SECTION 1

The material from this section comes from

College Mathematics for Business, Economics, Life Sciences & Social Sciences
Twelfth Edition
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PART 1

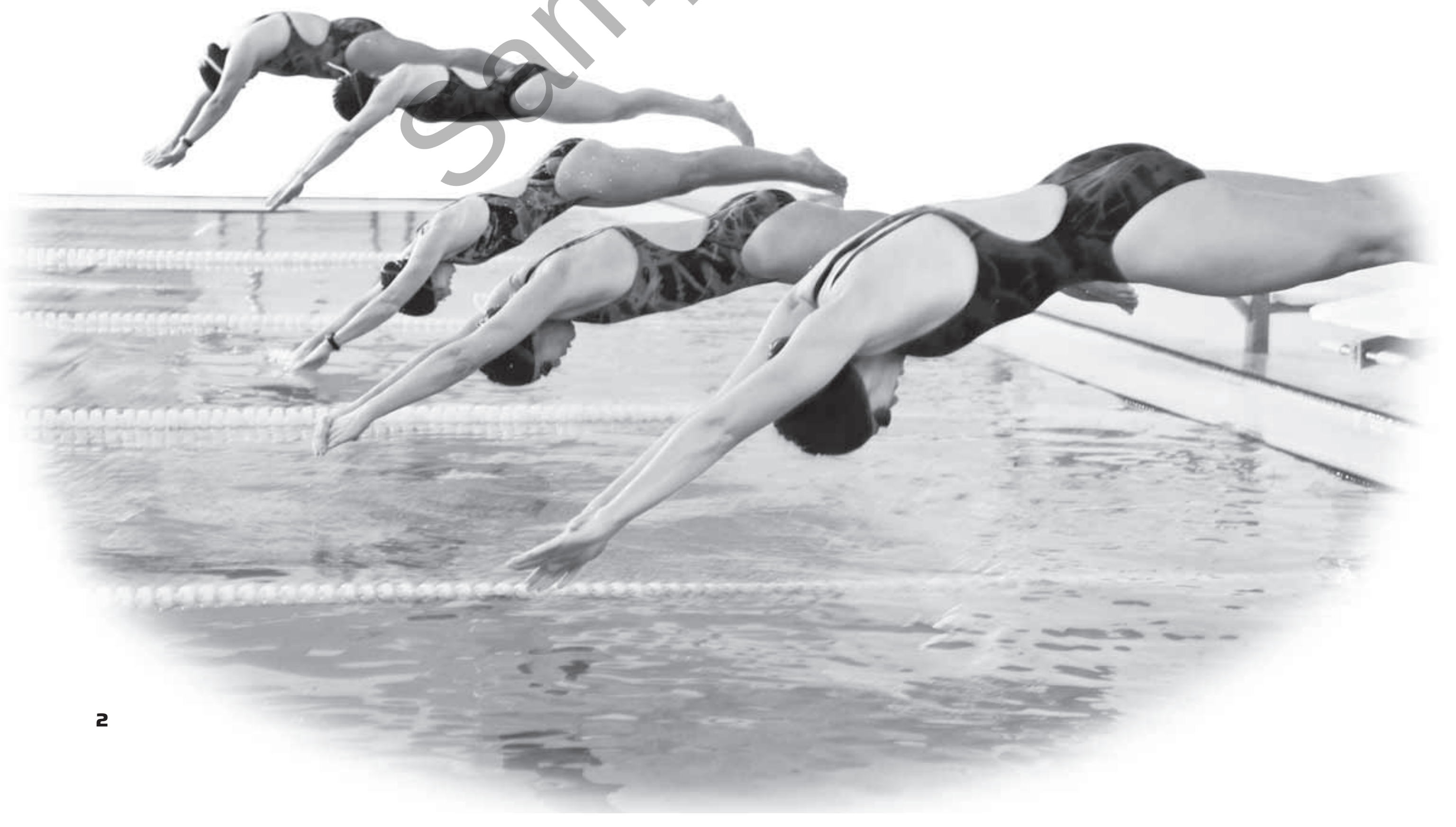
A LIBRARY OF ELEMENTARY FUNCTIONS

1

Linear Equations and Graphs

Introduction

We begin by discussing some algebraic methods for solving equations and inequalities. Next, we introduce coordinate systems that allow us to explore the relationship between algebra and geometry. Finally, we use this algebraic–geometric relationship to find equations that can be used to describe real-world data sets. For example, in Section 1-3 you will learn how to find the equation of a line that fits data on winning times in an Olympic swimming event (see Problems 27 and 28 on page 38). We also consider many applied problems that can be solved using the concepts discussed in this chapter.



