CHAPTER 2

Exploring what it means to know and do mathematics

LEARNING OBJECTIVES

After reading this chapter and engaging in the embedded activities and reflections, you should be able to:

LO 2.1 Describe what it means to do mathematics.

LO 2.2 Design and implement strategies for solving authentic mathematics tasks.

LO 2.3 Illustrate through content examples, what a mathematically proficient student knows and is able to do.

LO 2.4 Compare learning theories related to mathematics and connect the theories to effective teaching practices.

LO 2.5 Synthesise the important theoretical and content ideas related to learning mathematics.

This chapter explains how to help students learn mathematics. To get at how to help students learn, however, we must first consider what is important *to* learn. Let's look at a poorly understood topic, division of fractions, as an opening example. If a student has learned this topic well, what will they know and what should they be able to do? The answer is more than being able to successfully implement a procedure (e.g. commonly called the 'invert and multiply' procedure). There is much more to know and understand about division of fractions: What does $3 \div \frac{1}{4}$ mean conceptually? What is a situation that might be solved with such an equation? Will the result be greater than or less than 3 and why? What ways can we solve equations like this? What illustration or manipulative could illustrate this equation? What is the relationship of this equation to subtraction? To multiplication? All of these questions can be answered by a student who fully understands a topic such as division of fractions. We must lead students to this conceptual understanding.

This chapter can help you. It could be called the 'what' and 'how' of teaching mathematics. First, *what* does doing mathematics look like (be ready to experience this yourself through four tasks) and what is important to know about mathematics? Second, *how* do we help students develop a strong understanding of mathematics? By the end of this chapter, you will be able to draw strong connections between the what and the how of teaching mathematics.

▶ WHAT DOES IT MEAN TO DO MATHEMATICS?

Mathematics is more than completing sets of exercises or mimicking processes the teacher explains. Doing mathematics means generating strategies for solving a problem, applying that strategy and checking to see whether your answer makes sense. Finding and exploring regularity or order, and then making sense of it, is what doing mathematics in the real world is all about.

Doing mathematics in classrooms should closely model the act of doing mathematics in the real world. Even our youngest students can notice patterns and order. For example, post a series of problems and ask Year 1 or 2 students, 'What patterns do you notice?'

$6 + 7 =$
$5 + 8 =$
$4 + 9 =$

Think about the patterns students might notice: the first addend is going down 1, the second one is going up 1, and the sums are the same. How might exploring these patterns help students to learn about addition? Also consider the next situation related to multiplication that might be explored by Year 3 to Year 5 students.

Even × Even = Even	Always	Sometimes	Never
Odd × Odd = Even	Always	Sometimes	Never
Odd × Even = Even	Always	Sometimes	Never

Exploring generalisations such as these multiplication ones provides students with an opportunity to learn important relationships about numbers as they deepen their understanding of the operations. With each of these problems, you have the opportunity to have students debate which answer they think is correct and to justify (i.e., prove) their response.

In the middle years, students continue to explore more advanced patterns and order, extending to negative numbers and exponents, as well as using variables. You also might ask middle years students to look for patterns comparing two solutions, as in this example:

For a fundraiser, Annie and Mac decided to sell wristbands. They cost 75c each and they are going to sell them for $2.50. They sold 35 the first day. They each calculate the day 1 profit differently. Who is correct? Explain.

Annie: $(35 \times 2.50) - (35 \times 0.75) =$

Mac: $\$1.75 \times 35 =$

In comparing these two strategies for finding profit, students are seeing relationships between the equations and the situations, noticing properties of the operations 'in action', and discussing equivalencies (a major idea in mathematics).

Engaging in the science of pattern and order, as the previous two examples illustrate, is *doing* mathematics. Basic facts and basic skills such as computation of whole numbers, fractions and decimals are important in enabling students to be able to *do* mathematics. But if skills are taught by rote memorisation or isolated practice, students will not learn to *do* mathematics and will not be prepared to do the mathematics required in the 21st century.

VERBS OF DOING MATHEMATICS

Doing mathematics begins with posing worthwhile tasks and then creating an environment where students take risks, share and defend mathematical ideas. Students in traditional mathematics classes often describe mathematics as imitating what the teacher shows them. Instructions to students given by teachers or in textbooks ask students to listen, copy, memorise, drill and compute. These are lower-level thinking activities and do not adequately

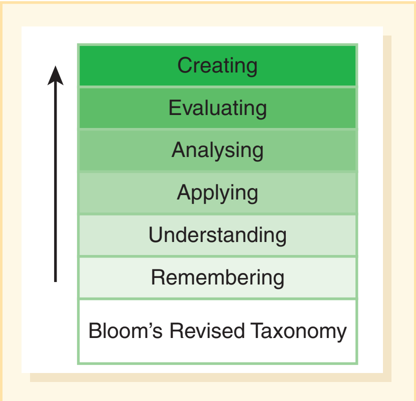


FIGURE 2.1 Bloom's (Revised) Taxonomy

Source: Anderson, L. W. & Krathwohl, D. R. (Eds.). (2001). *A taxonomy for learning, teaching, and assessing: A revision of Bloom's Taxonomy of educational objectives: Complete Edition.* New York, NY: Addison Wesley, Longman.

prepare students for the real act of doing mathematics. In contrast, the following verbs engage students in doing mathematics:

collaborate	describe	justify
communicate	develop	predict
compare	explain	represent
conjecture	explore	solve
construct	formulate	use
create	invent	verify
critique	investigate	

These verbs lead to opportunities for higher-level thinking and encompass 'making sense' and 'figuring out'. These verbs may look familiar to you, as they are on the higher level of Bloom's (revised) taxonomy (Anderson & Krathwohl, 2001) (see Figure 2.1).

In observing a Year 3 classroom where the teacher used this approach to teaching mathematics, researchers found that students became 'doers' of mathematics. In other words, the students began to take the maths ideas to the next level by (1) connecting to previous material, (2) responding with information beyond the required response and (3) conjecturing or predicting (Fillingim & Barlow, 2010). When this happens on a daily basis, students are getting an empowering message: 'You are capable of making sense of this – you are capable of *doing mathematics*.'

▶ AN INVITATION TO DO MATHEMATICS

The purpose of this section is to provide you with opportunities to engage in the science of pattern and order – to *do* some mathematics. For each problem posed, allow yourself to try to (1) make connections within the mathematics (i.e., make mathematical relationships explicit) and (2) engage in productive struggle.

We will explore four different problems. None requires mathematics beyond primary school mathematics – not even algebra. But the problems do require higher-level thinking and reasoning. As you read each task, stop and solve it first before reading the 'A few ideas' section. Then, you will be doing mathematics and seeing how others may think about the problem differently (or the same).

SEARCHING FOR PATTERNS

1. *Start and jump numbers*

Begin with a number (start) and add (jump) a fixed amount. For example, start with 3 and jump by 5s. Use the Start and jump numbers activity page or write the list on a piece of paper. Examine the list and record as many patterns as you see.

A few ideas. Here are some questions to guide your pattern search:

- Do you see at least one alternating pattern?
- Have you noticed an odd/even pattern? Why is this pattern true?
- What do you notice about the numbers in the tens place?
- Do the patterns change when the numbers are greater than 100?

Don't forget to think about what happens to your patterns after the numbers are more than 100. How are you thinking about 113? One way is as 1 hundred, 1 ten and 3 ones. But, of course, it could also be 'eleventy-three', where the tens digit has gone from 9 to 10 to 11. How do these different perspectives affect your patterns? What would happen after 999?

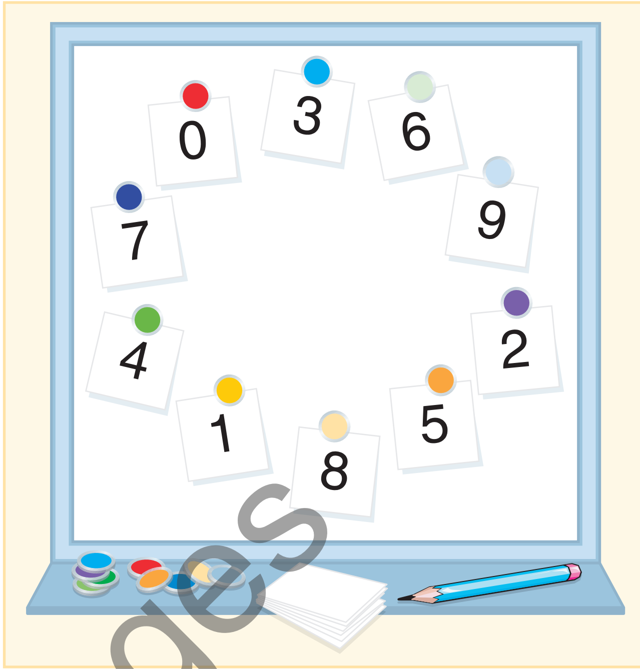


FIGURE 2.2 For jumps of 3, this cycle of digits will occur in the ones place. The start number determines where the cycle begins.

Next steps. Sometimes when you have discovered some patterns in mathematics, it is a good idea to make some changes and see how the changes affect the patterns. What changes might you make in this problem?

