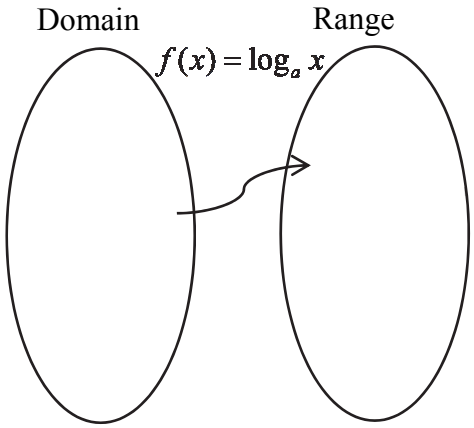
(a)* $\log_3(81)$ (b)* $\log_2 \frac{1}{8}$ (c) $\log_5(1)$ (d) $\log_2(16)$

(e) $\log_3(9)$ (f) $\log_4(2)$ (g) $\log_{1/3}(27)$ (h) $\log_{\sqrt{5}}(25)$

Let's recall the domain and range of an exponential function:



Since a logarithmic function is the *inverse* of an exponential function, fill in the domain and range below based on what we learned in Section 4.2.



Domain and Range of the Logarithmic Function $y = \log_a(x)$ (defining equation $x = a^y$)

Domain:_____ Range:_____

Example 4*: Determine the Domain of a Logarithmic Function
Find the domain of each logarithmic function.

(a) $f(x) = \log_3(x-2)$

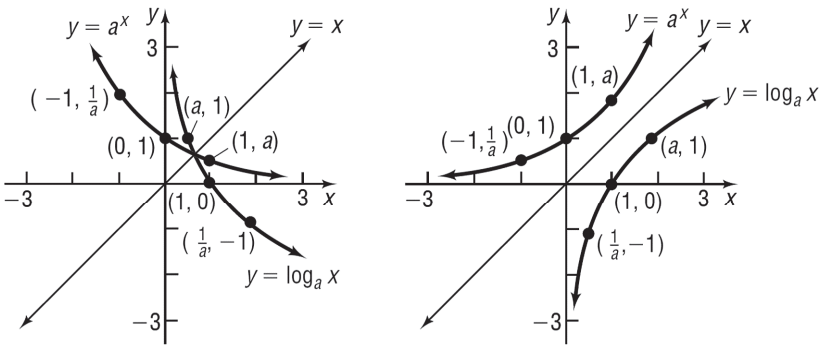
(b) $F(x) = \log_2\left(\dfrac{x+3}{x-1}\right)$

(c) $h(x) = \log_2|x-1|$

(d) $g(x) = \log_{\frac{1}{2}} x^2$

Properties of the Logarithmic Function $f(x) = \log_a(x)$

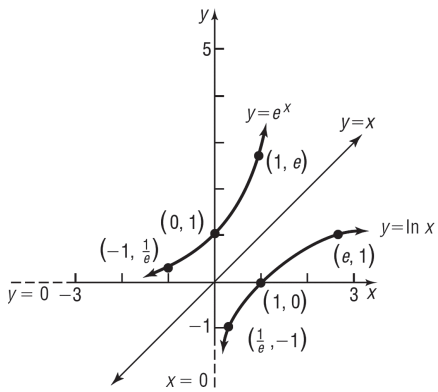
1. The domain _____; The range is _____.
2. The x-intercept is _____. There is _____ y-intercept.
3. The y-axis ($x = 0$) is a _____ asymptote of the graph.
4. A logarithmic function is decreasing if _____ and increasing if _____.
5. The graph of f contains the points _____.
6. The graph is _____, with no _____.



(a) $0 < a < 1$ **(b)** $a > 1$

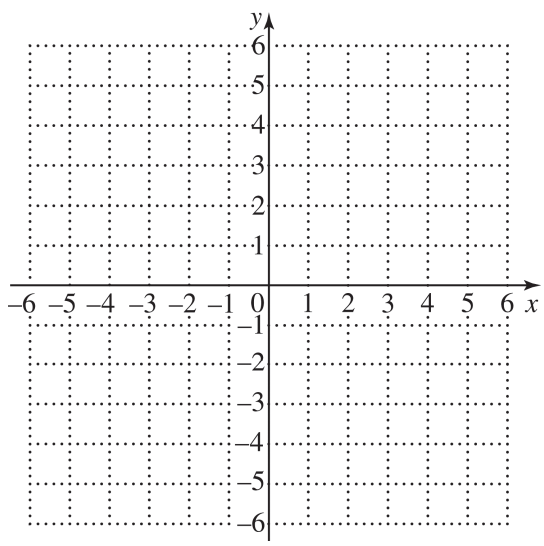
Fact

Natural Logarithm: $\ln(x)$ means $\log_e(x)$. It is derived from the Latin phrase, *logarithmus naturalis*. In other words, $y = \ln(x)$ if and only if $x = e^y$.