1-1 Linear Equations and Inequalities

- Linear Equations
- Linear Inequalities
- Applications

The equation

$$3 - 2(x + 3) = \frac{x}{3} - 5$$

and the inequality

$$\frac{x}{2} + 2(3x - 1) \geq 5$$

are both first degree in one variable. In general, a **first-degree**, or **linear**, **equation** in one variable is any equation that can be written in the form

$$\textbf{Standard form:} \quad ax + b = 0 \qquad a \neq 0 \tag{1}$$

If the equality symbol, =, in (1) is replaced by <, >, ≤, or ≥, the resulting expression is called a **first-degree**, or **linear**, **inequality**.

A **solution** of an equation (or inequality) involving a single variable is a number that when substituted for the variable makes the equation (or inequality) true. The set of all solutions is called the **solution set**. When we say that we **solve an equation** (or inequality), we mean that we find its solution set.

Knowing what is meant by the solution set is one thing; finding it is another. We start by recalling the idea of equivalent equations and equivalent inequalities. If we perform an operation on an equation (or inequality) that produces another equation (or inequality) with the same solution set, then the two equations (or inequalities) are said to be **equivalent**. The basic idea in solving equations or inequalities is to perform operations that produce simpler equivalent equations or inequalities and to continue the process until we obtain an equation or inequality with an obvious solution.

Linear Equations

Linear equations are generally solved using the following equality properties.

THEOREM 1 Equality Properties

An equivalent equation will result if

1. The same quantity is added to or subtracted from each side of a given equation.
2. Each side of a given equation is multiplied by or divided by the same nonzero quantity.

EXAMPLE 1 **Solving a Linear Equation** Solve and check:

$$8x - 3(x - 4) = 3(x - 4) + 6$$

SOLUTION

$$\begin{aligned} 8x - 3(x - 4) &= 3(x - 4) + 6 && \text{Use the distributive property.} \\ 8x - 3x + 12 &= 3x - 12 + 6 && \text{Combine like terms.} \\ 5x + 12 &= 3x - 6 && \text{Subtract } 3x \text{ from both sides.} \\ 2x + 12 &= -6 && \text{Subtract 12 from both sides.} \\ 2x &= -18 && \text{Divide both sides by 2.} \\ x &= -9 \end{aligned}$$

CHECK

$$\begin{aligned} 8x - 3(x - 4) &= 3(x - 4) + 6 \\ 8(-9) - 3[(-9) - 4] &\stackrel{?}{=} 3[(-9) - 4] + 6 \\ -72 - 3(-13) &\stackrel{?}{=} 3(-13) + 6 \\ -33 &\stackrel{\checkmark}{=} -33 \end{aligned}$$

Matched Problem 1 Solve and check: $3x - 2(2x - 5) = 2(x + 3) - 8$

EXPLORE & DISCUSS 1 According to equality property 2, multiplying both sides of an equation by a nonzero number always produces an equivalent equation. What is the smallest positive number that you could use to multiply both sides of the following equation to produce an equivalent equation without fractions?

$$\frac{x+1}{3} - \frac{x}{4} = \frac{1}{2}$$

EXAMPLE 2 **Solving a Linear Equation** Solve and check: $\frac{x+2}{2} - \frac{x}{3} = 5$