Your changes may be even more interesting than the following suggestions. But here are some ideas:

- Change the start number but keep the jump number equal to 5. What is the same and what is different?
- Keep the same start number and explore with different jump numbers.
- What patterns do different jump numbers make? For example, when the jump number was 5, the ones-digit pattern repeated every two numbers – it had a 'pattern length' of 2. But when the jump number is 3, the length of the ones-digit pattern is 10. Do other jump numbers create different pattern lengths?
  - For a jump number of 3, how does the ones-digit pattern relate to the circle of numbers in Figure 2.2? Are there similar circles of numbers for other jump numbers?
  - Using the circle of numbers for 3, find the pattern for jumps of multiples of 3; that is, jumps of 6, 9 or 12.

Using technology. Calculators facilitate exploration of this problem. Using the calculator makes the list generation accessible for young children who can't skip count yet and it opens the door for students to work with bigger jump numbers like 25 or 36. Most simple calculators have an automatic constant feature that will add the same number successively. For example, if you press 3 + 5 = and then keep pressing =, the calculator will keep counting by fives from the previous answer. This works for the other three operations. Consider demonstrating this with an online calculator or app for the whiteboard so the class can observe and discuss the counting.

ANALYSING A SITUATION

2. *Two machines, one job*

Ron's Recycle Shop started when Ron bought a used paper-shredding machine. Business was good, so Ron bought a new shredding machine. The old machine could shred a truckload of paper in 4 hours. The new machine could shred the same truckload in only 2 hours. How long will it take to shred a truckload of paper if Ron runs both shredders at the same time?

Use the **Two machines, one job** activity page to record your solution to this problem. Do not read on until you have an answer or are stuck. Can you check that you are correct? Can you approach the problem using a picture?

A few ideas. Have you tried to predict approximately how much time you think it should take the two machines? For example, will it be closer to 1 hour or closer to 4 hours? What facts about the situation led you to this estimated time? Is there a way to check your estimate? Checking a guess in this way sometimes leads to a new insight.

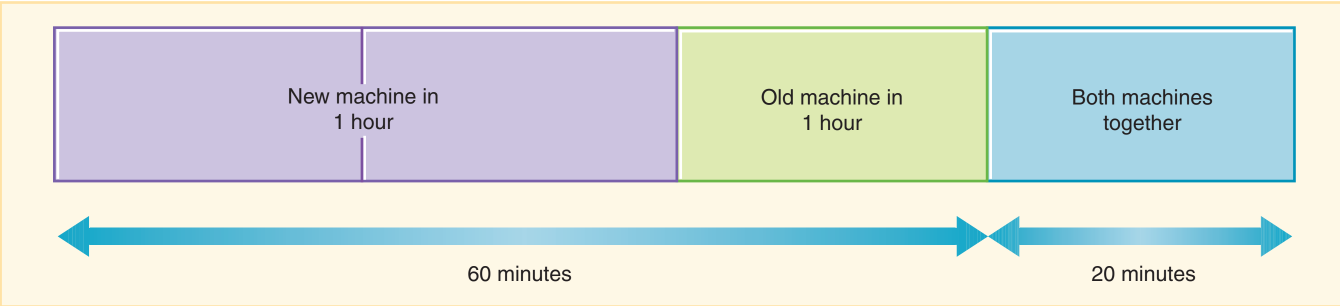


FIGURE 2.3 Cora's solution to the paper-shredding problem.

Some people draw pictures to solve problems. Others like to use something they can move or change. For example, you might draw a rectangle or a line segment to stand for the truckload of paper or you might get some counters (or cubes) and make a collection that stands for the truckload.

Consider the solutions of others. There are many ways to model and solve the problem, and understanding other people's ways can develop our own understanding. The following is one explanation for solving the problem, using strips (based on Schifter & Fosnot, 1993):

> 'This rectangle [see Figure 2.3] stands for the whole truckload. In 1 hour, the new machine will do half of this.' The rectangle is divided in half. 'In 1 hour, the old machine could shred $\frac{1}{4}$ of the paper.' The rectangle is divided accordingly. 'So, in 1 hour, the two machines have done $\frac{3}{4}$ of the truck and there is $\frac{1}{4}$ left. What is left is $\frac{1}{3}$ as much as what they have already done, so it should take the two machines $\frac{1}{3}$ as long to do that part as it took to do the first part. One-third of an hour is 20 minutes. That means it takes 1 hour and 20 minutes to do it all.

As with the teachers in these examples, it is important to decide whether your solution is correct through justifying why you did what you did; this reflects real problem-solving rather than checking with an answer key. After you are satisfied that you have solved the problem in a correct manner, try to find other ways that students might solve the problem; in considering multiple ways, you are making mathematical connections.

GENERALISING RELATIONSHIPS

3. *One up, one down*

Addition. When you add 7 + 7, you get 14. When you make the first number 1 more and the second number 1 less, you get the same answer:

$$\overset{\uparrow}{7} + \overset{\downarrow}{7} = 14 \quad \text{and} \quad 8 + 6 = 14$$

It works for 5 + 5 too:

$$\overset{\uparrow}{5} + \overset{\downarrow}{5} = 10 \quad \text{and} \quad 6 + 4 = 10$$

Does this work for any doubles? For what other addition problems does one up, one down work? Why does it work?

Explore this problem using the **One up, one down: Addition** activity page. Explore and develop your own conjectures. When you look at your calculations, can you see a pattern that you think might be true? When we do an investigation, we might see something we think is true: so we articulate a conjecture; that is, verbalise what we think is true. Next, we need to check the conjecture – is it true for all known cases and examples? Then we test the conjecture and either accept or modify the conjecture. If we modify the conjecture, we will need to go through the cycle again.

Multiplication. Explore this problem using the **One up, one down: Multiplication** activity page, which focuses on the question, 'How does one up, one down work with multiplication?'

$$\begin{array}{c}\uparrow \quad \downarrow \\ 7 \times 7 = 49 \\ 8 \times 6 = 48\end{array}$$

'One up, one down' results in an answer that is one less than the original problem. Does this work for any squares (e.g. 5 × 5)? Is it true for non-square multiplication problems? Explore and develop your own conjectures.

Explore the multiplication problem, responding to the questions posed. Notice that you are asked to develop conjectures. Developing and testing conjectures is an important aspect of mathematical reasoning (Lannin, Ellis & Elliott, 2011).

A few ideas. Multiplication is more complicated. Why? Use a physical model or picture to compare the before and after products. For example, draw rectangles (or arrays) with a length and width of each of the factors (see Figure 2.4(a)), then draw the new rectangle (e.g. 8-by-6-unit rectangle). See how the rectangles compare.

You may prefer to think of multiplication as equal sets. For example, using stacks of chips, 7 × 7 is seven stacks with seven counters in each stack (set) (see Figure 2.4(b)). The expression 8 × 6 is represented by eight stacks of six (though six stacks of eight is a possible interpretation). See how the stacks for each expression compare. Consider working with one or both of these approaches to gain insights and make conjectures.

Additional patterns to explore. Doing mathematics includes the tendency to extend beyond the problem posed. This problem lends itself to many 'what if?' questions. Here are a few. Try and find some other ones.

- Have you looked at how the first two numbers are related? For example, 7 × 7, 5 × 5 and 9 × 9 are all products with like factors. What if the product were two consecutive numbers (e.g. 8 × 7 or 13 × 12)? What if the factors differ by 2 or by 3?
- Think about adjusting by numbers other than one. What if you adjust two up and two down (e.g. 7 × 7 to 9 × 5)?
- What happens if you use big numbers instead of small ones (e.g. 30 × 30)?
- If both factors increase (i.e., one up, one up), is there a pattern?

Have you made some mathematical connections and conjectures in exploring this problem? In doing so you have hopefully felt a sense of accomplishment and excitement – one of the benefits of *doing* mathematics.

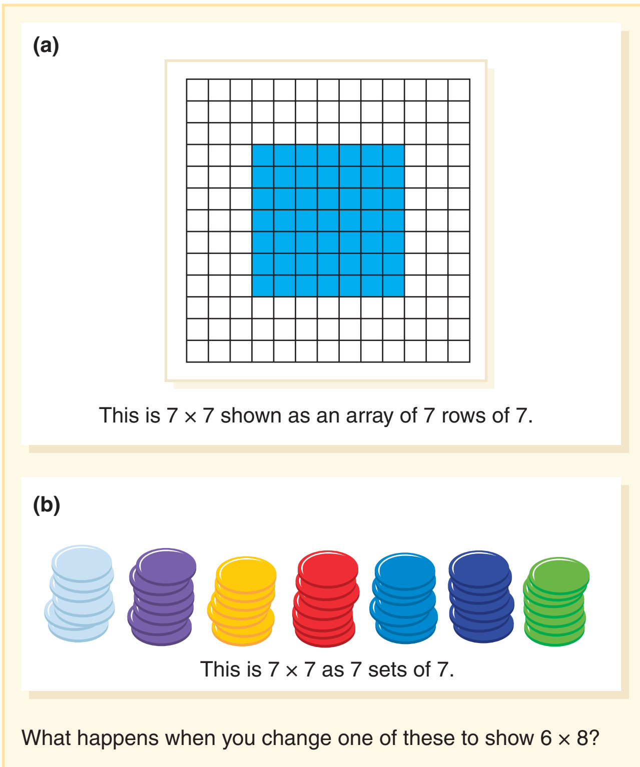


FIGURE 2.4 Two physical ways to think about multiplication that might help in the exploration.