Example 5*: Graph Logarithmic Functions

(a)* Graph $f(x) = 3\ln(x-1)$.



(b)* State the domain of $f(x)$.

(c)* From the graph, determine the range and vertical asymptote of *f*.

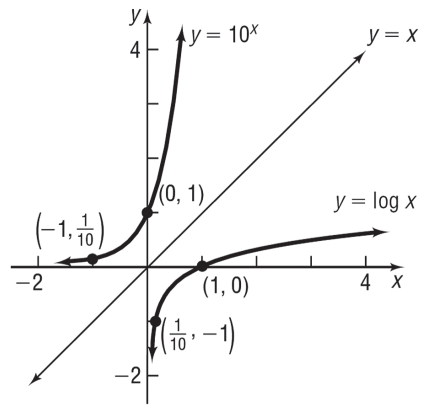
(d) Find f^{-1}, the inverse of *f*.

(e) Use f^{-1} to confirm the range of *f* found in part (c). From the domain of *f*, find the range of f^{-1}.

(f) Graph f^{-1} on the same set of axis as *f*.

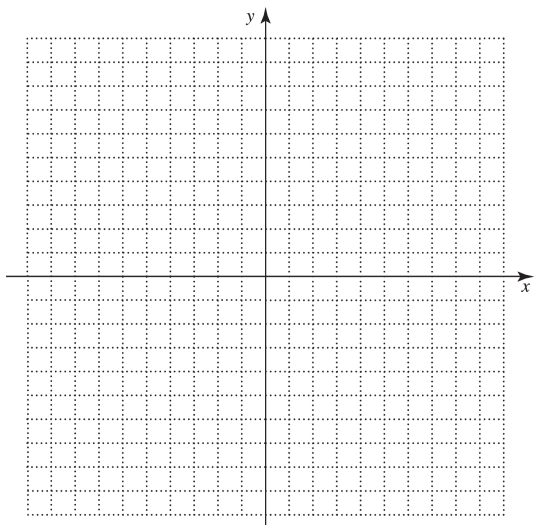
Fact

Common Logarithm: $\log(x)$ means $\log_{10}(x)$. In other words, $y = \log(x)$ if and only if $x = 10^y$.



Example 6: Graph a Logarithmic Functions

(a) Graph $f(x) = -2\log(x+2)$.



(b) State the domain of $f(x)$.

(c) From the graph, determine the range and vertical asymptote of *f*.

(d) Find f^{-1}, the inverse of *f*.

(e) Use f^{-1} to confirm the range of *f* found in part (c). From the domain of *f*, find the range of f^{-1}

(f) Graph f^{-1} on the same set of axis as *f*.

Solving Basic Logarithmic Equations

When solving simple logarithmic equations (they will get more complicated in Section 4.6) follow these steps:

1. Isolate the logarithm if possible.
2. Change the logarithm to exponential form and use the strategies learned in Section 4.3 to solve for the unknown variable.

Example 7*: Solve Logarithmic Equations

Solve the following logarithmic equations

(a)* $\log_2(2x+1)=3$ (b)* $\log_x 343=3$ (c) $6-\log(x)=3$

(d) $\ln(x)=2$ (e) $7\log_6(4x)+5=-2$ (f) $\log_6 36=5x+3$

Steps for solving exponential equations of base *e* or base 10

1. Isolate the exponential part
2. Change the exponent into a logarithm.
3. Use either the "log" key (if log base 10) or the "ln" (if log base *e*) key to evaluate the variable.

Example 8*: Using Logarithms to Solve Exponential Equations

Solve each exponential equation.