SOLUTION What operations can we perform on

$$\frac{x+2}{2} - \frac{x}{3} = 5$$

to eliminate the denominators? If we can find a number that is exactly divisible by each denominator, we can use the multiplication property of equality to clear the denominators. The LCD (least common denominator) of the fractions, 6, is exactly what we are looking for! Actually, any common denominator will do, but the LCD results in a simpler equivalent equation. So, we multiply both sides of the equation by 6:

$$6\left(\frac{x+2}{2} - \frac{x}{3}\right) = 6 \cdot 5 \quad *$$

$$\overset{3}{\cancel{6}} \cdot \frac{(x+2)}{\underset{1}{\cancel{2}}} - \overset{2}{\cancel{6}} \cdot \frac{x}{\underset{1}{\cancel{3}}} = 30$$

$$3(x+2) - 2x = 30 \quad \text{Use the distributive property.}$$

$$3x + 6 - 2x = 30 \quad \text{Combine like terms.}$$

$$x + 6 = 30 \quad \text{Subtract 6 from both sides.}$$

$$x = 24$$

CHECK

$$\frac{x+2}{2} - \frac{x}{3} = 5$$

$$\frac{24+2}{2} - \frac{24}{3} \stackrel{?}{=} 5$$

$$13 - 8 \stackrel{?}{=} 5$$

$$5 \stackrel{\checkmark}{=} 5$$

Matched Problem 2 Solve and check: $\frac{x+1}{3} - \frac{x}{4} = \frac{1}{2}$

In many applications of algebra, formulas or equations must be changed to alternative equivalent forms. The following example is typical.

EXAMPLE 3 **Solving a Formula for a Particular Variable** If you deposit a principal P in an account that earns simple interest at an annual rate r, then the amount A in the account after t years is given by $A = P + Prt$. Solve for

(A) r in terms of A, P, and t

(B) P in terms of A, r, and t

*Dashed boxes are used throughout the book to denote steps that are usually performed mentally.

SOLUTION (A)

$$A = P + Prt \quad \text{Reverse equation.}$$
$$P + Prt = A \quad \text{Subtract } P \text{ from both sides.}$$
$$Prt = A - P \quad \text{Divide both members by } Pt.$$
$$r = \frac{A - P}{Pt}$$

(B)

$$A = P + Prt \quad \text{Reverse equation.}$$
$$P + Prt = A \quad \text{Factor out } P \text{ (note the use of the distributive property).}$$
$$P(1 + rt) = A \quad \text{Divide by } (1 + rt).$$
$$P = \frac{A}{1 + rt}$$

Matched Problem 3 If a cardboard box has length L, width W, and height H, then its surface area is given by the formula $S = 2LW + 2LH + 2WH$. Solve the formula for

(A) L in terms of S, W, and H

(B) H in terms of S, L, and W

Linear Inequalities

Before we start solving linear inequalities, let us recall what we mean by < (less than) and > (greater than). If a and b are real numbers, we write

$$a < b \quad a \text{ is less than } b$$

if there exists a positive number p such that $a + p = b$. Certainly, we would expect that if a positive number was added to any real number, the sum would be larger than the original. That is essentially what the definition states. If $a < b$, we may also write

$$b > a \quad b \text{ is greater than } a.$$

EXAMPLE 4 **Inequalities**

(A) $3 < 5$ Since $3 + 2 = 5$

(B) $-6 < -2$ Since $-6 + 4 = -2$

(C) $0 > -10$ Since $-10 < 0$ (because $-10 + 10 = 0$)

Matched Problem 4 Replace each question mark with either < or >.

(A) $2 \, ? \, 8$ (B) $-20 \, ? \, 0$ (C) $-3 \, ? \, -30$

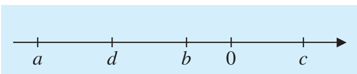


Figure 1 $a < b, c > d$

The inequality symbols have a very clear geometric interpretation on the real number line. If $a < b$, then a is to the left of b on the number line; if $c > d$, then c is to the right of d on the number line (Fig. 1). Check this geometric property with the inequalities in Example 4.

EXPLORE & DISCUSS 2 Replace **?** with < or > in each of the following:

(A) $-1 \, ? \, 3$ and $2(-1) \, ? \, 2(3)$

(B) $-1 \, ? \, 3$ and $-2(-1) \, ? \, -2(3)$

(C) $12 \, ? \, -8$ and $\frac{12}{4} \, ? \, \frac{-8}{4}$

(D) $12 \, ? \, -8$ and $\frac{12}{-4} \, ? \, \frac{-8}{-4}$

Based on these examples, describe verbally the effect of multiplying both sides of an inequality by a number.