EXPERIMENTING AND EXPLAINING

4. *The best chance of purple*

Samuel, Susan and Sandu are playing a game. The first one to spin 'purple' wins. Purple means that one spin lands on red and

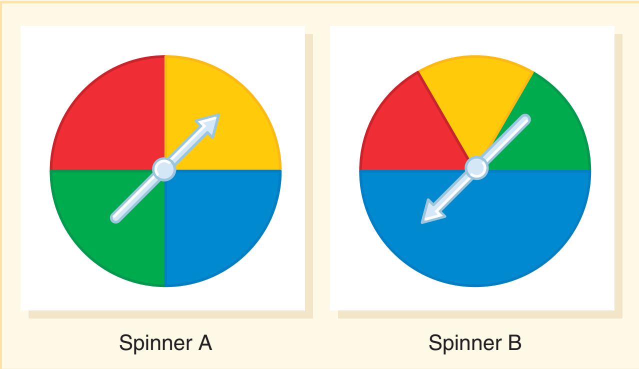


FIGURE 2.5 You may spin A twice, B twice or A then B. Which choice gives you the best chance of spinning a red and a blue (purple)?

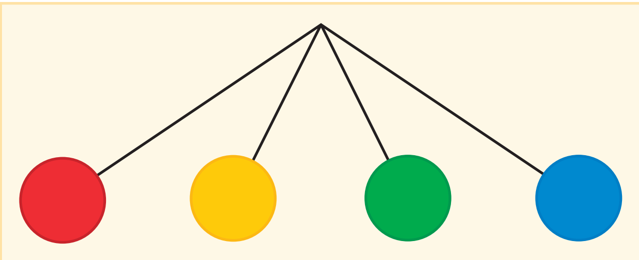


FIGURE 2.6 A tree diagram for spinner A in Figure 2.5.

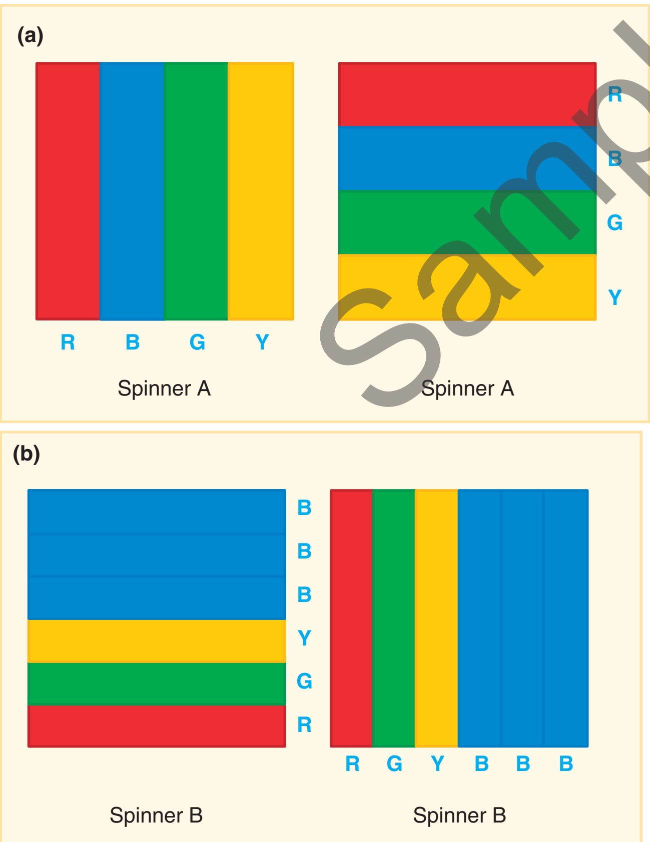


FIGURE 2.7 Grids can illustrate the chance of spinning purple with two spins.

one spin lands on blue (see Figure 2.5). Each person chooses to spin each spinner once or one of the spinners twice. Samuel chooses to spin spinner A twice; Susan chooses to spin spinner B twice; and Sandu chooses to spin each spinner once. Who has the best chance of purple? (based on Lappan & Even, 1989)

Think about the problem and what you know. Experiment. Use the **Best chance of purple** activity page to explore this problem.

A few ideas. A good strategy for learning is to first explore a problem concretely, then analyse it abstractly. This is helpful in situations involving chance or probability. Use a paper clip with the spinners on your activity page or use a virtual spinner (e.g. The NCTM Illuminations website has an Adjustable Spinner).

Consider these issues as you explore:

- Explain who you think is most likely to win and why.
- For Sandu's turn (spinner A, then spinner B), would it matter if he spun B first and then A? Why or why not?
- How might you change one spinner so that Susan has the best chance at purple?

Strategy 1: Tree diagrams. On spinner A, the four colours each have the same chance of coming up. You could make a tree diagram for A with four branches and all the branches would have the same chance (see Figure 2.6). Spinner B has different-sized sections, leading to the following questions:

- What is the relationship between the blue region and each of the others?
- How could you make a tree diagram for B with each branch having the same chance?
- How can you add to the diagram for spinner A so that it represents spinning A twice in succession?
- Which branches on your diagram represent getting purple?
- How could you make tree diagrams for each player's choices?
- How do the tree diagrams relate to the spinners?

Tree diagrams are only one way to approach this. If this strategy makes sense to you, stop reading and solve the problem. If tree diagrams do not seem like a strategy you want to use, read on.

Strategy 2: Grids. Partition squares to represent all the possible outcomes for spinner A and spinner B. Although there are many ways to divide a square into four equal parts, if you use lines going all in the same direction, you can make comparisons of all the outcomes of one event (one whole square) with the outcomes of another event (drawn on a different square). For two independent events, you can then create lines going in the other direction for the second event. Samuel's two spins are represented in Figure 2.7(a). If these two squares are overlapped, you can visually see that two parts (two-sixteenths) are 'blue on red' or 'red on blue'. Susan's probability can be determined by layering the squares in Figure 2.7(b); and Sandu's from layering one square from Figure 2.7(a) with one from Figure 2.7(b).

Why are there four parts for spinner A and 6 parts for spinner B? How is the grid strategy alike and different from the tree diagram? One strategy may make more sense to you and one may make more sense to another. Hearing other students' explanations and reasoning for both strategies is important in building a strong understanding of mathematics.

Interesting mathematics problems such as the four presented here are plentiful. The Math Forum from NCTM, for example, has a large collection of classic problems along with discussion, solutions and extensions. *The Australian Mathematics Education Journal* is published by the Australian Association of Mathematics Teachers (AAMT) and the *Discovery* section includes problems. The Queensland Association of Mathematics Teachers' journal includes problems for which students can submit their solutions. The solutions are then given in the next issue.

WHERE ARE THE ANSWERS?

Did you notice that no answers were shared for these four rich tasks? How do you feel about not being able to check your answer? You may be wondering if your answer is correct or if there are other answers. We did this intentionally, because one aspect of becoming mathematically proficient is to be able to rely on one's own justification and reasoning to determine if an answer is correct.

Consider the message students receive when the textbook or the teacher is the source of whether an answer is correct: 'Your job is to find the answers that the teacher already has'. In the real world of problem-solving and doing mathematics, there are no answer books. Doing mathematics includes using justification as a means of determining whether an answer is correct.

▶ WHAT DOES IT MEAN TO BE MATHEMATICALLY PROFICIENT?

In setting learning objectives for students, we often ask, 'What will students know? What will students be able to do?' The previous section addressed what they should be able to do, here we focus on several important points related to what students need to know. An important aspect of knowing is understanding.

Let's go back to fractions as an example. What is important for a student to know about fractions such as $\frac{6}{8}$? What might a Year 4 student know about $\frac{6}{8}$? At what point do they know enough that they can claim they 'understand' fractions? It is more complicated than it might first appear. Here is a short list of what they might know or be able to do:

- Read the fraction.
- Identify the 6 and 8 as the numerator and denominator, respectively.
- Recognise it is equivalent to $\frac{3}{4}$.
- Say that it is more than $\frac{1}{2}$ (recognise relative size).
- Draw a region that is shaded in a way to show $\frac{6}{8}$.
- Find $\frac{6}{8}$ on a number line.
- Illustrate $\frac{6}{8}$ of a set of 48 coins or counters.
- Know that there are infinitely many equivalencies to $\frac{6}{8}$.
- Recognise $\frac{6}{8}$ as a fraction or rational number.
- Realise $\frac{6}{8}$ might also be describing a ratio (girls to boys, for example).
- Be able to represent $\frac{6}{8}$ as a decimal fraction.

For any item on this list, how much and what a student understands will vary. For example, a student may know that $\frac{6}{8}$ can be simplified to $\frac{3}{4}$ but not understand that $\frac{3}{4}$ and $\frac{6}{8}$ represent equal quantities, thinking that three-quarters is actually smaller. Or, they may be able to find

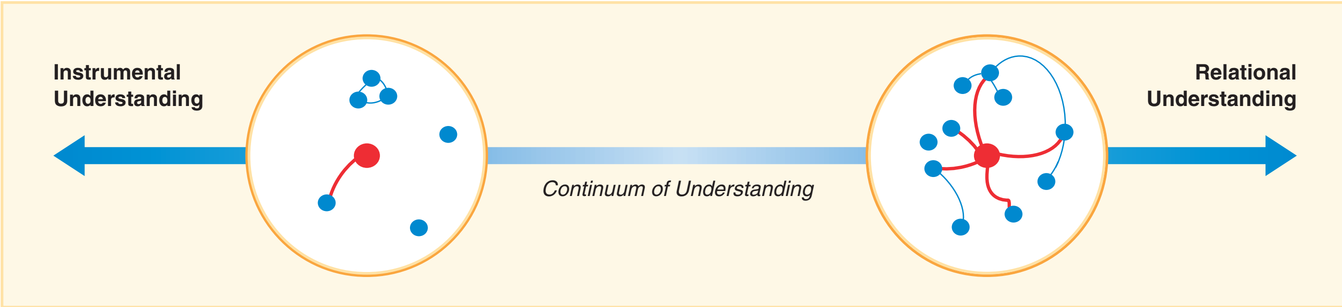


FIGURE 2.8 Understanding is a measure of the quality and quantity of connections that a new idea has with existing ideas. The greater the number of connections to a network of ideas, the better the understanding.

one fraction between $\frac{1}{2}$ and $\frac{6}{8}$, but not be able to find others or think there is only one fraction between these two fractions.