(a) $e^x = 7$

(b)* $2e^{3x} = 6$

(c) $e^{5x-1} = 9$

(d) $4(10^{2x}) + 1 = 21$

(e) $3e^{2x+1} - 2 = 10$

Chapter 4: Exponential and Logarithmic Functions
Section 4.5: Properties of Logarithms

Exploration 1: Product Law of Logarithms

Make and test a conjecture about the product law of logarithms $\log_a(xy)$

(a) Complete the table below

$\log_a(x)$	$\log_a(y)$	$\log_a(xy)$
$\log_2(4) =$	$\log_2(8) =$	$\log_2(32) =$
$\log_3(9) =$	$\log_3(27) =$	$\log_3(243) =$
$\log_2 16 =$	$\log_2(32) =$	$\log_2(512) =$
$\log_4(64) =$	$\log_4(16) =$	$\log_4(1024) =$
$\log_5(25) =$	$\log_5(5) =$	$\log_5(125) =$

(b) Examine the results of each row. Make a conjecture about the product law for logarithms.

Exploration 2: Quotient Law of Logarithms

Make and test a conjecture about the quotient law of logarithms: $\log_a(\frac{x}{y})$

(a) Complete the table below

$\log_a(x)$	$\log_a(y)$	$\log_a(\frac{x}{y})$
$\log_2(32) =$	$\log_2(8) =$	$\log_2(\frac{32}{8}) =$
$\log_3(27) =$	$\log_3(9) =$	$\log_3(\frac{27}{9}) =$
$\log_2(32) =$	$\log_2 16 =$	$\log_2(\frac{32}{16}) =$
$\log_4(64) =$	$\log_4(16) =$	$\log_4(\frac{64}{16}) =$
$\log_5(25) =$	$\log_5(5) =$	$\log_5(\frac{25}{5}) =$

(b) Examine the results of each row. Make a conjecture about the quotient law for logarithms.

Exploration 3: Make and test a conjecture about what $\log_a(1)$ equals for any base a. Why is this?

Exploration 4: Make and test a conjecture about what $\log_a(a)$ equals for any base a. Why is this?

Properties of Logarithms:

In the following properties, M, N, and a are positive real numbers, where $a \neq 1$, and r is any real number :

1. $a^{\log_a M} =$ _____
2. $\log_a a^r =$ _____
3. $\log_a (MN) =$ _____
4. $\log_a \left(\frac{M}{N}\right) =$ _____
5. $\log_a M^r =$ _____
6. $a^x =$ _____

Example 1*: Work with the Properties of Logarithms

Use the laws of logarithms to simplify the following:

(a) $3^{\log_3 18}$ (b) $2^{\log_2(-5)}$ (c) $\log_{\frac{1}{2}}\left(\frac{1}{2}\right)^{20}$ (d) $\ln(e^3)$

Example 2: Work with the Properties of Logarithms
Use the laws of logarithms to find the exact value without a calculator.

(a) $\log_3(24) - \log_3(8)$

(b) $\log_8(2) - \log_8(32)$

(c) $6^{\log_6(3)+\log_6(5)}$

(d) $e^{\log_{e^2}(25)}$

Example 3*: Write a Logarithmic Expression as a Sum or Difference of Logarithms
Write each expression as a as a sum or difference of logarithms. Express all powers as factors.

(a) $\log_3\left[(x-1)(x+2)^2\right], x > 1$

(b) $\log_5\left[\dfrac{x^2y^3}{\sqrt{z}}\right]$

Example 4*: Write a Logarithmic Expression as a Single Logarithm
Write each of the following as a single logarithm.

(a) $\log_2 x + \log_2(x-3)$

(b) $3\log_6 z - 2\log_6 y$

(c) $\ln(x-2) + \dfrac{1}{2}\ln x - 5\ln(x+3)$

Properties of Logarithms continued:

In the following properties, M, N, and a are positive real numbers where $a \neq 1$:

7. If $M = N$, then _____
8. If $\log_a M = \log_a N$, then _____

Let a ≠ 1, and b ≠ 1 be positive real numbers. Then the **change of base formula** says:

9. $\log_a M =$ _____

Why would we want to use the change of base formula?