The procedures used to solve linear inequalities in one variable are almost the same as those used to solve linear equations in one variable, but with one important exception, as noted in item 3 of Theorem 2.

THEOREM 2 Inequality Properties

An equivalent inequality will result, and the **sense or direction will remain the same** if each side of the original inequality

1. has the same real number added to or subtracted from it.
2. is multiplied or divided by the same *positive* number.

An equivalent inequality will result, and the **sense or direction will reverse** if each side of the original inequality

3. is multiplied or divided by the same *negative* number.

NOTE: Multiplication by 0 and division by 0 are not permitted.

Therefore, we can perform essentially the same operations on inequalities that we perform on equations, with the exception that **the sense of the inequality reverses if we multiply or divide both sides by a negative number**. Otherwise, the sense of the inequality does not change. For example, if we start with the true statement

$$-3 > -7$$

and multiply both sides by 2, we obtain

$$-6 > -14$$

and the sense of the inequality stays the same. But if we multiply both sides of $-3 > -7$ by -2, the left side becomes 6 and the right side becomes 14, so we must write

$$6 < 14$$

to have a true statement. The sense of the inequality reverses.

If $a < b$, the **double inequality** $a < x < b$ means that $\boldsymbol{a < x}$ **and** $\boldsymbol{x < b}$; that is, x is between a and b. **Interval notation** is also used to describe sets defined by inequalities, as shown in Table 1.

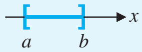
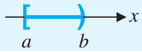
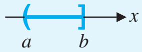
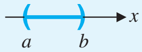
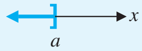
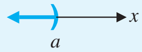
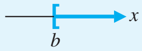
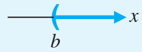
The numbers a and b in Table 1 are called the **endpoints** of the interval. An interval is **closed** if it contains all its endpoints and **open** if it does not contain any of its endpoints. The intervals $[a, b]$, $(-\infty, a]$, and $[b, \infty)$ are closed, and the intervals (a, b), $(-\infty, a)$, and (b, ∞) are open. Note that the symbol ∞ (read infinity) is not a number. When we write $[b, \infty)$, we are simply referring to the interval that starts at b and continues indefinitely to the right. We never refer to ∞ as an endpoint, and we never write $[b, \infty]$. The interval $(-\infty, \infty)$ is the entire real number line.

Note that an endpoint of a line graph in Table 1 has a square bracket through it if the endpoint is included in the interval; a parenthesis through an endpoint indicates that it is not included.

CONCEPTUAL INSIGHT

The notation $(2, 7)$ has two common mathematical interpretations: the ordered pair with first coordinate 2 and second coordinate 7, and the open interval consisting of all real numbers between 2 and 7. The choice of interpretation is usually determined by the context in which the notation is used. The notation $(2, -7)$ could be interpreted as an ordered pair but not as an interval. In interval notation, the left endpoint is always written first. So, $(-7, 2)$ is correct interval notation, but $(2, -7)$ is not.

Table 1 Interval Notation

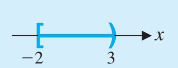
Interval Notation	Inequality Notation	Line Graph
$[a, b]$	$a \le x \le b$	
$[a, b)$	$a \le x < b$	
$(a, b]$	$a < x \le b$	
(a, b)	$a < x < b$	
$(-\infty, a]$	$x \le a$	
$(-\infty, a)$	$x < a$	
$[b, \infty)$	$x \ge b$	
(b, ∞)	$x > b$	

EXAMPLE 5 **Interval and Inequality Notation, and Line Graphs**

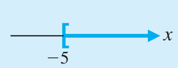
(A) Write $[-2, 3)$ as a double inequality and graph.

(B) Write $x \ge -5$ in interval notation and graph.

SOLUTION (A) $[-2, 3)$ is equivalent to $-2 \le x < 3$.



(B) $x \ge -5$ is equivalent to $[-5, \infty)$.



Matched Problem 5 (A) Write $(-7, 4]$ as a double inequality and graph.

(B) Write $x < 3$ in interval notation and graph.

EXPLORE & DISCUSS 3 The solution to Example 5B shows the graph of the inequality $x \ge -5$. What is the graph of $x < -5$? What is the corresponding interval? Describe the relationship between these sets.