Understanding can be defined as a measure of the quality and quantity of connections that an idea has with existing ideas. Understanding is not an all-or-nothing proposition. It depends on the existence of appropriate ideas and on the creation of new connections, varying with each person (Backhouse, Haggarty, Pirie & Stratton, 1992; Davis, 1986; Hiebert & Carpenter, 1992).

RELATIONAL UNDERSTANDING

One way that we can think about understanding is that it exists along a continuum from an *instrumental understanding* – doing something without understanding (see Figure 2.8) – to a *relational understanding* – knowing what to do and why. These two terms were introduced by Richard Skemp in 1978 and continue to be an important distinction related to what is important for students to know about mathematics.

In the $\frac{6}{8}$ example, a student who only knows a procedure for simplifying $\frac{6}{8}$ to $\frac{3}{4}$ has an understanding near the instrumental end of the continuum, while a student who can draw diagrams, give examples, find equivalencies and tell the approximate size of $\frac{6}{8}$ has an understanding towards the relational end of the continuum. Here we briefly share two important ways to nurture a relational understanding.

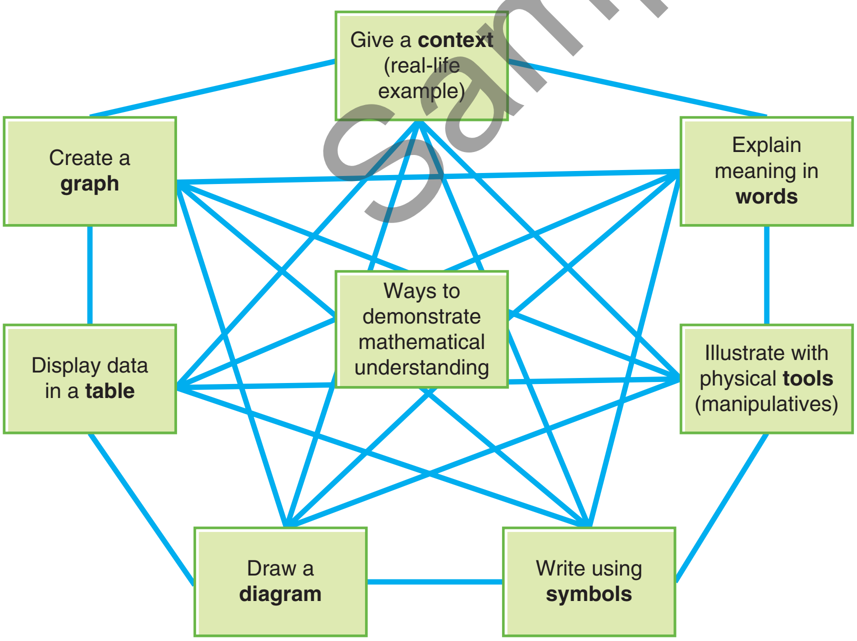
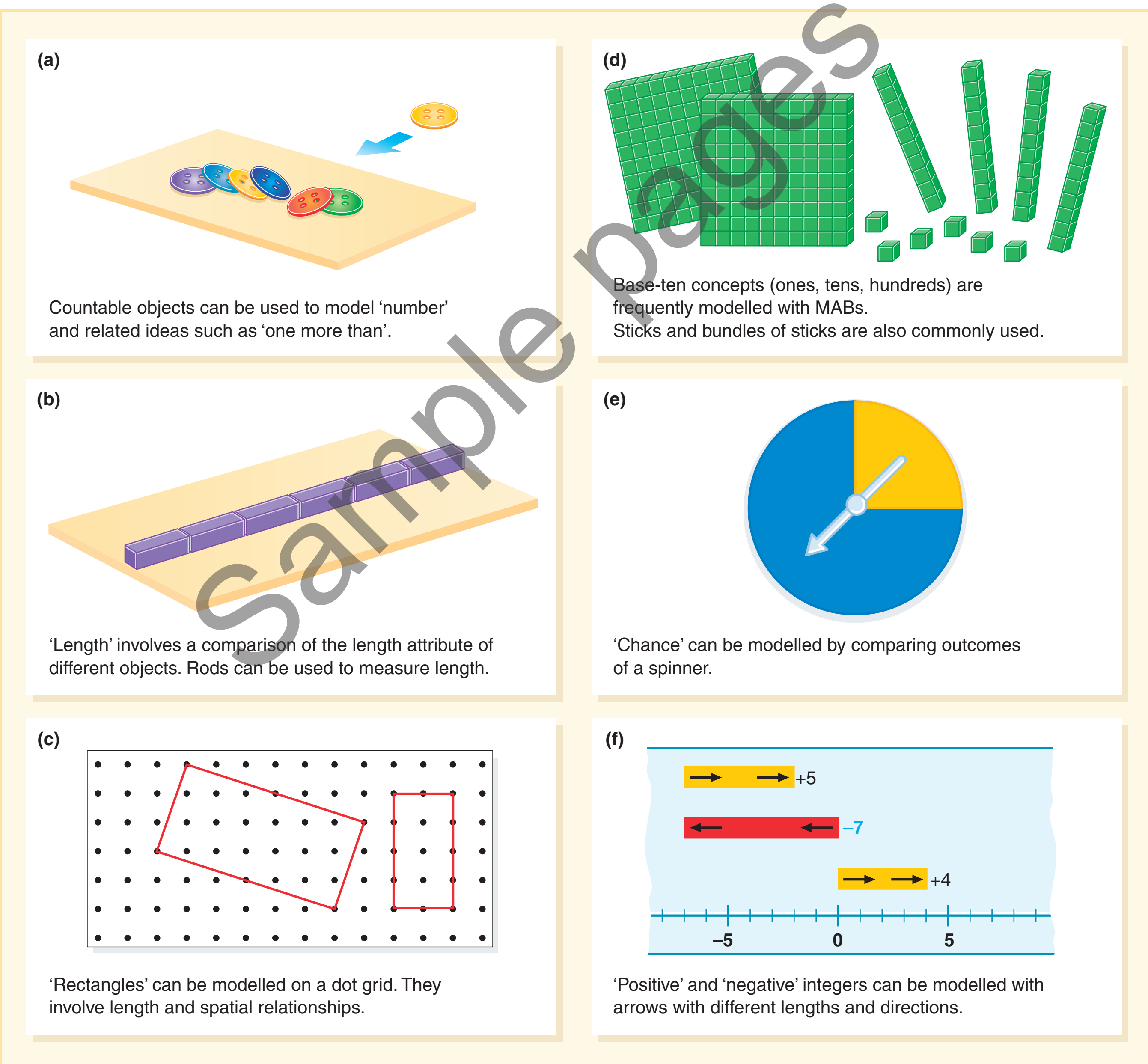


FIGURE 2.9 Web of representations: Translations between and within each representation of a mathematical idea can help students build a relational understanding of a mathematical concept.

Use and connect different representations. In order for students to build connections among ideas, different representations must be included in teaching and opportunities must be provided for students to make connections among the representations. Figure 2.9 illustrates a 'web of representations' that apply to any mathematical concept. Students who have difficulty translating a concept from one representation to another also have difficulty solving problems and understanding computations (Lesh, Cramer, Doerr, Post & Zawojewski, 2003). Strengthening the ability to move between and among these representations improves student understanding and retention. For any topic you teach, you can give students the **Think board** activity page (Gunningham, 2002) to complete. You can fill out one box and ask them to insert the other representations or you can invite a group to work on all four representations for a given topic (e.g. multiplication of whole numbers).

Explore with tools. A *tool* is any object, picture or drawing that can be used to explore a concept. Examples of tools for doing mathematics include calculators and manipulatives. *Manipulatives* are physical objects that students and teachers can use to illustrate and discover mathematical concepts, whether made specifically for mathematics (e.g. connecting cubes) or for other purposes (e.g. buttons). Choices for manipulatives (including virtual manipulatives) abound – from common objects such as small stones to commercially produced materials such as Pattern Blocks. Figure 2.10 shows six examples, each representing a different concept, just to give a glimpse (Part II of this book is full of more options). More and more of these manipulatives and others (e.g. geoboards, MABs (base-ten blocks), spinners, number lines) are available in a virtual format, for example, on the Scootle website and the NCTM Illuminations website. Each has a range of manipulatives available.



(a) Countable objects can be used to model 'number' and related ideas such as 'one more than'.

(b) 'Length' involves a comparison of the length attribute of different objects. Rods can be used to measure length.

(c) 'Rectangles' can be modelled on a dot grid. They involve length and spatial relationships.