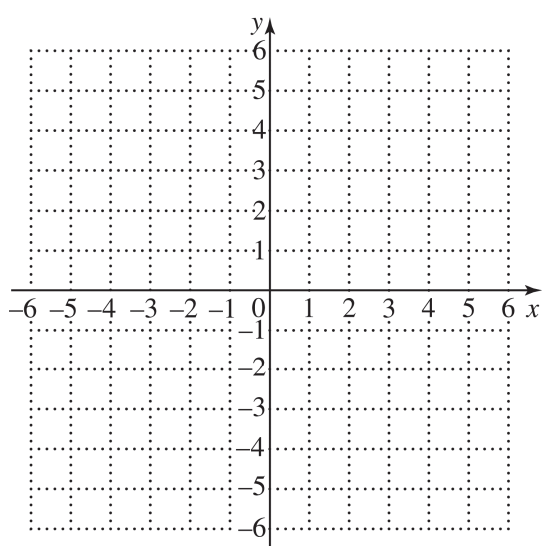
Example 5*: Evaluate a Logarithm Whose Base is Neither 10 nor *e*.

Approximate the following. Round your answers to four decimal places.

(a) $\log_3 12$ (b) $\log_7 325$

Example 6*: Graph a Logarithmic Function Whose Base is Neither 10 Nor *e*

Use a graphing utility to graph $y = \log_5 x$.



Chapter 4: Exponential and Logarithmic Functions
Section 4.6: Logarithmic and Exponential Equations

We will use the properties of logarithms found in Section 4.5 to solve all types of equations where a variable is an exponent. The following definition and properties that we've seen in previous sections will be particularly useful and provided here for your review:

The **logarithmic function to the base a**, where $a > 0$ and $a \neq 1$, is denoted by $y = \log_a x$ (read as "y is the logarithm to the base a of x") and is defined by:

$$y = \log_a x \text{ if and only if } x = a^y$$

The domain of the logarithmic function $y = \log_a x$ is $x > 0$.

Properties of Logarithms:

In the following properties, M, N, and a are positive real numbers, where $a \neq 1$, and r is any real number :

1. $a^{\log_a M} = M$
2. $\log_a a^r = r$
3. $\log_a (MN) = \log_a M + \log_a N$
4. $\log_a \left(\frac{M}{N}\right) = \log_a M - \log_a N$
5. $\log_a M^r = r \log_a M$
6. $a^x = e^{x \ln a}$
7. If $M = N$, then $\log_a M = \log_a N$
8. If $\log_a M = \log_a N$, then $M = N$.

Strategy for Solving Logarithmic Equations Algebraically

1. Rewrite the equation using properties of logarithms so that it is written in one of the following two ways: $\log_a x = c$ or $\log_a(\text{something}) = \log_a(\text{something else})$.
2. If the equation is of the form $\log_a x = c$ change it to exponential form to undo the logarithm and solve for x.
3. If the equation is of the form $\log_a(\text{something}) = \log_a(\text{something else})$ use property 8 to get rid of the logarithms and solve.
4. Check your solutions. Remember that The domain of the logarithmic function $y = \log_a x$ is $x > 0$.

Example 1*: Solve Logarithmic Equations
Solve the following equations:

(a) $\log_3 4 = 2\log_3 x$

(b) $\log_2(x+2) + \log_2(1-x) = 1$

Example 2: Solve Logarithmic Equations
Solve the following equations:

(a) $\ln(x-1) + \ln x = \ln(x+2)$

(b) $\log_4(h+3) - \log_4(2-h) = 1$

(c) $\log(1-c) = 1 + \log(1+c)$

(d) $\ln(3m+1) = 2 + \ln(m-3)$

Example 3*: Solve Exponential Equations
Solve the following equations:

(a) $9^x - 3^x - 6 = 0$

(b) $3^x = 7$

(c) $5 \cdot 2^x = 3$

(d) $2^{x-1} = 5^{2x+3}$

So far we have solved exponential and logarithmic equations algebraically. Another method we can use is to solve by graphing. Here is a list of steps for how to do this:

Solving by Graphing

1. Put one side of the equation in y_1.
2. Put one side of the equation in y_2.
3. Graph the equations and find the point at which they intersect.
4. The x value is your solution.

Example 4*: Solving Logarithmic and Exponential Equations Using a Graphing Utility
Solve $e^x = -x$ using a graphing utility.

Chapter 4: Exponential and Logarithmic Functions
Section 4.7: Financial Models

Many financial models use exponential functions. Before we introduce these models, let's define some terms. **Interest** is money paid for the use of money. The total amount borrowed (whether by an individual from a bank in the form of a loan or by a bank from an individual in the form of a savings account) is called the **principal.** The **rate of interest**, expressed as a percent, is the amount charged for the use of the principal for a given period of time, usually on a yearly (that is, per annum) basis. One such formula used to calculate interest is the simple interest formula:

Simple Interest Formula Thereom: If a principal of P dollars is borrowed for a period of t years at a per annum interest rate, r, expressed as a decimal, the interest I, charged is:_____

In working with problems involving interest, the term **payment period** is defined as follows:

Annually:	**Monthly:**	**Semiannually:**
Daily:	**Quarterly:**	

Example 1: Compute Simple Interest
Use the simple interest formula to calculate the interest you would receive if you invested $10,000 at 12% interest for 1 year.