EXAMPLE 6 **Solving a Linear Inequality** Solve and graph:

$$2(2x + 3) < 6(x - 2) + 10$$

SOLUTION

$2(2x + 3) < 6(x - 2) + 10$ *Remove parentheses.*

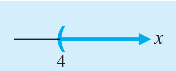
$4x + 6 < 6x - 12 + 10$ *Combine like terms.*

$4x + 6 < 6x - 2$ *Subtract 6x from both sides.*

$-2x + 6 < -2$ *Subtract 6 from both sides.*

$-2x < -8$ *Divide both sides by −2 and reverse the sense of the inequality.*

$x > 4$ or $(4, \infty)$



Notice that in the graph of $x > 4$, we use a parenthesis through 4, since the point 4 is not included in the graph.

Matched Problem 6 Solve and graph: $3(x - 1) \le 5(x + 2) - 5$

EXAMPLE 7 **Solving a Double Inequality** Solve and graph: $-3 < 2x + 3 \le 9$

SOLUTION We are looking for all numbers x such that $2x + 3$ is between -3 and 9, including 9 but not -3. We proceed as before except that we try to isolate x in the middle:

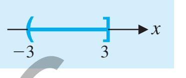
$$-3 < 2x + 3 \le 9$$

$$-3 - 3 < 2x + 3 - 3 \le 9 - 3$$

$$-6 < 2x \le 6$$

$$\frac{-6}{2} < \frac{2x}{2} \le \frac{6}{2}$$

$$-3 < x \le 3 \quad \text{or} \quad (-3, 3]$$



Matched Problem 7 Solve and graph: $-8 \le 3x - 5 < 7$

Note that a linear equation usually has exactly one solution, while a linear inequality usually has infinitely many solutions.

Applications

To realize the full potential of algebra, we must be able to translate real-world problems into mathematics. In short, we must be able to do word problems.

Here are some suggestions that will help you get started:

Procedure for Solving Word Problems

1. Read the problem carefully and introduce a variable to represent an unknown quantity in the problem. Often the question asked in a problem will indicate the best way to introduce this variable.
2. Identify other quantities in the problem (known or unknown), and whenever possible, express unknown quantities in terms of the variable you introduced in Step 1.
3. Write a verbal statement using the conditions stated in the problem and then write an equivalent mathematical statement (equation or inequality).
4. Solve the equation or inequality and answer the questions posed in the problem.
5. Check the solution(s) in the original problem.

EXAMPLE 8 **Purchase Price** John purchases a computer from an online store for $851.26, including a $57 shipping charge and 5.2% state sales tax. What is the purchase price of the computer?

SOLUTION **Step 1** **Introduce a variable for the unknown quantity.** After reading the problem, we decide to let x represent the purchase price of the computer.

Step 2 **Identify quantities in the problem.**

Shipping charges:	$57
Sales tax:	$0.052x$
Total cost:	$851.26

Step 3 **Write a verbal statement and an equation.**

$$\text{Price} + \text{Shipping Charges} + \text{Sales Tax} = \text{Total Order Cost}$$
$$x + 57 + 0.052x = 851.26$$

Step 4 **Solve the equation and answer the question.**

$$x + 57 + 0.052x = 851.26 \quad \text{Combine like terms.}$$
$$1.052x + 57 = 851.26 \quad \text{Subtract 57 from both sides.}$$
$$1.052x = 794.26 \quad \text{Divide both sides by 1.052.}$$
$$x = 755$$

The price of the computer is \$755.

Step 5 **Check the answer in the original problem.**

Price =	\$755.00
Shipping charges =	\$57.00
Tax $0.052 \cdot 755$ =	\$39.26
Total =	\$851.26

Matched Problem 8 Mary paid 8.5% sales tax and a \$190 title and license fee when she bought a new car for a total of \$28,400. What is the purchase price of the car?

The next example involves the important concept of **break-even analysis**, which is encountered in several places in this text. Any manufacturing company has **costs**, ***C***, and **revenues**, ***R***. The company will have a **loss** if $R < C$, will **break even** if $R = C$, and will have a **profit** if $R > C$. Costs involve **fixed costs**, such as plant overhead, product design, setup, and promotion, and **variable costs**, which are dependent on the number of items produced at a certain cost per item.