(d) Base-ten concepts (ones, tens, hundreds) are frequently modelled with MABs. Sticks and bundles of sticks are also commonly used.

(e) 'Chance' can be modelled by comparing outcomes of a spinner.

(f) 'Positive' and 'negative' integers can be modelled with arrows with different lengths and directions.

FIGURE 2.10 Examples of tools to illustrate mathematics concepts.

A tool does not 'illustrate' a concept. The tool is used to visualise a mathematical concept and only your mind can impose the mathematical relationship on the object (Suh, 2007b; Thompson, 1994). As noted in the 'Best chance of purple' problem, manipulatives can be a testing ground for emerging ideas. They are more concrete and provide insights into new and abstract relationships. A variety of tools should be accessible for students to select and use appropriately as they engage in doing mathematics.

Before you continue, consider each of the concepts and the corresponding model in Figure 2.10. Try to separate the physical tool from the relationship that you must impose on the tool in order to 'see' the concept.

The examples in Figure 2.10 are models that can show the following concepts:

- The concept of '6' is a relationship between sets that can be matched to the words *one*, *two*, *three*, *four*, *five* or *six*. Changing a set of counters by adding one changes the relationship. The difference between the set of 6 and the set of 7 is the relationship 'one more than'.
- The concept of 'measure of length' is a comparison. The length measure of an object is a comparison relationship of the length of the object to the length of the unit.
- The concept of 'rectangle' includes both spatial and length relationships. The opposite sides are of equal length and parallel and the adjacent sides meet at right angles.
- The concept of 'hundred' is not in the larger square but in the relationship of that square to the strip (ten) and to the little square (one).
- 'Chance' is a relationship between the frequency of an event happening compared with all possible outcomes. The spinner can be used to create relative frequencies. These can be predicted by observing relationships of sections of the spinner.
- The concept of a 'negative integer' is based on the relationships of 'magnitude' and 'is the opposite of'. Negative quantities exist only in relation to positive quantities. Arrows on the number line model the opposite of relationship in terms of direction and size or magnitude relationship in terms of length.

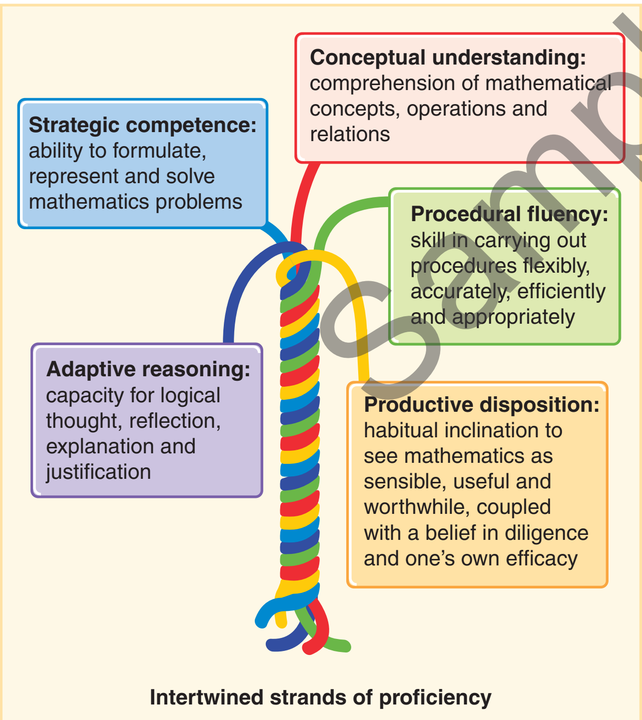


FIGURE 2.11 *Adding it up* describes five strands of mathematical proficiency.

Source: National Research Council. (2001). *Adding it up: Helping children learn mathematics*, p. 5. Reprinted with permission from the National Academy of Sciences, courtesy of the National Academies Press, Washington, DC.

While tools can be used to support relational understanding, they can be used ineffectively and not accomplish this goal. One of the most widespread misuses of tools occurs when the teacher tells students, 'do as I do'. There is a natural temptation to get out the materials and show students how to use them to 'show' the concept. It is just as possible to move blocks around mindlessly as it is to 'invert and multiply' mindlessly. Neither promotes thinking or aids in the development of concepts (Ball, 1992; Clements & Battista, 1990; Stein & Bovalino, 2001). At the other extreme, it is ineffective to provide no focus or purpose for using the tools. This will result in non-productive and un-systematic investigation (Stein & Bovalino, 2001).

MATHEMATICAL PROFICIENCY

The proficiency strands (or key ideas) of the Australian Curriculum: Mathematics (see Appendix A) describe what a mathematically proficient student can do. As discussed in Chapter 1, they are understanding, fluency, problem-solving and reasoning. These are *daily expectations* for doing mathematics, beginning in Foundation and continuing throughout school. Figure 2.11 illustrates the 'intertwined strands of proficiency' that was developed by the National Research Council (NRC) in *Adding it up* (NRC, 2001). These are very similar to the proficiency strands of the Australian Curriculum: Mathematics. Conceptual understanding in the diagram is similar to understanding in the Australian Curriculum; strategic competence is similar to problem-solving;

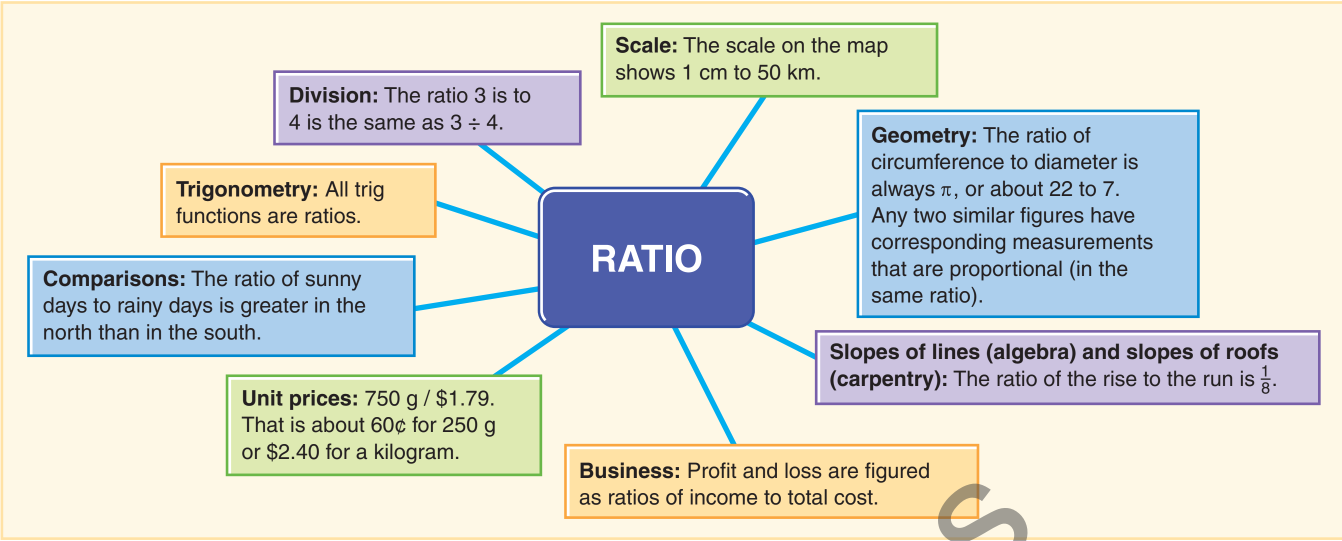


FIGURE 2.12 Potential web of ideas that could contribute to the understanding of 'ratio'.

procedural fluency is similar to fluency; and adaptive reasoning is similar to reasoning and highlights the intertwined nature of the strands. The productive disposition acknowledges the value of having general capability in numeracy (ACARA, 2016).

Conceptual understanding. Conceptual understanding is a flexible web of connections and relationships within and between ideas, interpretations and images of mathematical concepts – a relational understanding. Consider the web of associations for ratio as shown in Figure 2.12.

Students with a conceptual understanding will connect what they know about division and numbers to make sense of scaling, unit prices and so on. Note how much is involved in having a relational understanding of ratio.

Conceptual understanding includes the network of representations and interpretations of a concept through the use of pictures, manipulatives, tables, graphs, words and so on (see Figure 2.9). An illustration for ratios across these representations is provided in Figure 2.13.