Rarely is money put in an account and left to earn interest at the end of its life. Typically, your money earns interest, and then that interest earns interest, and so on and so on. This model is called **compound interest**.

Let's derive a formula for compound interest using Example 1 above. Let's say we invest our money in an account that earns interest that is compounded semi-annually. How much would we have at the end of 1 year? What if the account was compounded quarterly? Monthly?

Do you see a pattern that we can generalize?

Compound Interest Formula Theorem: The amount A after t years due to a principal P invested at an annual interest rate r compounded n times per year is:

$A =$ _____

Example 2*: Find the Future Value of a Lump Sum of Money

Use the compound interest formula to calculate the amount of money you would have after 1 year if you invest $1000 at an annual rate of 10% compounded:

(a) Annually

(c) Monthly

(b) Quarterly

(d) Daily

(e) What do you notice as you increase n?

Example 3: Compounding Interest

Use the compound interest formula to calculate the amount of money you would have after 1 year if you invest $1 at 100% interest compounded:

(a) Annually

(c) Monthly

(b) Quarterly

(d) Daily

(e) What do you notice as you increase n?

Compound Continuously

The act of compounding without bound is expressed as continuous compounding. Recall from Section 4.3, the number $e \approx 2.718281828459$ which was defined as $e = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$.

How does this definition of *e* relate to what we did in Example 3?

This leads up to our next formula:

Continuous Compounding Theorem: The amount *A* after *t* years due to a principal *P* invested at an annual interest rate *r* compounded continuously is:

_____

Example 4: Continuous Compounding

Find the amount *A* that results from investing a principal *P* of \$2000 at an annual rate *r* of 8% compounded continuously for a time *t* of 1 year.

Example 5: **Continuous Compounding**

You have \$1000 to invest in a bank that offers 4.2% annual interest on a savings account compounded monthly. What annual interest rate do you need to earn to have the same amount at the end of the year if the interest is compounded continuously?

The comparable interest rate found in Example 5 is called the **effective rate of interest**. It tells you the equivalent annual simple interest rate that would yield the same amount as compounding *n* times per year, or continuously, after 1 year.

Effective Rate of Interest Theorem: The effective rate of interest, r_e, of an investment earning an annual interest rate *r* is given by

Compounding *n* times per year: $r_e =$

Continuous Compounding: $r_e =$

Example 6*: Find the Effective Rate of Interest
Find the effective rate for 5% compounded quarterly

Example 7: Find the Effective Rate of Interest– Which is the Best Deal?
Suppose you want to open a money market account. You visit three banks to determine their money market rates. Bank A offers you 5% compounded monthly and Bank B offers you 5.04% compounded quarterly. Bank C offers 4.9% compounded continuously. Determine which bank is offering the best deal.

The **present value** of A dollars to be received at a future date is the principal that you would need to invest now so that it will grow to A dollars in a specified time period. Inflation is a perfect example of this. The formula for present value actually comes from solving the compound interest formula for P.

Present Value Formula Theorem: The present value of P of A dollars to be received after t years, assuming a per annum interest rate r compounded n times per year, is:

$P =$ _____

If the interest is compounded continuously, then:

$P =$ _____

Example 8*: Find the Present Value of a Lump Sum of Money
Find the principal needed now to get $100 after 2 years at 6% compounded monthly.

Example 9*: Find the Time Required to Double an Investment

(a) How long does it take for an investment to double in value if it is invested in 5 years?

(b) How long does it take if the interest is compounded continuously?

Example 10: Rate of Interest Required to Double an Investment

What annual rate of interest compounded quarterly should you seek if you want to double your investment in 6 years?

Chapter 4: Exponential and Logarithmic Functions
Section 4.8: Exponential Growth and Decay Models; Newton's Law; Logistical Growth

Just like money can grow continuously, so can other natural phenomena demonstrate uninhibited growth or decay. Some examples are cell division of many living organisms which demonstrate the growth process and radioactive substances that have a specific half-life and demonstrate decay.