EXAMPLE 9 **Break-Even Analysis** A multimedia company produces DVDs. One-time fixed costs for a particular DVD are \$48,000, which include costs such as filming, editing, and promotion. Variable costs amount to \$12.40 per DVD and include manufacturing, packaging, and distribution costs for each DVD actually sold to a retailer. The DVD is sold to retail outlets at \$17.40 each. How many DVDs must be manufactured and sold in order for the company to break even?

SOLUTION **Step 1** Let x = number of DVDs manufactured and sold.

Step 2

$$C = \text{cost of producing } x \text{ DVDs}$$
$$R = \text{revenue (return) on sales of } x \text{ DVDs}$$
$$\text{Fixed costs} = \$48{,}000$$
$$\text{Variable costs} = \$12.40x$$
$$C = \text{Fixed costs} + \text{variable costs}$$
$$= \$48{,}000 + \$12.40x$$
$$R = \$17.40x$$

Step 3 The company breaks even if $R = C$; that is, if

$$\$17.40x = \$48{,}000 + \$12.40x$$

Step 4

$$17.4x = 48{,}000 + 12.4x \quad \text{Subtract 12.4x from both sides.}$$
$$5x = 48{,}000 \quad \text{Divide both sides by 5.}$$
$$x = 9{,}600$$

The company must make and sell 9,600 DVDs to break even.

Step 5 Check:

Costs	**Revenue**
48,000 + 12.4(9,600)	17.4(9,600)
= \$167,040	= \$167,040

Matched Problem 9 How many DVSs would a multimedia company have to make and sell to break even if the fixed costs are \$36,000, variable costs are \$10.40 per DVD, and the DVDs are sold to retailers for \$15.20 each?

EXAMPLE 10 **Consumer Price Index** The Consumer Price Index (CPI) is a measure of the average change in prices over time from a designated reference period, which equals 100. The index is based on prices of basic consumer goods and services. Table 2 lists the CPI for several years from 1960 to 2005. What net annual salary in 2005 would have the same purchasing power as a net annual salary of \$13,000 in 1960? Compute the answer to the nearest dollar. (*Source*: U.S. Bureau of Labor Statistics)

Table 2 CPI (1982–1984 = 100)

Year	Index
1960	29.6
1975	53.8
1990	130.7
2005	195.3

SOLUTION

Step 1 Let $x =$ the purchasing power of an annual salary in 2005.

Step 2 Annual salary in 1960 = \$13,000

CPI in 1960 = 29.6

CPI in 2005 = 195.3

Step 3 The ratio of a salary in 2005 to a salary in 1960 is the same as the ratio of the CPI in 2005 to the CPI in 1960.

$$\frac{x}{13{,}000} = \frac{195.3}{29.6}$$ Multiply both sides by 13,000.

Step 4

$$x = 13{,}000 \cdot \frac{195.3}{29.6}$$
$$= \$85{,}774 \text{ per year}$$

Step 5

Salary Ratio	**CPI Ratio**
$\frac{85{,}774}{13{,}000} = 6.598$	$\frac{195.3}{29.6} = 6.59797$

Note: The slight difference in these ratios is due to rounding the 2005 salary to the nearest dollar.

Matched Problem 10 What net annual salary in 1975 would have had the same purchasing power as a net annual salary of \$100,000 in 2005? Compute the answer to the nearest dollar.

Exercises 1-1

A

Solve Problems 1–6.

1. $2m + 9 = 5m - 6$

2. $3y - 4 = 6y - 19$

3. $2x + 3 < -4$

4. $5x + 2 > 1$

5. $-3x \geq -12$

6. $-4x \leq 8$

Solve Problems 7–10 and graph.

7. $-4x - 7 > 5$

8. $-2x + 8 < 4$

9. $2 \leq x + 3 \leq 5$

10. $-4 < 2y - 3 < 9$

Solve Problems 11–24.

11. $\frac{x}{4} + \frac{1}{2} = \frac{1}{8}$

12. $\frac{m}{3} - 4 = \frac{2}{3}$