Procedural fluency. Procedural fluency is sometimes confused with being able to do standard algorithms correctly and quickly, but it is much bigger than that. Look at the four descriptors of procedural fluency in Figure 2.11. Fluency includes having the ability to be flexible and to choose an appropriate strategy for a particular problem. Let's look at the problem 37 + 28. Younger students might be able to count all (see Figure 2.14(a)), or even start with the larger and count on, to reach a total. Eventually, skip counting can be used as a more efficient strategy and students are

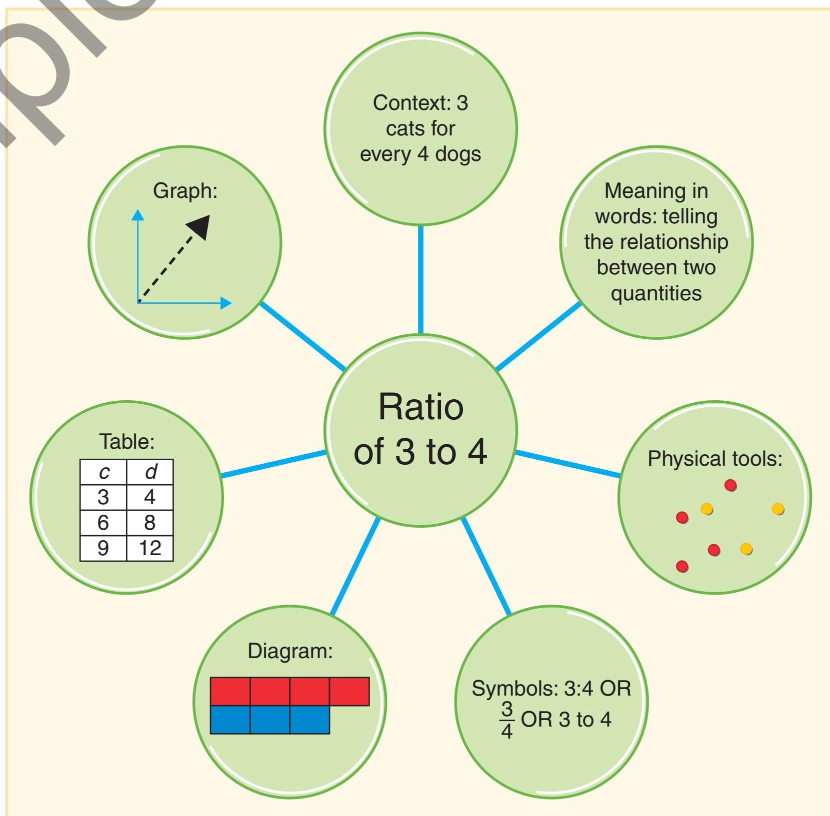


FIGURE 2.13 Multiple representations for ratio of 3 to 4.

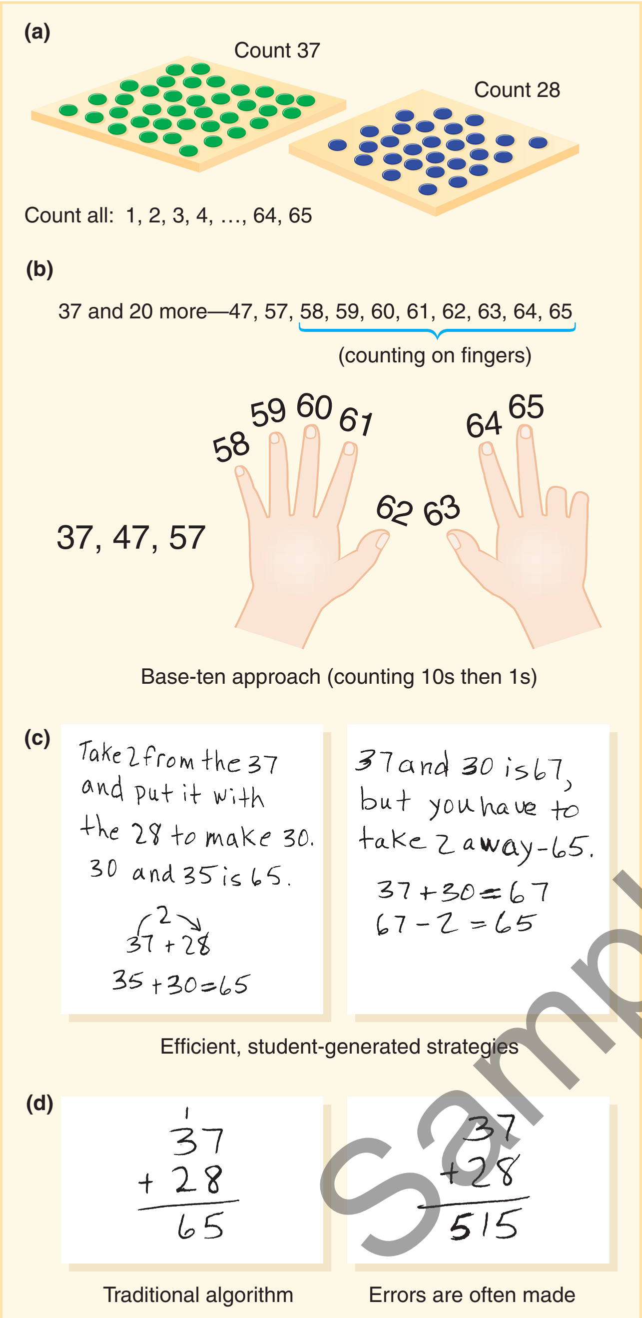
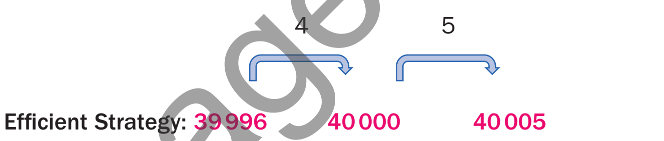


FIGURE 2.14 A range of levels of procedural fluency for 37 + 28.

able to count up by 10s and 1s (see Figure 2.14(b)). At a higher level of fluency, students are able to select a strategy that is efficient, for example, moving two from the 37 to the 28 to create a benchmark number or adding two on to 28 to add, and then taking it off again (see Figure 2.14(c)). Notice that to use these efficient and appropriate strategies requires a conceptual understanding of place value and addition.

The ineffective practice of teaching procedures in the absence of conceptual understanding results in a lack of retention and increased errors, rigid approaches and inefficient strategy use (Clarke, 2005) (Figure 2.14(d)).

Think about the following problem: 40 005 − 39 996. A student with rigid procedural skills may launch into the standard algorithm, regrouping across zeros (often with difficulty), rather than notice that the number 39 996 is just 4 away from 40 000, and therefore that the difference between the two numbers is 9.



Developing conceptual understanding alongside procedural proficiency is crucial to becoming mathematically proficient (Baroody, Feil & Johnson, 2007; Bransford, Brown & Cocking, 2000; NCTM, 2014). You can use the **Observation tool** that focuses on evidence of mathematical proficiency or the application of the proficiency strands in a classroom setting.

Perseverance and a productive disposition.

Being proficient at mathematics is not just what you know, but how you go about solving problems. Consider this short list of reflective prompts. Which ones might a proficient student say yes to often?

- When you read a problem you don't know how to solve, do you think, 'Cool, something challenging. I can solve this. Let me think how.'
- Do you consider several possible approaches before diving in to solve a problem?
- Do you have a way of convincing yourself or your peer that your solution to a problem is correct?
- Do you recognise a wrong path and try something else?
- When you finish a problem, do you wonder whether it is right? If there are other answers?
- Do you look for patterns across examples and try to see a new shortcut or approach that might work?
- As you work, do you decide to draw a picture, use a calculator or model the problem with a manipulative?

When students are in classrooms where they are able to *do* mathematics, these proficiencies develop and students build a stronger understanding of the mathematics they need to know, both the concepts and procedures, and are able to become mathematically proficient.

▶ HOW DO STUDENTS LEARN MATHEMATICS?

Now that you have had the chance to experience doing mathematics, you may have a series of questions: Can students solve such challenging tasks? Why take the time to solve these problems – isn't it better to do a lot of shorter problems? Why should students be doing problems like this, especially if they are reluctant to do so? In other words, how does 'doing mathematics' relate to student learning? The answer lies in learning theory and research on how people learn.

In mathematics education there is no consensus about what it means to know and understand mathematics. Theories such as behaviourism, cognitivism, constructivism and sociocultural theory have influenced the way in which mathematics is taught. Even within these theories, there are different interpretations of what they mean and what the interpretation of that theory into classroom practice might look like. As a teacher, you rely on your own beliefs and theories as you decide what you think will most help your students learn. Your beliefs may be influenced by theorists and come from your own pragmatic experiences. It is important for you to attend to your own beliefs and how they relate to your teaching practice (Davis & Sumara, 2012).

Learning theories have been developed through analysis of students and adults as they develop new understandings. They can be thought of as tools or lenses for interpreting how a person learns (Simon, 2009). Here we describe two theories, constructivism and sociocultural theory, that are commonly used by researchers to understand how students learn mathematics. These theories are not competing and are compatible with each other (Norton & D'Ambrosio, 2008).

CONSTRUCTIVISM

Constructivism is rooted in Jean Piaget's work, which was developed in the 1930s and translated to English in the 1950s. At the heart of constructivism is the notion that learners are not blank slates but rather creators or constructors of their own learning. Integrated *networks*, or *cognitive schemas*, are both the product of constructing knowledge and the tools with which new knowledge can be constructed. As learning occurs, the networks are rearranged, added to or modified. This is an active endeavour on the part of the learner (Baroody, 1987; Cobb, 1988; Fosnot, 1996; von Glasersfeld, 2014).

All people construct or give meaning to things they perceive or think about. As you read these words you are giving meaning to them. Whether listening passively to a lecture or actively engaging in synthesising findings in a project, your brain is applying prior knowledge to your existing schemas to make sense of the new information.

Through *reflective thought* – the effort needed to connect existing ideas to new information – people modify their existing schemas to incorporate new ideas (Fosnot, 1996). This can happen through *assimilation* or *accommodation*. Assimilation occurs when a new concept 'fits' with prior knowledge and the new information expands an existing network. Accommodation takes place when the new concept does not 'fit' with the existing network, causing what Piaget called *disequilibrium*, so the brain revamps or replaces the existing schema.