Uninhibited Growth of Cells

A model that gives the number N of cells in a culture after a time t has passed is

$$N(t) = \_\_\_\_\_ \qquad k > 0$$

where N_0 is _____ and k is a _____ constant that represents the growth rate of the cells.

Example 1*: Find Equations of Uninhibited Growth

A colony of bacteria grows according to the law of uninhibited growth according to the function $N(t) = 100e^{0.045t}$, where N is measured in grams and t is measured in days.

(a) Determine the initial amount of bacteria.

(b) What is the growth rate of the bacteria?

(c) What is the population after 5 days?

(d) How long will it take the population to reach 140 grams?

(e) What is the doubling time for the population?

Example 2: Find Equations of Uninhibited Growth
A colony of bacteria increases according to the law of uninhibited growth. According to the formula on the previous page, if N is the number of bacteria in the culture and t is the time in hours, then $N(t) = N_0 e^{kt}$.

(a) If 10,000 bacteria are present initially and the number of bacteria doubles in 5 hours, how many bacteria will there be in 24 hours?

(b) How long is it until there are 500,000 bacteria?

Uninhibited Radioactive Decay

The amount A of a radioactive material present at time t is given by

$A(t) =$ _____ $k < 0$

where A_0 is _____ and k is a _____ number that represents the growth rate of the cells.

Note: All radioactive substances have a specific **half – life,** which is the time required for half of the radioactive substance to decay.

Example 3: Find Equations of Decay
Iodine 131 is a radioactive material that decays according to the function $A(t) = A_0 e^{-0.087t}$, where A_0 is the initial amount present and A is the amount present at time t (in days). Assume that the scientist initially has a sample of 200 grams of iodine 131.

(a) What is the decay rate of iodine 131?

(b) How much iodine 131 is left after 5 days?

(c) When will 100 grams of iodine be left?

Fact: Living things contain 2 kinds of carbon—carbon 12 and carbon 14. When a person dies, carbon 12 stays constant, but carbon 14 decays. In fact, carbon 14 is said to have a half-life of 5730 years. This change in the amount of carbon 14 makes it possible to calculate when the organism died.

Example 4*: Find Equations of Decay
Traces of burned wood along with ancient stone tools in an archeological dig in Chile were found to contain approximately 1.67% of the original amount of carbon 14. If the half – life of carbon 14 is 5730 years, approximately when was the tree cut and burned?

Newton's Law of Cooling
The temperature u of a heated object at a given time t can be modeled by the following function:

$$u(t) = \_\_\_\_\_ \quad k < 0$$

where T is the _____ of the surrounding medium, u_0 is the _____ of the heated object, and k is a _____ constant.

Example 5: Using Newton's Law of Cooling
An object is heated to 75°C and is then allowed to cool in a room whose air temperature is 30°C.

(a) If the temperature of the object is 60° after 5 minutes, when will its temperature be 50°?

(b) Using a graphing utility, graph the relation found between the temperature and time.

(c) Using a graphing utility, verify the results from part (a).

(d) Using a graphing utility, determine the elapsed time before the object is 35°C.

(e) What do you notice about the temperature as time passes?

The exponential growth model $N(t) = N_0 e^{kt} \quad k > 0$ assumes uninhibited growth, meaning that the value of the function grows without limit. We can use this function to model cell division, assuming that no cells die and no by – products are produced. However, cell division eventually is limited by factors such as living space and food supply. The **logistic model**, given next, can describe situations where the growth or decay factor of the dependent variable is limited.

Logistic Model

In a logistic growth model, the population *P* after time *t* is given by the function

$$P(t) = \text{———————}$$

where *a, b,* and *c* are constants with $c > 0$. The model is a growth model if _____; the model is a decay model if _____.

Properties of the Logistic Growth Model

1. The domain is _____. The range is the interval ____, where *c* is the **carrying capacity.**
2. There are no *x* – intercepts; the *y* – intercept is _____.
3. There are two horizontal asymptotes:_____
4. $P(t)$ is an increasing function if _____ and a decreasing function if _____.
5. There is an **inflection point** where $P(t)$ equals ____ of the carrying capacity. The inflection point is the point on the graph where the graph changes from being curved upward to curved downward for growth functions and the point where the graph changes from being curved downward to curved upward for decay functions.
6. The graph is smooth and continuous, with no corners or gaps.

Example 6*: Use Logistic Models

The EFISCEN wood product model classifies wood products according to their life – span. There are four classifications: short (1 year), medium short (4 years), medium long (16 years), and long (50 years). Based on data obtained from the European Forest Institute, the percentage of remaining wood products are *t* years for wood products with long life – spans (such as those used in the building industry) is given by $P(t) = \dfrac{100.3952}{1 + 0.0316e^{0.0581t}}$.

(a) What is the decay rate?

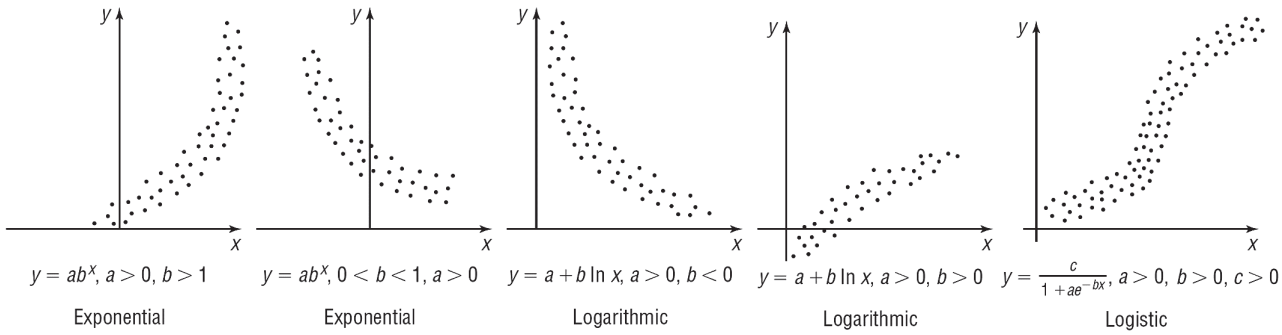
(b) What is the percentage of remaining wood products after 10 years?

(c) How long does it take for the percentage of remaining wood products to reach 50%?

(d) Explain why the numerator given in the model is reasonable.

Chapter 4: Exponential and Logarithmic Functions
Section 4.9: Building Exponential, Logarithmic, and Logistical Models from Data

In earlier chapters we discussed how to find the linear, quadratic, and cubic functions of best fit. In this section we will discuss how to find the exponential, logarithmic, and logistic models of best fit. The diagrams below show how data will be typically observed for the three models as well as any restrictions on the values of the parameters.



Example 1*: Build an Exponential Model from Data
The data in the table shows the closing price of Harley Davidson Stock from the years 1987 to 2000.

Year	Closing Price
1987 ($x = 1$)	0.392
1988 ($x = 2$)	0.7652
1989 ($x = 3$)	1.1835
1990 ($x = 4$)	1.1609
1991 ($x = 5$)	2.6988
1992 ($x = 6$)	4.5381
1993 ($x = 7$)	5.3379
1994 ($x = 8$)	6.8032
1995 ($x = 9$)	7.0328
1996 ($x = 10$)	11.5585
1997 ($x = 11$)	13.4799
1998 ($x = 12$)	23.5424
1999 ($x = 13$)	31.9342
2000 ($x = 14$)	39.7277

(a) Using a graphing utility, draw a scatter diagram with year as the independent variable.

(b) Using a graphing utility build an exponential model from the data.

(c) Express the function found in part (b) in the form $A = A_0 e^{kt}$.

(d) Graph the exponential function found in part (b) or (c) on the scatter diagram.

(e) Using the solution to part (b) or (c) predict the closing stock price for the year.

Example 2*: Build a Logistic Model from Data
The following data represent the population of the United States. An ecologist is interested in building a model that describes the population of the United States.

(a) Using a graphing utility, draw a scatter diagram of the data using years since 1900 as the independent variable and population as the dependent variable.

(b) Using a graphing utility, build a logistic model from the data.

Year	Population
1900	76,212,168
1910	92,228,496
1920	106,021,537
1930	123,202,624
1940	132,164,568
1950	151,325,798
1960	179,323,175
1970	203,302,031
1980	226,542,203
1990	248,708,873
2000	281,421,906

(c) Using a graphing utility, draw the function found in part (b) on the scatter diagram.

(d) Based on the function found in part (b), what is the carrying capacity of the United States?

(e) Use the function found in part (b) to predict the population of the United States in 2004.

(f) When will the United States Population be 300,000,000?