Construction of ideas. To construct or build something in the physical world requires tools, materials and effort. The tools we use to build understanding are our existing ideas and knowledge. The materials we use may be things we see, hear or touch, or our own thoughts and ideas. The effort required to connect new knowledge to old knowledge is reflective thought.

Consider the picture in Figure 2.15 to be a small section of our cognitive makeup. The blue dots represent existing ideas. The lines joining the ideas represent our logical connections or relationships that have developed between and among ideas. The red dot is an emerging idea, one that is being constructed. Whatever existing ideas (blue dots) are used in the construction will be connected to the new idea (red dot) because those were the ideas that gave meaning to it. If a potentially relevant idea (blue dot) is not accessed by the learner when learning a new concept (red dot), then that potential connection will not be made. For more

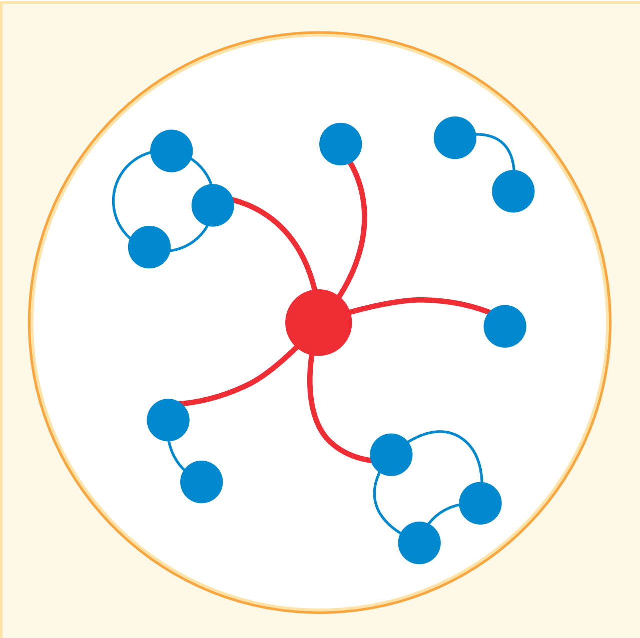


FIGURE 2.15 We use the ideas we already have (blue dots) to construct a new idea (red dot), in the process developing a network of connections between ideas. The more ideas used and the more connections made, the better we understand.

information on how constructivism applies to mathematics education, the Math Forum offers links to numerous sites and articles.

SOCIOCULTURAL THEORY

In the 1920s and 1930s, Lev Vygotsky, a Russian psychologist, began developing what is now called sociocultural theory. There are many theoretical ideas that sociocultural theory shares with constructivism (for example, the learning process as active meaning-seeking on the part of the learner), but sociocultural theory has several unique features. One is that mental processes exist between and among people in social learning settings, and that from these social settings the learner moves ideas into their own psychological realm (Forman, 2003).

An important aspect of sociocultural theory is that the way in which information is internalised, or learned, depends on whether it was within a learner's zone of proximal development (ZPD) (Vygotsky, 1978). Simply put, the ZPD refers to a range of knowledge that may be out of reach for a person to learn on their own, but is accessible if the learner has support from peers or more knowledgeable others. Researchers Cobb (1994) and Goos (2004) suggest that in a true mathematical community of learners there is something of a common ZPD that emerges across learners and there are also the ZPDs of individual learners.

Another major component in sociocultural theory is *semiotic mediation*. Semiotic refers to the use of language and other tools, such as diagrams, pictures and actions, to convey cultural practices. Mediation means that these semiotics are exchanged between and among people. So, semiotic mediation is the way in which an individual's beliefs, attitudes and goals affect and are affected by sociocultural practices (Forman & McPhail, 1993). In mathematics, semiotics include mathematical symbols (e.g. the equal sign) and it is through classroom interactions and activities that the meaning of these symbols are developed.

Social interaction is essential for learning to occur. The nature of the community of learners is affected by not just the culture the teacher creates, but the broader social and historical culture of the members of the classroom (Forman, 2003). In summary, from a sociocultural perspective, learning is dependent on the new knowledge falling within the ZPD of the learner who must have access to the assistance, and occurs through interactions that are influenced by tools of mediation and the culture within and beyond the classroom.

IMPLICATIONS FOR TEACHING MATHEMATICS

It is not necessary to choose between a social constructivist theory that favours the views of Vygotsky and a cognitive constructivism built on the theories of Piaget (Cobb, 1994; Simon, 2009). In fact, when considering classroom practices that maximise opportunities to construct ideas, or to provide tools to promote mediation, they are quite similar. Classroom discussion based on students' own ideas and solutions to problems is essential to learning (Wood & Turner-Vorbeck, 2001).

Remember that learning theory is not a teaching strategy – theory *informs* teaching. This section outlines teaching strategies that are informed by constructivist and sociocultural perspectives. You will see these strategies revisited in more detail in Chapters 3 and 4, where a problem-based model for instruction is discussed, and in Section II of this book, where you learn how to apply these ideas to specific areas of mathematics.

Importantly, if these strategies are grounded in how people learn, it means *all* people learn this way – students with additional learning needs, English as an additional language or dialect (EAL/D) students, students who struggle and students who are gifted. Too often, when teachers make adaptations and modifications for particular learners, they trade in strategies that align with learning theories and research for methods that seem 'easier' for students. These strategies, however, provide fewer opportunities for students to connect ideas and build knowledge, thereby impeding, not supporting, learning.

Build new knowledge from prior knowledge. If you are teaching a new concept, like division, it must be developed using what students already know about sharing and repeated subtraction. Consider the following task.

Goodies Toy Shop is creating bags with 3 squishy balls in each. If they have 24 squishy balls, how many bags will they be able to make?

Here, consider how you might introduce division to Year 3 students and what your expectations might be for this problem as a teacher grounding your work in constructivist or sociocultural learning theory.

From a constructivist and sociocultural perspective, this classroom culture allows students to access their prior knowledge, use cultural tools and build new knowledge. You might ask students to use manipulatives or to draw pictures to solve this problem. As they work, they might have different ways of thinking about the problem (e.g. skip counting up by 3s, or skip counting down by 3s). These ideas become part of a classroom discussion, connecting what they know about repeated subtraction and repeated addition, and connecting that to multiplication and division. Interestingly, this practice of connecting ideas is not only grounded in learning theory, but has been established through research studies.

Recall that making mathematical relationships explicit is connected with improving student conceptual understanding (Hiebert & Grouws, 2007). The teacher's role in making mathematical relationships explicit is to be sure that students are making the connections that are implied in a task. For example, asking students to relate today's topic to one they investigated last week and asking 'How is Lisa's strategy like Marco's strategy?' when the two students have picked different ways to solve a problem are both ways to be 'explicit' about mathematical relationships. Students apply their prior knowledge, test ideas, make connections, compare and make conjectures. The more students see the connections among problems and among mathematical concepts, the more deeply they understand mathematics.

Provide opportunities to communicate about mathematics. Learning is enhanced when the learner is engaged with others working on the same ideas. The rich interaction in such a classroom allows students to engage in reflective thinking and to internalise concepts that may be out of reach without the interaction and input from peers and their teacher. In discussions with peers, students will be adapting and expanding on their existing networks of concepts.

Create opportunities for reflective thought. Classrooms need to provide structures and supports to help students make sense of mathematics in light of what they know. For a new idea you are teaching to be interconnected in a rich web of inter-related ideas, children must be mentally engaged. They must find the relevant ideas they possess and bring them to bear on the development of the new idea. In terms of the dots in Figure 2.15, we want to activate every blue dot students have that is related to the new red dot we want them to learn. It is through thinking, talking and writing that we can help students reflect on how mathematical ideas are connected to each other.

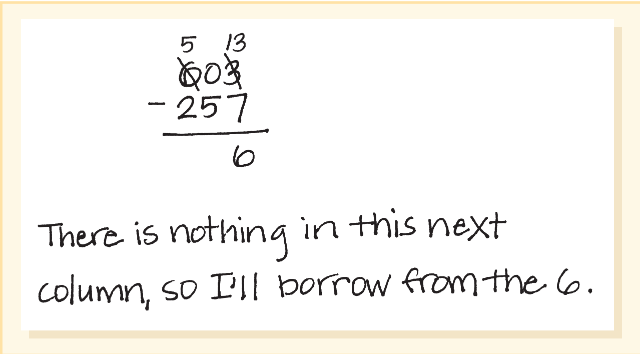


FIGURE 2.16 This student's work indicates that she has a misconception about place value and regrouping.

Encourage multiple approaches. Encourage students to use strategies that make sense to them. The student whose work is presented in Figure 2.16 may not understand the algorithm she used. If instead she were asked to use her own approach to find the difference, she might be able to get to a correct solution and build on her understanding of place value and subtraction.

Even learning a basic fact like 7×8 can have better results if a teacher promotes multiple strategies. Imagine a class where students discuss and share clever ways to work out the product. One student might think of 5 eights (40) and then 2 more eights (16) to equal 56. Another may have learned 7×7 (49) and added on 7 more to get 56. Still another might think '8 sevens' and take half of the sevens (4×7) to get 28 and double 28 to get 56. A class discussion sharing these ideas brings to the fore a wide range of useful mathematical 'dots' relating addition and multiplication concepts.

Engage students in productive struggle. Have you ever just wanted to think through something yourself without being interrupted or told how to do it? Yet, how often in mathematics class does this happen? As soon as a student pauses in solving a problem, the teacher steps in to show or explain. While this may initially get the student to an answer faster, it does not help the student learn mathematics – engaging in productive struggle is what helps students learn mathematics. As Piaget describes, learners are going to experience disequilibrium in developing new ideas. Let students know this disequilibrium is part of the process. This is also one of the findings mentioned earlier as key to developing conceptual understanding (Hiebert & Grouws, 2007).

Notice the importance of both words in 'productive struggle'. Students must have the tools and prior knowledge to solve a problem and not be given a problem that is out of reach, or they will struggle without being productive; yet students should not be given tasks that are straightforward and easy or they will not be struggling with mathematical ideas. When students, even very young students, know that struggle is expected as part of the process of doing mathematics, they embrace the struggle and feel success when they reach a solution (Carter, 2008).

This means redefining what it means to 'help' students. Rather than showing students how to do something, your role is to ask probing questions that keep students engaged in the productive struggle until they reach a solution. This communicates high expectations and maximises students' opportunities to learn with understanding.

Treat errors as opportunities for learning. When students make errors, it can mean a misapplication of their prior knowledge in the new situation. Remember that from a constructivist perspective, the mind is sifting through what it knows in order to find useful approaches for the new situation. Students rarely give random responses, so their errors are insight into misconceptions they might have. For example, students comparing decimals may incorrectly apply 'rules' of whole numbers, such as 'the more digits, the bigger the number' (Steinle & Stacey, 2004b). Often one student's misconception is shared by others in the class and discussing the problem publicly can help other students understand (Hoffman, Breyfogle & Dressler, 2009). You can introduce errors and ask students to imagine what might have led to that answer (Rathouz, M., 2011). This public negotiation of meaning allows students to construct deeper meaning for the mathematics they are learning.

Scaffold new content. The practice of *scaffolding*, often associated with sociocultural theory, is based on the idea that a task otherwise outside of a student's ZPD can become accessible if it is carefully structured. For concepts completely new to students, the learning requires more structure or assistance, including the use of tools like manipulatives or more assistance from peers. As students become more comfortable with the content, the scaffolds are removed and the student becomes more independent. Scaffolding can provide support for those students who may not have a robust collection of 'blue dots'.

Honour diversity. Finally, and importantly, these theories emphasise that each learner is unique, with a different collection of prior knowledge and cultural experiences. Since new knowledge is built on existing knowledge and experience, effective teaching incorporates and builds on what the students bring to the classroom, honouring those experiences. Thus, lessons begin with eliciting prior experiences, and understandings and contexts for the lessons are selected based on students' knowledge and experiences. Some students will not have all the 'blue dots' they need – it is your job to provide experiences where those blue dots are developed and then connected to the concept being learned.

Classroom culture influences the individual learning of your students. As stated previously, you should support a range of approaches and strategies for doing mathematics. Students' ideas should be valued and included in classroom discussions of mathematics. This shift in practice, away from the teacher outlining one way to work on the problem, establishes a classroom culture where ideas are valued. This approach values the uniqueness of each individual.

Create a classroom environment for doing mathematics. Classrooms where students are making sense of mathematics do not happen by accident – they happen because the teacher establishes practices and expectations that encourage risk-taking, reasoning, sharing and so on. The list below provides expectations that are often cited as ones that support students in doing mathematics (Clarke & Clarke, 2004; Hiebert et al., 1997; NCTM, 2007).

1. *Persistence, effort and concentration are important in learning mathematics.* Engaging in productive struggle is important in learning. The more a student stays with a problem, the more likely they are to get it right. Getting a tough problem right leads to a stronger sense of accomplishment than getting a quick, easy problem correct.
2. *Students share their ideas.* Everyone's ideas are important and hearing different ideas helps students to become strategic in selecting good strategies.
3. *Students listen to each other.* All students have something to contribute and these ideas should be considered and evaluated for whether they will work in that situation.
4. *Errors or strategies that didn't work are opportunities for learning.* Mistakes are opportunities for learning – why did that approach not work? Could it be adapted and work or is a completely different approach needed? Doing mathematics involves monitoring and reflecting on the process – catching and adjusting errors along the way.
5. *Students look for and discuss connections.* Students should see connections between different strategies to solve a particular problem, as well as connections to other mathematics concepts and to real contexts and situations. When students look for and discuss connections, they see mathematics as worthwhile and important, rather than an isolated collection of facts.

These five features are evident in what teachers do and what students do. You can visit classrooms or record your own teaching and use the **Observation tool: Classroom environment for doing mathematics** to see the extent to which students are becoming mathematically proficient. This is a way to see how you can establish such an environment in your own classroom.

▶ CONNECTING THE DOTS

It seems appropriate to close this chapter by connecting some dots, especially because the ideas represented here are the foundation for the approach to each topic in the content chapters. This chapter began with discussing what *doing* mathematics is and challenging you to do some mathematics. Each of these tasks offered opportunities to make connections between mathematics concepts – connecting the blue dots.

Second, you read about what is important to *know* about mathematics – that having relational knowledge (knowledge in which blue dots are well connected) requires conceptual and procedural understanding as well as other proficiencies. The problems that you solved in the first section emphasised concepts and procedures while placing you in a position to use strategic competence, adaptive reasoning and a productive disposition.

Finally, you read how learning theory – the importance of having opportunities to connect the dots – connects to mathematics learning. The best learning opportunities, according to constructivism and sociocultural theories, are those that engage learners in using their own knowledge and experience to solve problems through social interactions and reflection. This is what you were asked to do in the four tasks. Did you learn something new about mathematics? Did you connect an idea that you had not previously connected?

This chapter focused on connecting the dots between theory and practice – building a case that your teaching must focus on opportunities for students to develop their own networks of blue dots. As you plan and design instruction, you should constantly reflect on how to elicit prior knowledge by designing tasks that reflect the social and cultural backgrounds of students, to challenge students to think critically and creatively, and to include a comprehensive treatment of mathematics.

REFLECTIONS ON CHAPTER 2

WRITING TO LEARN

Assess your understanding and application of chapter content by answering the following questions.

1. How would you describe what it means to *do* mathematics?
2. Select three of the verbs for doing mathematics. For each, think about what it looks like when a student is 'doing' it, then explain or draw a picture of what it might look like.
3. What is important to know about relational understanding?
4. Using the task 'One up, one down' as the example, describe how to implement it with students in a way that reflects constructivist and/or sociocultural learning theory.

FOR DISCUSSION AND EXPLORATION

Consider the task below and respond to the following questions.

- Some people say that to add four consecutive numbers, you add the first and the last numbers and multiply by 2. (Stoessiger & Edmunds, 1992)
  - **a.** Is this always true? How do you know?
  - **b.** What features of 'doing mathematics' does it have?
  - **c.** What web of ideas do you need to draw on to make sense of the problem?
  - **d.** To what extent does the task have the potential to develop mathematical proficiency?
- Not every educator believes in the constructivist-orientated approach to teaching mathematics. Some of their reasons include the following: there is not enough time to let kids discover everything; basic facts and ideas are better taught through quality explanations; students should not have to 'reinvent the wheel'. How would you respond to these arguments?

RESOURCES FOR CHAPTER 2

RECOMMENDED READINGS

Articles

Berkman, R. M. (2006). One, some, or none: Finding beauty in ambiguity. *Mathematics Teaching in the Middle School, 11*(7), 324–327.

This article offers a great teaching strategy for nurturing relational thinking. Examples of the engaging 'one, some or none' activity are given for geometry, number and algebra activities.

Carter, S. (2008). Disequilibrium & questioning in the primary classroom: Establishing routines that help students learn. *Teaching Children Mathematics, 15*(3), 134–137.

This is a wonderful teacher's story of how she infused the constructivist notion of disequilibrium and the related idea of productive struggle to support learning in her Year 1 class.

Hedges, M., Huinker, D. & Steinmeyer, M. (2005). Unpacking division to build teachers' mathematical knowledge. *Teaching Children Mathematics, 11*(9), 478–483.

This article describes the many concepts related to division.

Suh, J. (2007). Tying it all together: Classroom practices that promote mathematical proficiency for all students. *Teaching Children Mathematics, 14*(3), 163–169.

As the title implies, this is a great resource for connecting the NRC's Mathematics Proficiencies (National Research Council, 2001) to teaching.

Books

Lampert, M. (2001). *Teaching problems and the problems of teaching*. New Haven, CT: Yale University Press.

Lampert reflects on her personal experiences in teaching Year 5 and shares her perspectives on the many issues and complexities of teaching. It is wonderfully written and easily accessed at any point in the book.

Mason, J., Burton, L. & Stacey, K. (2010). *Thinking mathematically* (2nd ed.). Harlow, England: Pearson Education.

This classic book is about doing mathematics. There are excellent problems to explore along the way, with strategy suggestions. It is an engaging book that will help you learn more about your own problem-solving and become a better teacher of mathematics.

Sullivan, P. (2011) *Teaching mathematics: Using research-informed strategies*. Victoria: Australian Council for Educational Research.

In this review, Sullivan describes the goals of school mathematics in terms of the intertwined strands of proficiency and suggests ways to be effective teachers of mathematics using meaningful tasks.

Websites

The Australian Mathematics Teacher: www.aamt.edu.au.

NCTM Illuminations: https://illuminations.nctm.org

NCTM Maths Forum: www.nctm.org/mathforum.

Queensland Association of Mathematics Teachers' journal: http://qamt.qld.edu.au/resources/publications/?

Scootle: www.scootle.edu.au/ec/p/home.